TURNED IVORY POLYHEDRA

T. Tarnai
Budapest

Contents

CHINESE BALLS

Contemporary Chinese balls

Made of jade and sandalwood, with 12 and 14 holes

Ivory Chinese balls

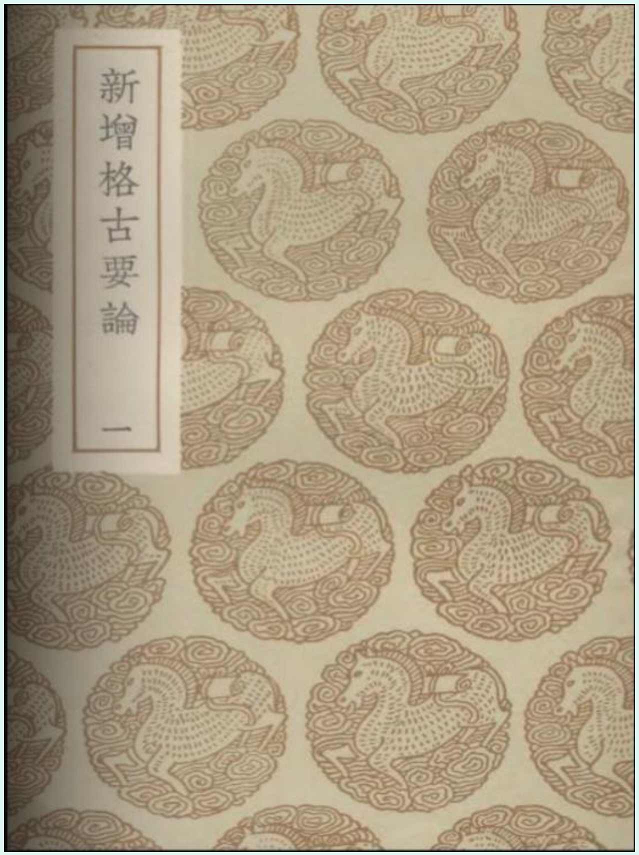
14-layer ball, 19th century, British Museum, London

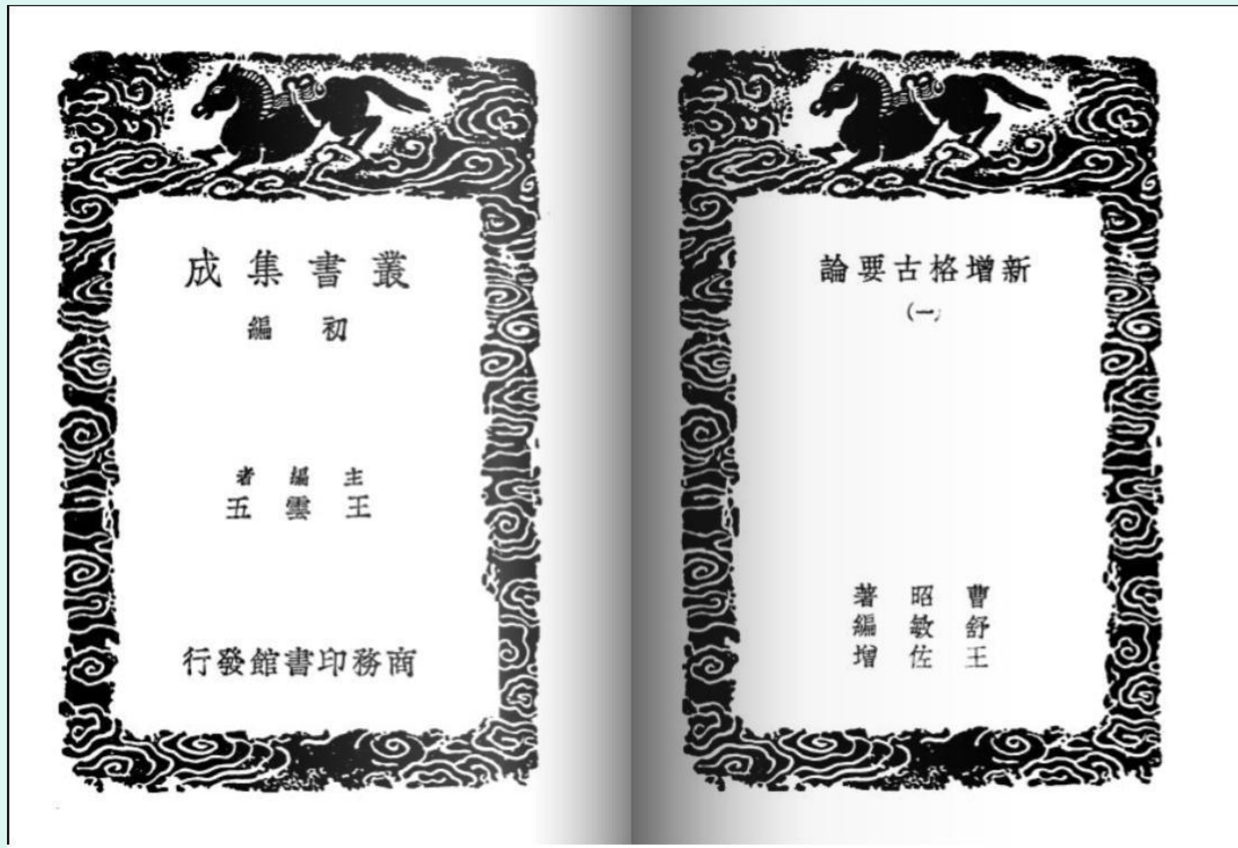
43-layer ball, 19th-20th century, Chen Family Temple, Guangzhou

HISTORY

China

Cao Zhao, *Ge gu yao lun,* Nanjing, 1388





Title page of the 1938 edition

China

- 

  Cao Zhao, *Ge gu yao lun* (Essential Criteria of Antiquities), Nanjing, 1388.

  „I have seen a hollow-centred ivory ball, which had two concentric balls inside it, which can both revolve. It is called 'witch ball'. I was told that it was made for the Palaces of the Song dynasty."

- To our knowledge, there is no trace of „Devil's work ball" in the 15th, 16th and 17th centuries.
- In the 18th and 19th centuries, these balls were made in the Canton region for the European trade, after the first European lathe was imported in 1722 (Shih Ching-fei 2007).

Europe

- The method of making multi-layer spheres and polyhedra was developed in Southern Germany in the second half of the 16th century.

