Hőegyenlet newtoni mechanikából?

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"Ha egy matematikai diszciplína messzire távolodik tapasztalati forrásától, az súlyos veszélyeket rejt magában. A forrásától eltávolodott folyó jelentéktelen ágak sokaságává különül el és a diszciplína részletek és bonyodalmak szervezetlen tömegévé válik."

(in The Works of the Mind (1947))

Contents

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2. Gaspard-Gilbert model and two-step strategy
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4. Step2: Mesoscopic model and gap bound of Grigo-Khanin-Sz.
Heat Equation (no mass flow)

\[
\frac{\partial T(x, t)}{\partial t} = \frac{1}{c} \nabla [\kappa \nabla T(x, t)]
\]

\(c\) - specific heat/unit volume (= 1)
\(\kappa = \kappa(T)\) - thermal conductivity

For a wide class of models: \(\kappa(T) = \text{const.} \sqrt{T}\)
(insulating materials, or gas of weakly/rarely interacting particles)

However: de Roeck-IP Tóth: (in progress)

\[
\kappa(T) = \frac{\text{const.}}{T^{3/2}}
\]

Oscillating interest since late 60’s. Some surveys:

Recent wave:

- **Ruelle, CMP, 1996** - **Dolgopyat, Inv. Math, 2004**: Linear response theory vs. differentiability of SRB states

- **Eckmann-Young, CMP, 2006** (also Lin-Young, 2010) non-equilibrium steady states under phenomenological assumptions
  (Krámli-Simányi-Sz., JSP, 1987: RWwIS, temperature profile by postulating local equilibrium)

- **Gaspard-Gilbert, PhRL, 2008–**: model of localized hard disks (balls) - a two step approach:
  1. derive a **mesoscopic master equ.** from the **microscopic kinetic equ.** of the Hamiltonian model
     - in the **rare (but strong) interaction limit**
     - it is a **Markov jump process**
  2. derive the **macroscopic heat equ.** from the **mesoscopic master equ.**
No mass transport (Coquard et al., J. Non-Crystalline Solids, 2013: Modelling of conductive heat transfer though nano-structured porous silica materials)
Parameter choice of Gaspard-Gilbert, ’08.

- box size: 1; periodic b. c.’s along \( y \)-axis
- chain length = \( N \);
- radius of **fixed** scatterers (shaded circles) = \( \rho_f \)
- radius of **moving** disks (empty circles) = \( \rho_m \)
- condition of **confinement**: \( \rho_f + \rho_m > 1/2 \)
- condition of **conductivity**:
  \[
  \rho_m > \rho_{\text{crit}} = \sqrt{(\rho_f + \rho_m)^2 - (1/2)^2}
  \]
- small parameter \( \varepsilon = \rho_m - \rho_{\text{crit}} > 0 \)

Gaspard-Gilbert’s trick:

- **Keep** \( \rho_f + \rho_m =: \rho \) **fixed**
- If \( \rho_m = \rho_{\text{crit}} \), then we have \( N \) non-interacting billiards. Moreover, their phase spaces only depend on \( \rho \)!
Ernst-Dorfman, ’89: The kinetic equ. for the $N$-particle density $p_N(q_1, v_1, \ldots, q_N, v_N; t)$ is

$$\partial_t p_N = \sum_{j=1}^{N} \left(-v_j \partial q_j + K_{wall,j} + C_{j,j+1}\right) p_N$$

- the first two terms on the RHS describe the billiard dynamics of each disk within its cell (denote wall collision rate by $\nu_{\text{wall},\varepsilon}$)
- the third one: the interaction of neighboring disks provides the energy transfer (denote binary collision rate by $\nu_{\text{bin},\varepsilon}$)
Scale separation

Gaspard-Gilbert '08. '09: Scale separation at
\[ \varepsilon \downarrow 0, \quad \text{i. e.} \quad \nu_{\text{wall,} \varepsilon}(\sim \nu_{\text{wall, crit}} > 0) \gg \nu_{\text{bin,} \varepsilon} \to 0 \]