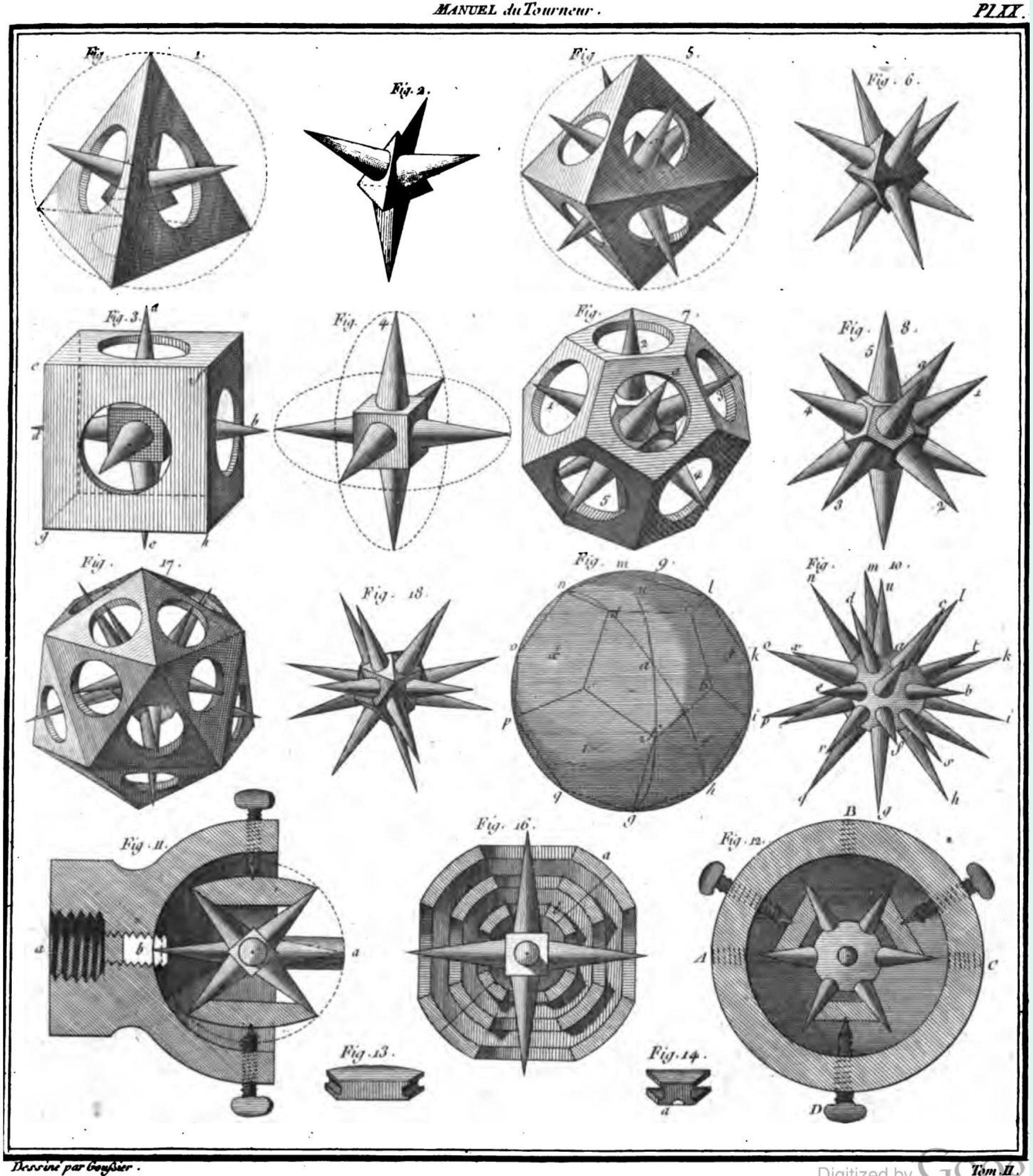
- Earliest works: Georg Wecker, 1581

  Giovanni Ambrogio Maggiore, 1582

  (invented contrefait sphere,1574-77)

- Techniques: Joseph Moxon, *Mechanick Excercises or the Doctrine of Handy-Works*, London, 1678

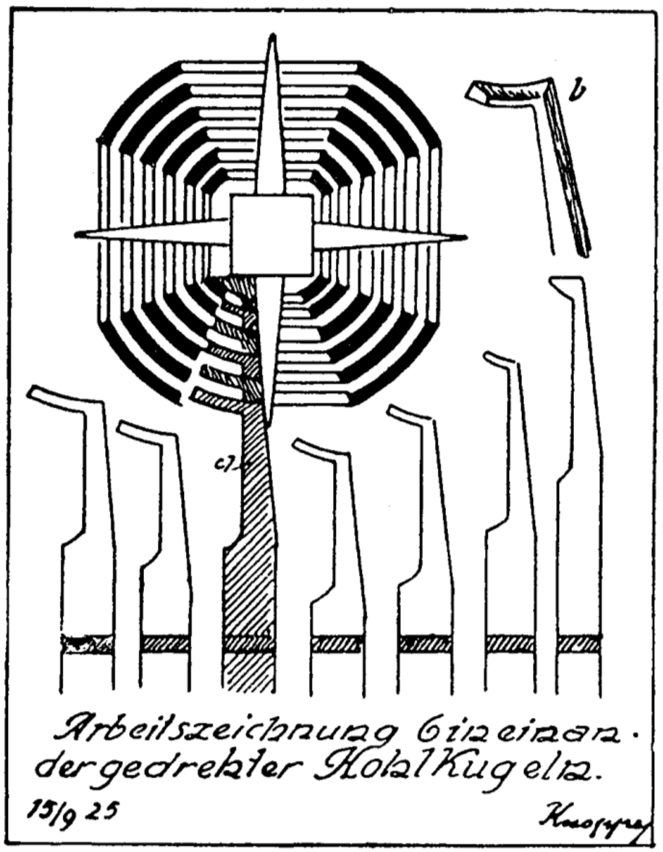
  Charles Plumier, *L'Art du Tourneur*, Lyon, 1701

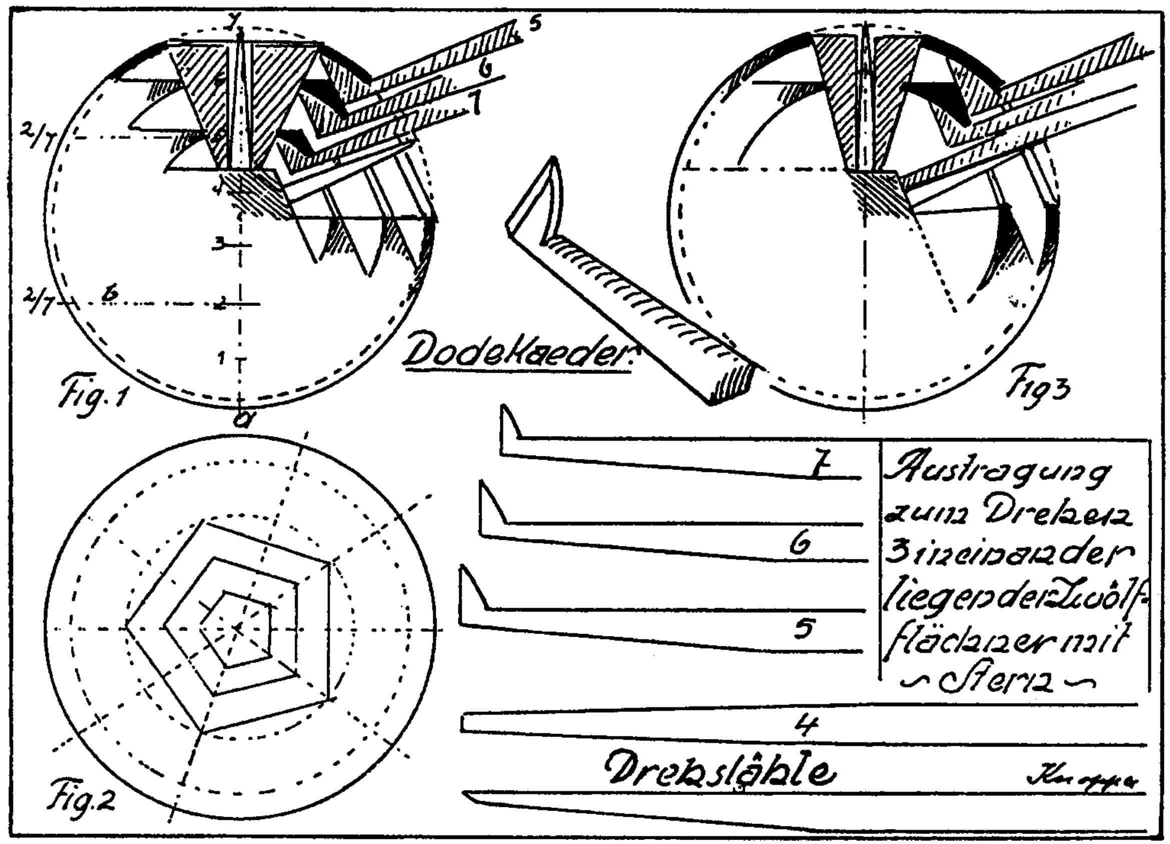


Turned Platonic solids

Bergeron, L.-E., *Manuel du Tourneur*, Paris, 1816.

Gouges applied on a lathe for a multi-layer ball and dodecahedron





Hugo Knoppe, *Meistertechniken der Dechslerkunst*,
Verlag Th. Schäfer, Hannover, 1986

GREAT CARVED IVORY COLLECTIONS

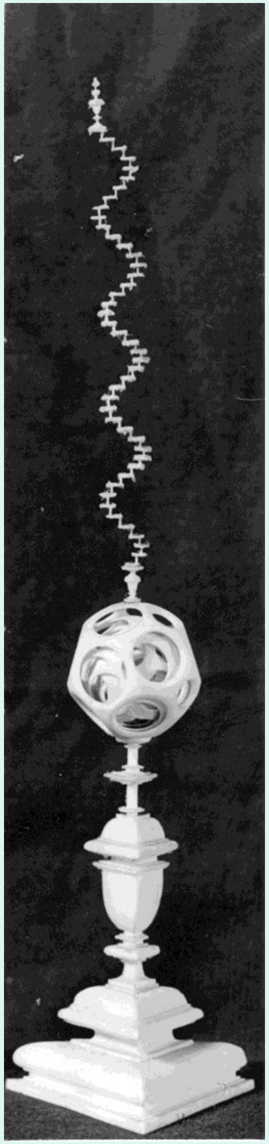
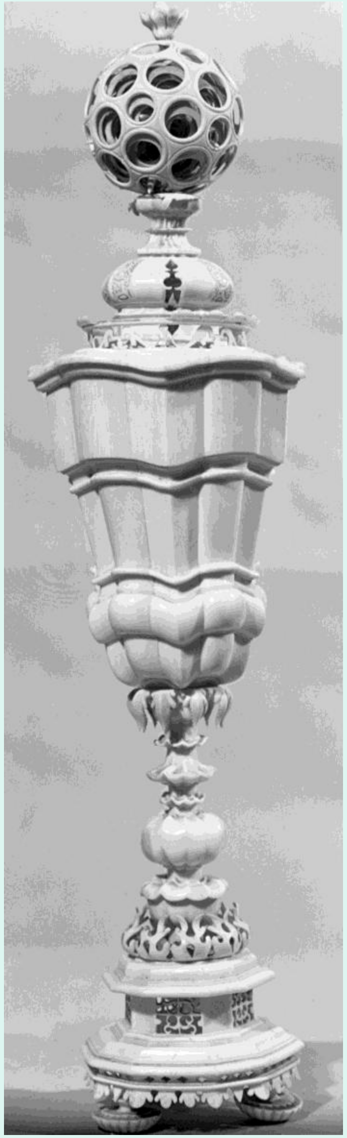
Grünes Gewölbe Dresden