1. they derive a master equation for the density
\[ P_N(E_1, \ldots, E_N; t) \quad (E_j = v_j^2: 1 \leq j \leq N) \]

2. HDL: from the master equation they obtain the coefficient of heat conductivity: \( \kappa = \text{const.} \sqrt{T} \) (??, but Sasada, ms in progress)

2013, B. Fernandez: True understanding

2011, S. Olla: GG’s derivation of HDL uses an incorrect symmetry argument
2013, M. Sasada: correction, based on Green-Kubo (still heuristic)
Challenge: Rigorous theory for GG

Keller-Liverani, CMP, 2009: rare interaction limit. CML, i.e. interval maps **coupled by collisions**. Result: Uniqueness of SRB and exponential space-time corr. decay.


Dolgopyat-Nándori (in progress): Heat equ. from deterministic dynamics BUT with stochastic bry conditions corresponding to different temperatures

Li Yau-LS Young (in progress): stochastic dynamics in slab w. different bry temperatures. Task: definition of local temperature!?
By Hirata-Saussol-Vaienti, CMP, 1999 (also Collet-Eckmann, Springer, 2006; Chazottes-Collet, EThDS, 2013): If

- a dynamical system \((M, T, \mu)\) is mixing in a controlled way (e.g., \(\alpha\)-mixing)
- and \(A_\varepsilon\) is a sequence of nice subsets (to avoid e.g., neighborhoods of periodic points) with \(\lim_{\varepsilon\to 0} \mu(A_\varepsilon) = 0\)

then the successive entrance times of the dynamics into \(A_\varepsilon\) form a Poisson process on the time scale of \(\mu(A_\varepsilon)^{-1}\).

For simplicity let \(N = 2\) with free boundary conditions along \(x\)-axis. The model is isomorphic to a 4D semi-dispersing billiard. It is K-mixing, but no mixing rate is known. (Bálint-Tóth, ’08 is for dispersive billiards, only, and, moreover, it is hypothetical).
Conjecture for 2-disk chain, with IP Tóth

$N = 2$, free boundary along $x$-axis. Dynamics: $(\varepsilon = \rho_m - \rho_{\text{crit}})$

$\left( M_\varepsilon = \{ q_1, v_1; q_2, v_2 | \text{dist}(q_1,q_2) \geq 2\rho_m, v_1^2 + v_2^2 = 1 \} \right), S^\mathbb{R}, \mu_\varepsilon )$.

Denote by $0 < \tau_{1,\varepsilon} < \tau_{2,\varepsilon} < \ldots$ successive binary collision times of the two disks. Then, as $\varepsilon \to 0$

- $E_1(\nu_{\text{bin}}, \varepsilon, t), E_2(\nu_{\text{bin}}, \varepsilon, t)$ converges to a jump Markov process on the state space $E_1 + E_2 = 1$ where $E_j(t) = \frac{1}{2}v_j^2(t); j = 1, 2$
- the transition kernel $k(E_1^+|E_1^-)$ is calculated by verifying Boltzmann’s 'microscopic chaos' property (cf. scattering cross section)

Note: $\nu_{\text{bin},\varepsilon} \sim \text{const.}\varepsilon^3$. 
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Envisioned proof

- since binary collisions are rare, **most of the time the two disks evolve independently**
- between two binary collisions - with an overwhelming probability - there is **averaging in each of the in-cell, 2D billiard dynamics**
- for these typically long time intervals it is natural to **apply Chernov-Dolgopyat averaging**
- for that purpose
  - one checks that for an **incoming proper family of stable pairs, so is the outgoing family ???**
  - one applies **martingale approximation for jump processes (á la Ethier-Kurtz)**
Paradigm: in-cell dynamics is billiard in cpct constant negative curvature

P Bálint-P Nándori-D. Sz.-T. Tasnády-IP Tóth; in progress
Geodesics in Poincaré model of hyperbolic geometry
Phase space: \( \text{dim } M = 7 \). Scatterers 16 boundaries, + 1 "cylinder"