Kunsthistorisches Museum Vienna

Habsburg Kunstkammer Collection

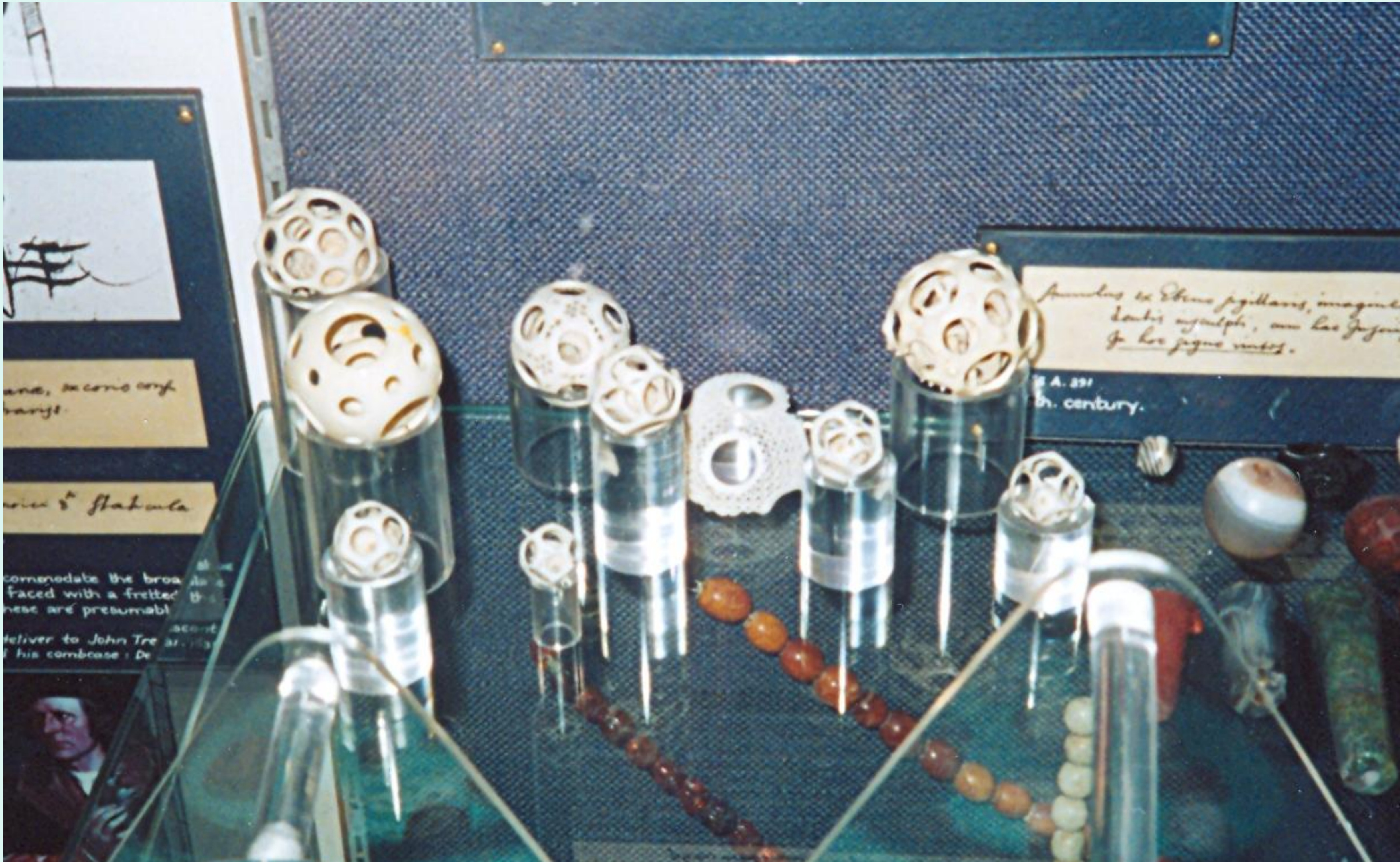
Museo degli Argenti, Florence




Royal Danish Collections

Ashmolean Museum, Oxford



Tradescant Collection, 1656

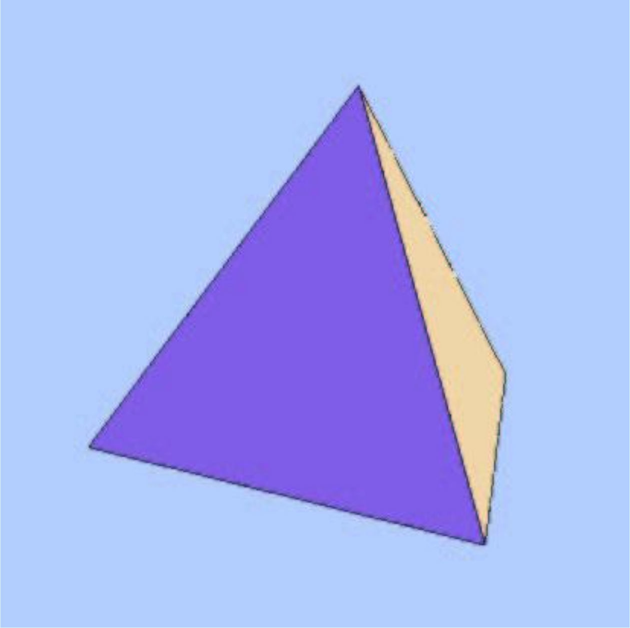
Musée des Arts et Métiers, Paris

François Barreau (1731-1814)

IDENTIFIED POLYHEDRA

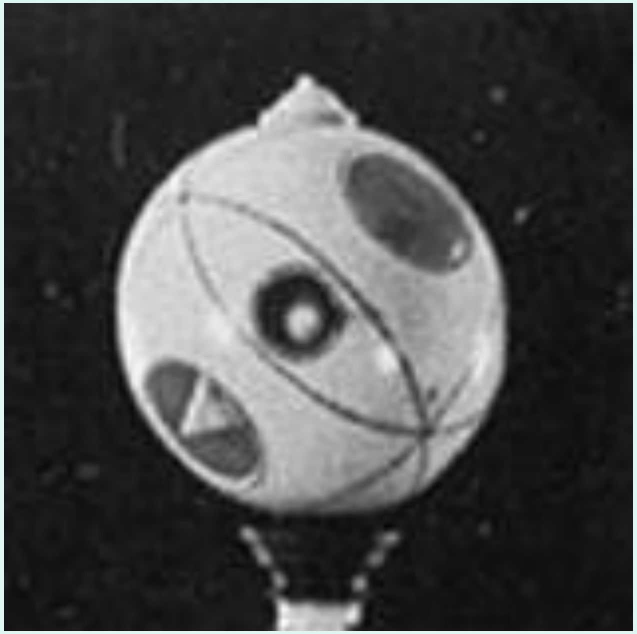
Platonic solids

Tetrahedron



Regular tetrahedron

F=4



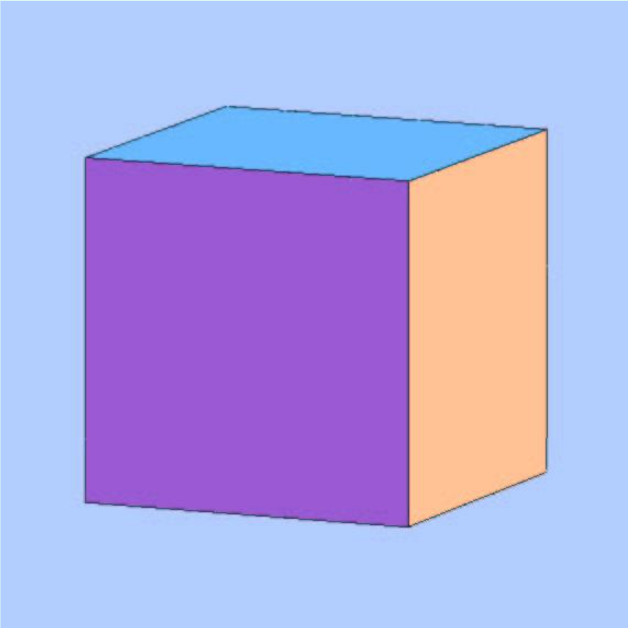
Contrefait sphere

with 4 main openings,
Egidius Lobenigk, ~1590
Grünes Gewölbe, Dresden

Spiked tetrahedron

François Barreau, ~1800
CNAM Paris

Cube



Regular hexahedron

F=6

Contrefait sphere

with 6 circular openings,

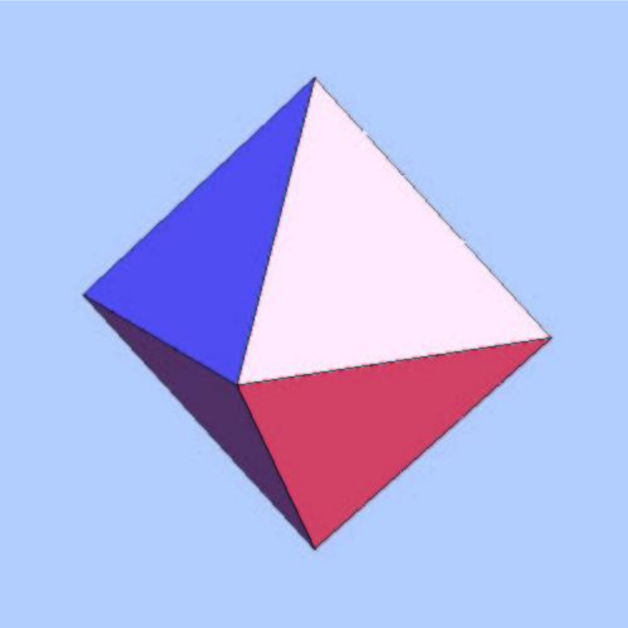
Grünes Gewölbe Dresden

Spiked cube

Probably Dresden, ~ 1600

Grünes Gewölbe Dresden

Octahedron

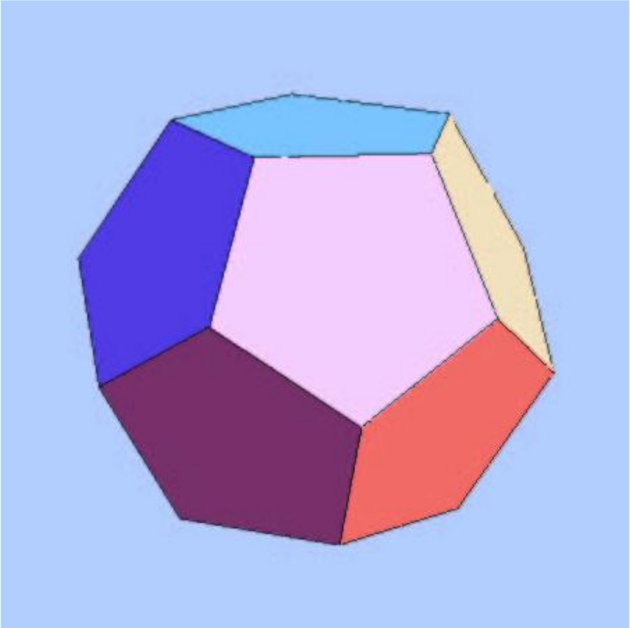


Regular octahedron
F=8

Contrefait sphere
with 8 circular openings,
Egidius Lobenigk, 1589
Grünes Gewölbe Dresden

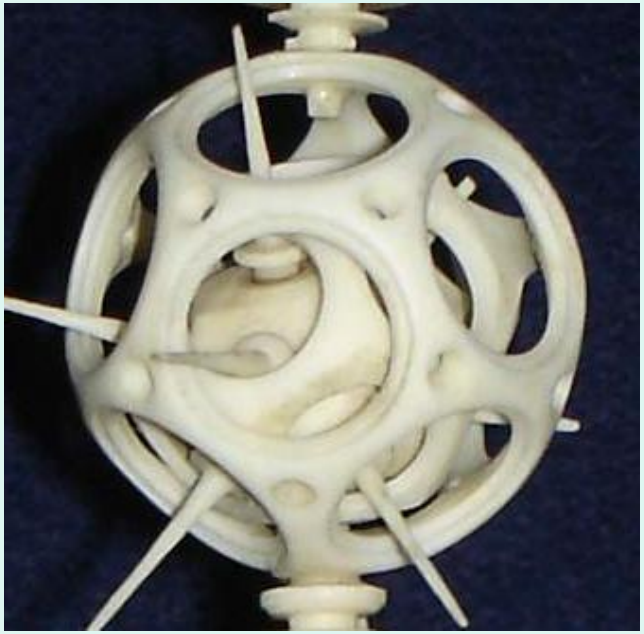
Spiked octahedron
Probably Dresden, ~ 1600
Grünes Gewölbe Dresden

Dodecahedron



Regular dodecahedron

F=12



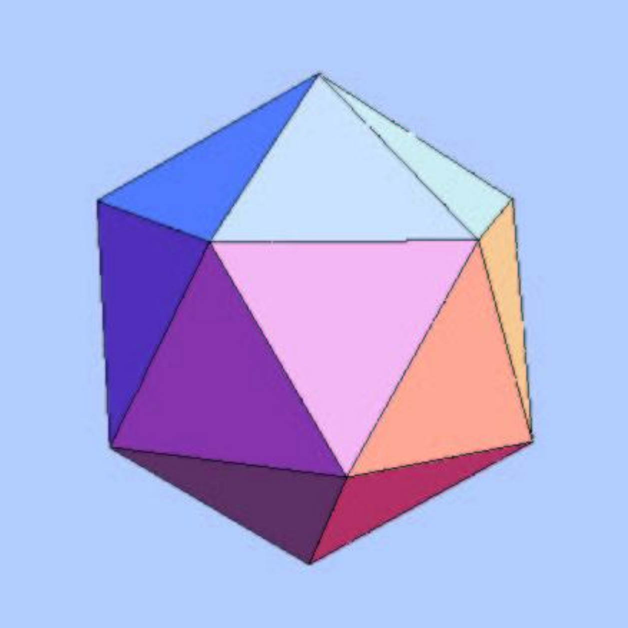
Nested contrefait sphere

with 12 circular openings,
made in Nuremberg,
1600-1650,
Royal Danish Collections,
Copenhagen

Spiked dodecahedron

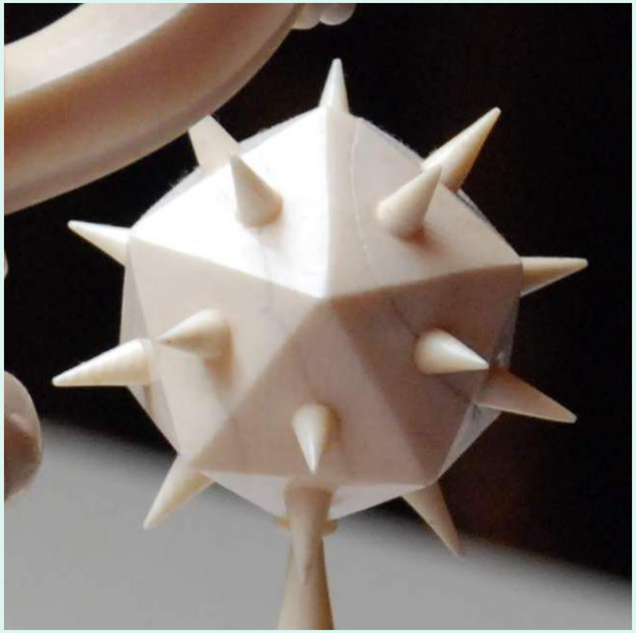
Probably Dresden, ~ 1600
Grünes Gewölbe Dresden