Goal: Mimic the geometry of GG model.
For a reasonable set of $0 < \rho_{cr} < \rho_m$ the billiard model is ergodic and K-mixing. **Tools:**

- (un)stable invariant manifolds of the billiard in the octagon with rectangular geodesic edges are (un)stable invariant manifolds of the non-compact model reflected through the edges of the octagon
- Chernov-Sinai, 1987 type local ergodicity theorem in the form of Liverani-Wojtkowski, 1995
- Krámlí-Simányi-Sz., 1989: method for semi-dispersing billiards
Second fundamental form of the superficies

Skew cylinder: superficies set (i.e. generator in Euclidean geometry): $A = \{(x, y)|d(x, y) = 2\rho_m\}$

Notation:
- $\mathbf{n}$ = collision unit normal pointing to the right
- $\mathbf{t}$ = collision unit tangent ($\mathbf{n}$ rotated with angle $\pi/2$)

Local orthogonal "basis":

\[
N = \begin{pmatrix} -\mathbf{n} \\ \mathbf{n} \end{pmatrix} \quad T_1 = \begin{pmatrix} \mathbf{n} \\ \mathbf{n} \end{pmatrix} \quad T_2 = \begin{pmatrix} \mathbf{t} \\ \mathbf{t} \end{pmatrix} \quad T_3 = \begin{pmatrix} \mathbf{t} \\ -\mathbf{t} \end{pmatrix}
\]

Eigenvalues: 0, $\tanh \rho_m$, $\coth \rho_m$
Dimension reduction: Billiard coupled to a piston

Collision rule: $v_x^+ = v^-, v_x^+ = v_x^-, v_y^+ = v_y^-.$

$(q_x, q_y) \in Q,$

$q \in [-\varepsilon, L - \varepsilon]$
Transition kernel

\[ k(E'_L|E_L) \sim \varepsilon^2 \frac{\tan \beta}{2\pi L|Q|} \frac{\sqrt{1 - \min\{E_L, E'_L\}}}{\sqrt{1 - E'_L\sqrt{E_L + E'_L} - 1}} \mathbb{1}\{E_L + E'_L > 1\} \]

thus for the rate \( \Lambda(E_L) = \int_0^1 k(e|E_L)de \) one has \( \varepsilon^{-2} \Lambda(E_L) = \)

\[ \begin{cases} \frac{\tan \beta}{L|Q|} \sqrt{1 - E_L} & \text{if } E_L < 1/2, \\ \frac{\tan \beta}{L|Q|} \left[ \sqrt{2E_L - 1} + \frac{\sqrt{1 - E_L}}{2} \left( \frac{\pi}{2} - \arcsin \left( 3 - \frac{2}{E_L} \right) \right) \right] & \text{if } E_L \geq 1/2. \end{cases} \]
Step 2: A (mesoscopic) stochastic model of energies


State space: \( x = (x_1, \ldots, x_N) \in \mathbb{R}_+^N \)

Generator \( \mathcal{L} \) of the continuous time Markov jump process \( X(t) \) (given on \( \mathbb{R}_+^N \)) acting on bounded functions \( A : \mathbb{R}_+^N \rightarrow \mathbb{R} \) is

\[
\mathcal{L} A(x) = \sum_{i=1}^{N-1} \Lambda(x_i, x_{i+1}) \int P(x_i, x_{i+1}; d\alpha) [A(T_{i,\alpha} x) - A(x)]
\]

where \( P(x_i, x_{i+1}; d\alpha) \) is a probability measure on \([0, 1]\).