Icosahedron



Regular icosahedron
F=20

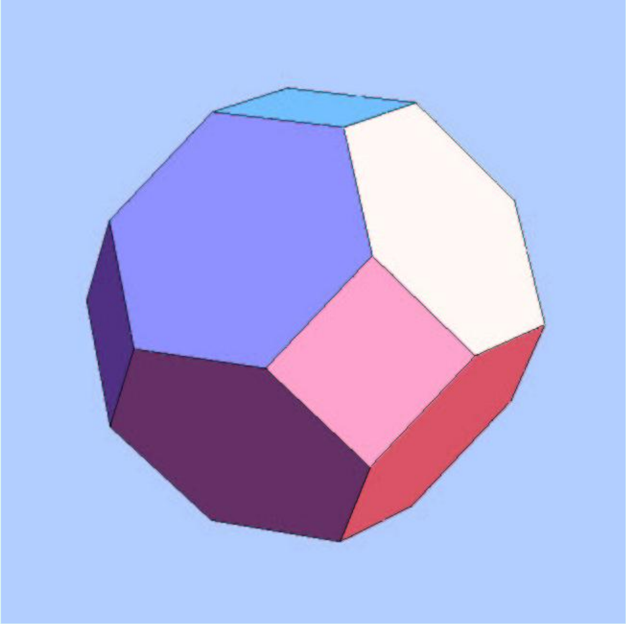
Contrefait sphere
with 20 circular openings
Egidius Lobenigk, 1591
Grünes Gewölbe Dresden



Spiked icosahedron
Probably Dresden, ~ 1600
Grünes Gewölbe Dresden

Archimedean solids

Truncated octahedron



Truncated octahedron

F=14

Nested truncated octahedron

with a circular opening on each face,

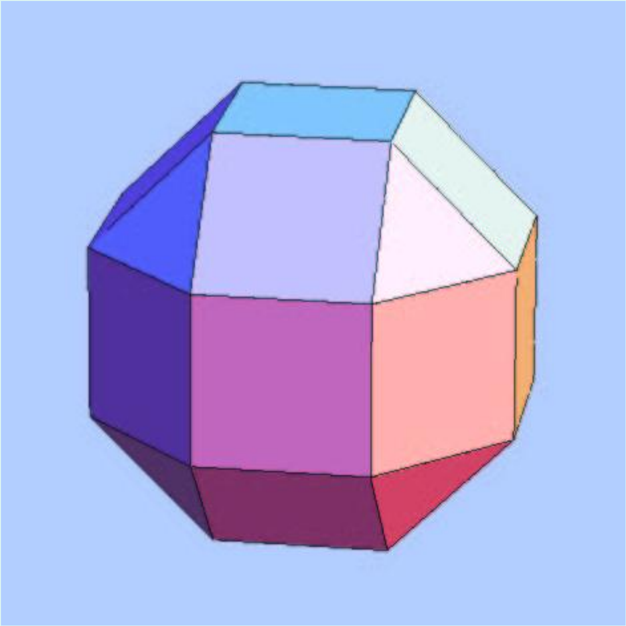
German, around 1650,

KHM Vienna

Spike direction

is the same as that for the cuboctahedron

Rhombicuboctahedron

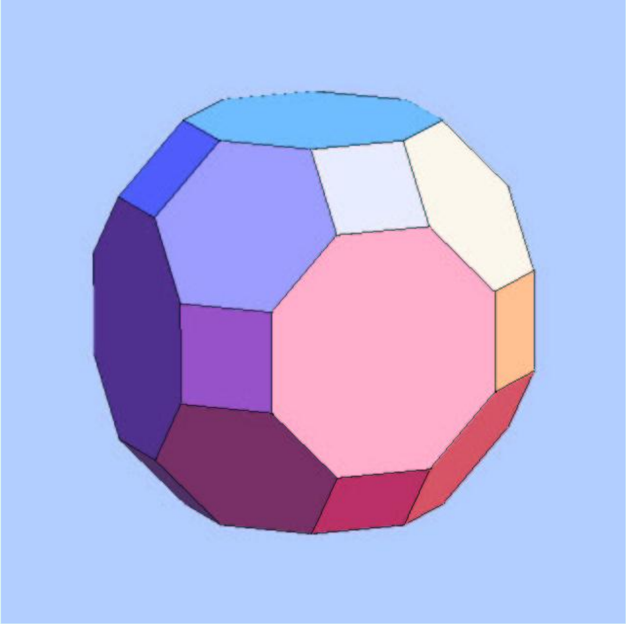


Rhombicuboctahedron
F=26

Contrefait sphere
with 26 circular openings,
François Barreau, ~1800
CNAM Paris

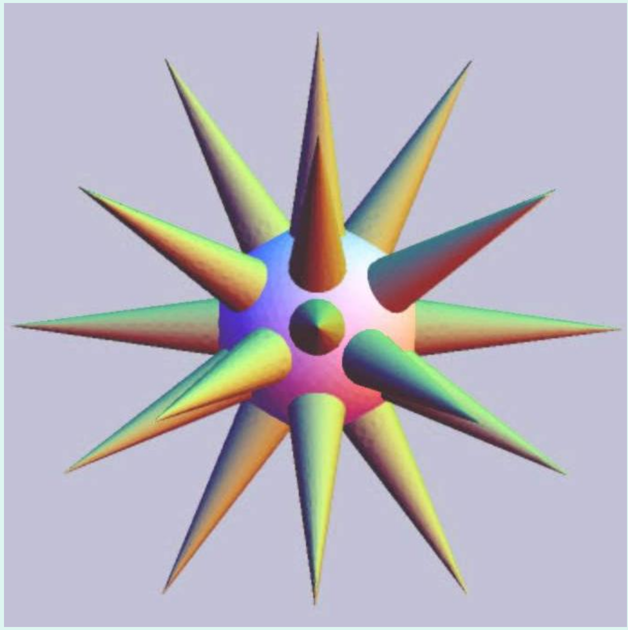
Sphere with 26 spikes
(Spire star,
Reformed church,
Frangepán Str, Budapest)

Great rhombicuboctahedron



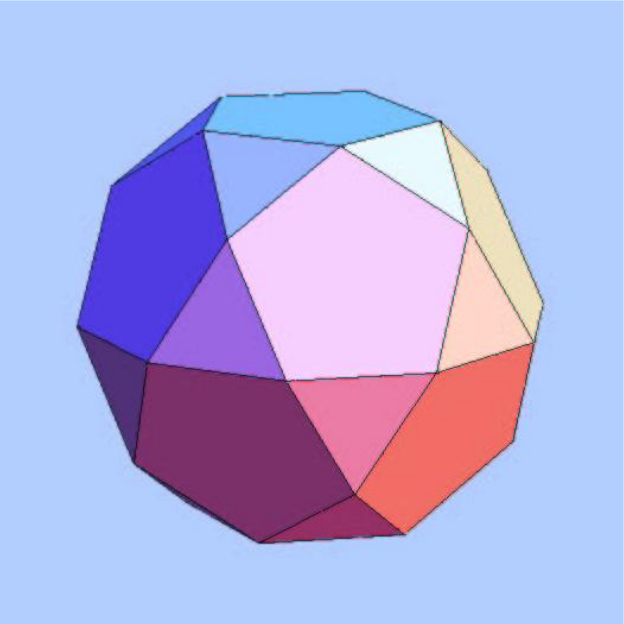
Great rhombicubocta-
hedron
F=26

Nested contrefait sphere
with 26 circular openings,
German, 1600-1650,
Tradescant Collection,
Ashmolean Museum Oxford