The maps \( T_{i,\alpha} \), modelling energy exchange between the neighboring sites \( i \) and \( i+1 \), are defined by

\[
T_{i,\alpha}(x_i) = \alpha(x_i + x_{i+1})
\]

\[
T_{i,\alpha}(x_{i+1}) = (1 - \alpha)(x_i + x_{i+1})
\]
Remarks

- Total energy is invariant, i.e.

\[ S_{\epsilon,N} = \left\{ x \in \mathbb{R}^N_+ \mid \sum_{i=1}^{N} \frac{1}{N} x_i = \epsilon \right\} \]

is invariant wrt dynamics;

- Standing assumptions:
  1. for any \( E, E' \), the kernel \( P(E, E', d\alpha) \)
     1. is symmetric wrt 1/2;
     2. is never equal to \( \frac{1}{2}(\delta_0 + \delta_1) \) (i.e. \( \{E_1^+, E_2^+\} \neq \{E_1, E_2\} \))
  2. plus an appropriate condition for \( \Lambda \).
Mesoscopic generator in the GG model, case $d = 3$

$$\Lambda(E_1, E_2) = \Lambda_s(E_1 + E_2) \Lambda_r\left(\frac{E_1}{E_1 + E_2}\right)$$

(factorization property!) where

$$\Lambda_s(s) = \sqrt{s} \quad \Lambda_r(\beta) = \frac{2\pi}{6} \frac{\frac{1}{2} + \beta \vee (1 - \beta)}{\sqrt{\beta \vee (1 - \beta)}}$$

and

$$P(x_1, x_2; d\alpha) = P\left(\frac{x_1}{x_1 + x_2}; d\alpha\right) = P(\beta; d\alpha)$$

with $\beta = \frac{x_1}{x_1 + x_2}$ (simple dependence!), where

$$\frac{P(\beta; d\alpha)}{d\alpha} = 3 \frac{1 \wedge \sqrt{\frac{\alpha \wedge (1 - \alpha)}{\beta \wedge (1 - \beta)}}}{2 \frac{1}{2} + \beta \vee (1 - \beta)}.$$
In the limit, as \( N \to \infty \) and \( \xi = i/N, \ t = N^2 \tau \), the empirical process

\[
\sum_{i=1}^{N} \frac{1}{N} \delta x_i(t)
\]

should converge to a process with density \( T(x, t) \) solving

\[
\frac{\partial T(x, t)}{\partial t} = \text{const.} \nabla \left[ \sqrt{T(x, t)} \nabla T(x, t) \right]
\]
Main result for GG

**Theorem (G-Kh-Sz, 12)**

If \( \Lambda_s(s) \) is replaced by any non-negative continuous function, which is bounded away from zero, then, for any \( N \) and \( \epsilon \),

1. For \( d \geq 2 \),
   - The product measure \( \mu(dx) = \nu(dx_1) \cdots \nu(dx_N) \) with \( \nu(dx_1) = \Gamma\left(\frac{d}{2} - 1\right) \) is the unique non-degenerate reversible product measure for \( X(t) \).
2. On every \( S_{\epsilon,N} \) there exists a unique stationary distribution \( \pi_{\epsilon,N} \). This measure is obtained by conditioning \( \mu(dx) \).

2. For \( d \geq 3 \), the spectrum \( \sigma(L) \) of the generator \( L \) acting on \( L^2_{\pi_{\epsilon,N}} \) satisfies

\[
\sigma(L) \subset \left( -\infty, -C \sin^2 \left( \frac{\pi}{N+2} \right) \right) \cup \{0\}
\]

for some constant \( C \), which may depend on the choice of \( \Lambda_s \).
Assume that the rate function $\Lambda$ factorizes and satisfies $\Lambda(a, b) \geq \Lambda^*(a + b)^m$ for some $\Lambda^* > 0$ and $m \geq 0$. (cf. porous medium equ.)

Denote the spectral gap for the $N$-chain by $\gamma^{(m)}(\varepsilon, N)$ wrt reversible measure $\bigotimes_1^N \Gamma(g)$ conditioned to $S_{\varepsilon, N}$ ($g = \frac{d}{2} - 1$ from before).

**Theorem (Sasada: arXiv:1305.4066)**

There exists a positive constant $C$ depending only on $\Lambda^*$, $m$ and $g$ such that $\forall \varepsilon > 0$ and $N \geq 2$

$$\gamma^{(m)}(\varepsilon, N) \geq C\varepsilon^m \frac{1}{N^2}.$$ 

Cf. Kac’ model for heat exchange and Boltzmann equation
THANKS