Respective spikes

Icosidodecahedron

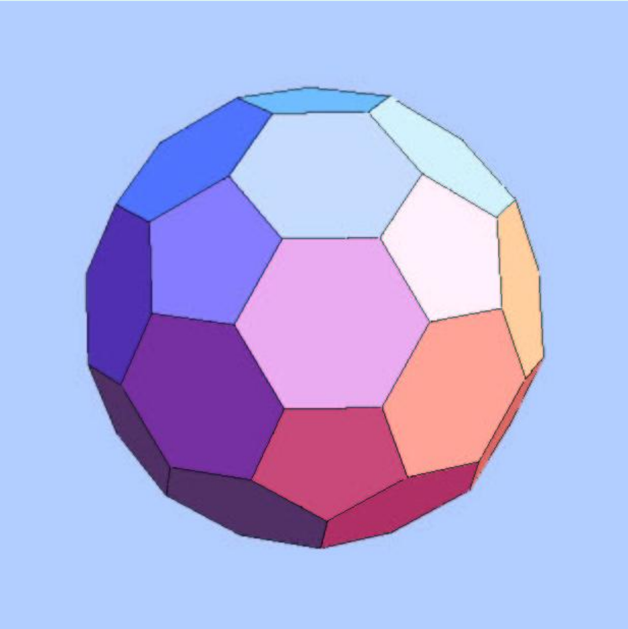


Icosidodecahedron
F=32

Contrefait sphere
with 32 circular openings,
François Barreau, ~1800,
CNAM Paris

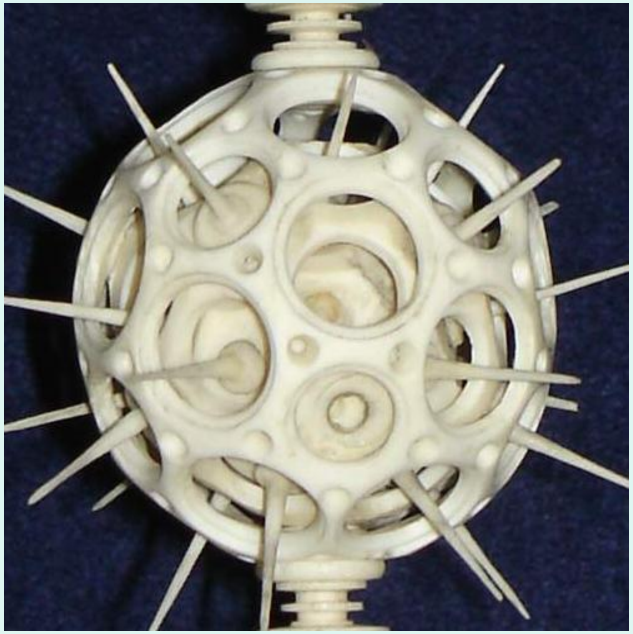
Spiked icosidodeca-
hedron
around 1600,
Grünes Gewölbe Dresden

Truncated icosahedron



Truncated icosahedron

F=32



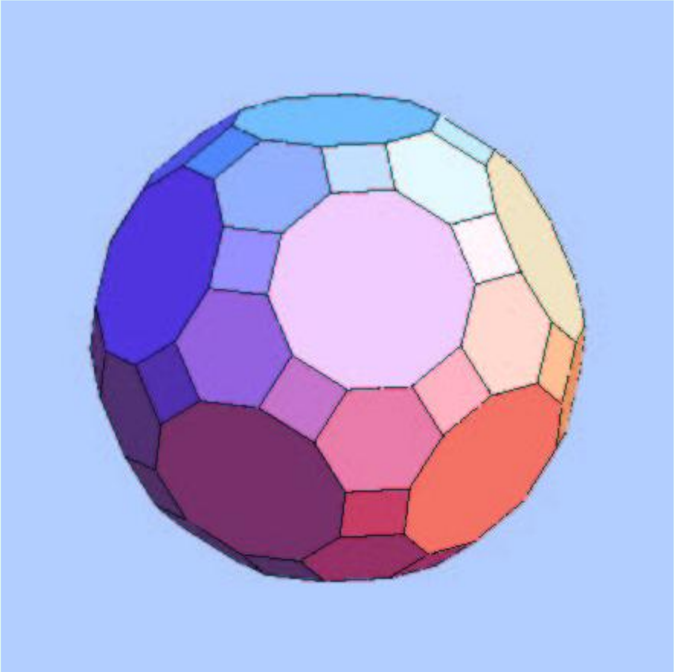
Contrefait sphere

with 32 circular openings,
made in Nuremberg ~1650
Royal Danish Collections,
Copenhagen

Spiked truncated icosahedron

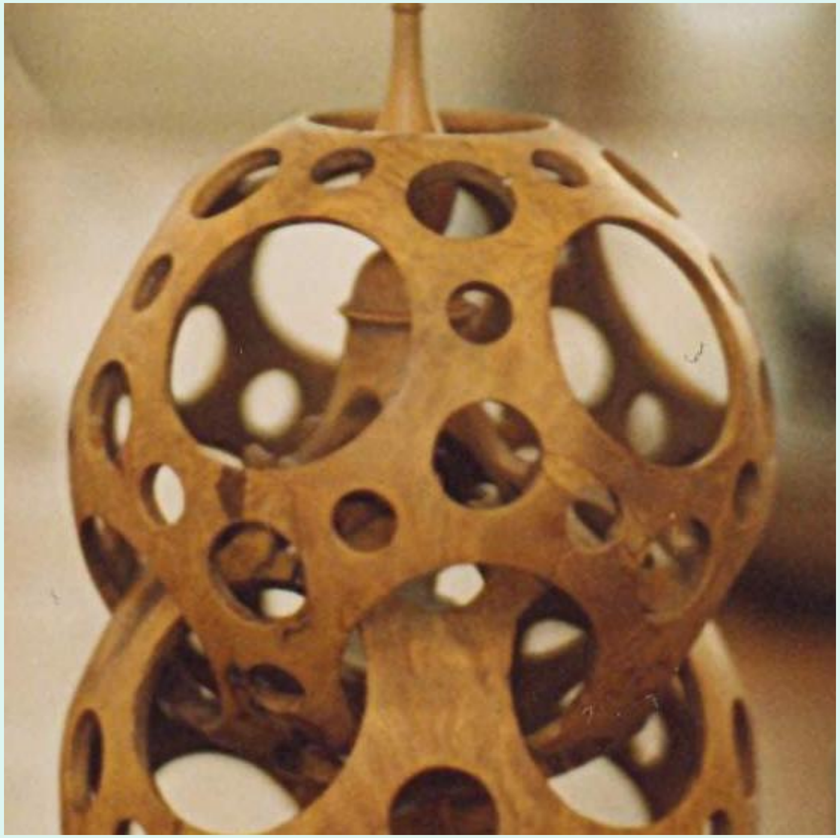
François Barreau, ~1800
CNAM Paris

Great rhombicosidodecahedron



Great rhombicosidodecahedron

F=62

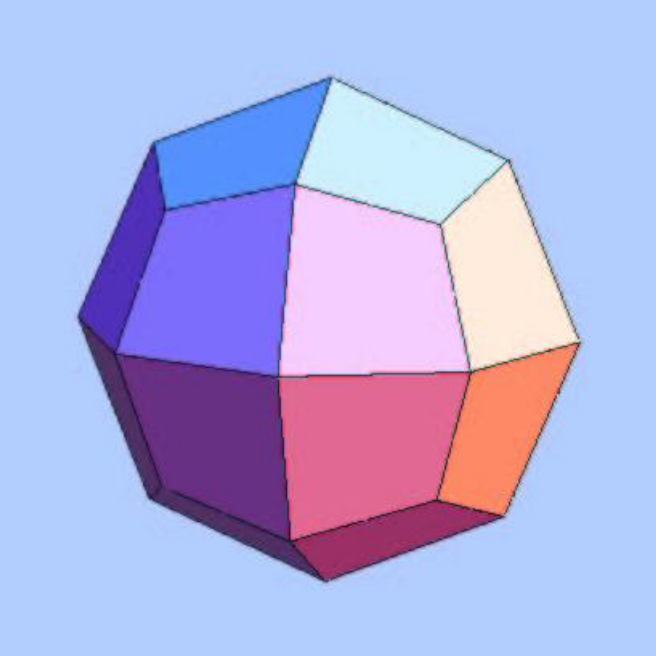


Contrefait sphere

with 62 circular openings,
François Barreau, ~1800
CNAM Paris

Catalan solids

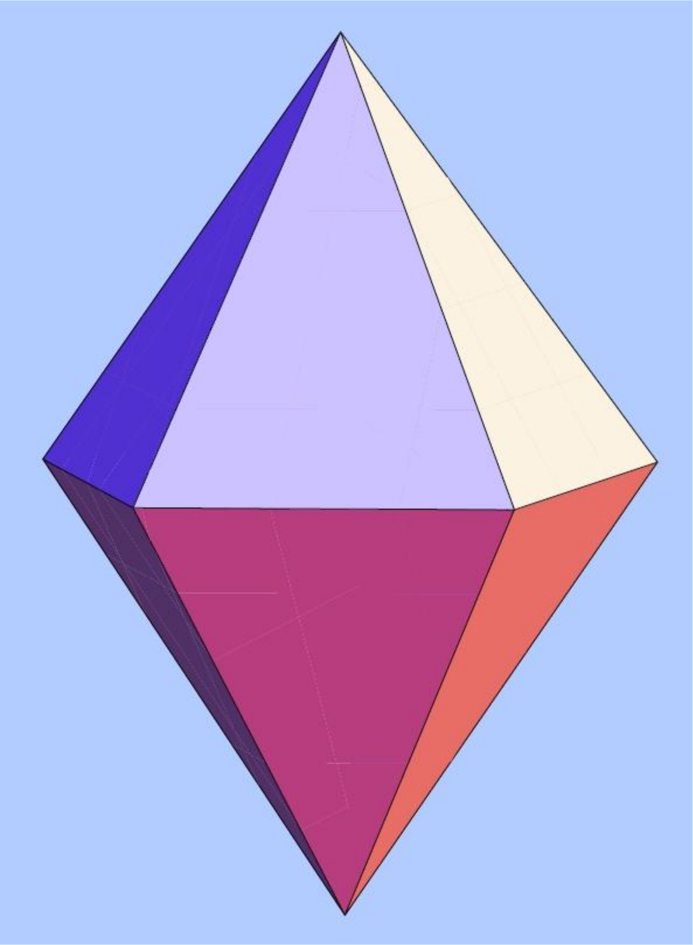
Deltoidal icositetrahedron



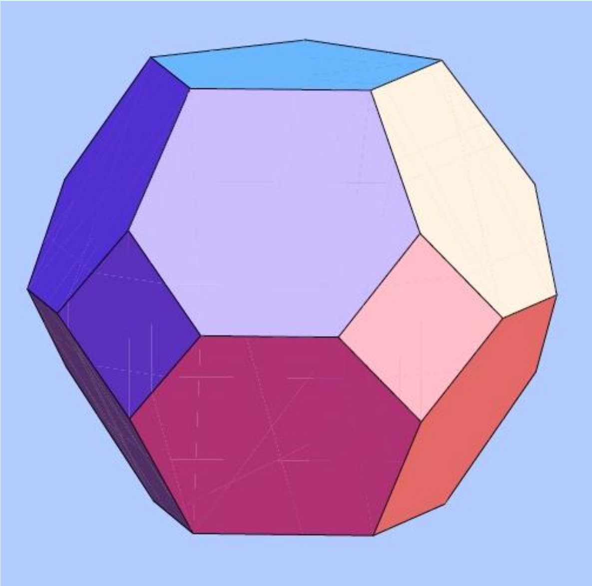
Deltoidal icositetrahedron
F=24

Nested contrefait sphere
with 24 circular openings
Historisches Museum Frankfurt, cca 1580?

Pentagonal dipyramid



Pentagonal dipyramid



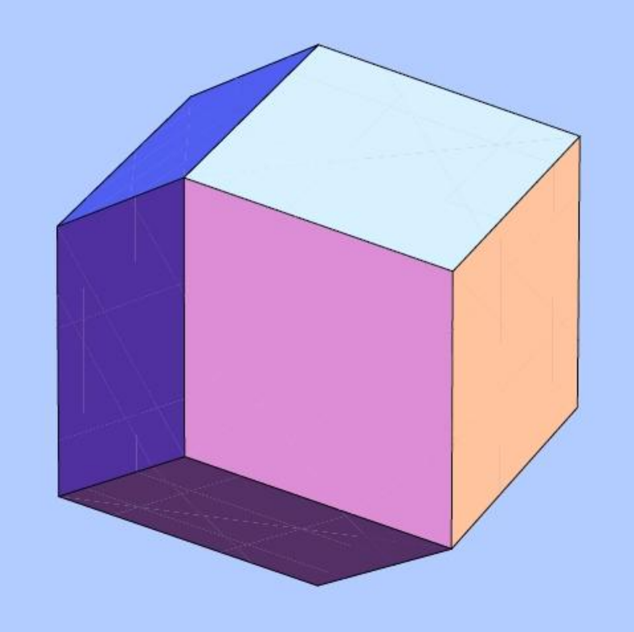
Truncated pentagonal dipyramid

F=17

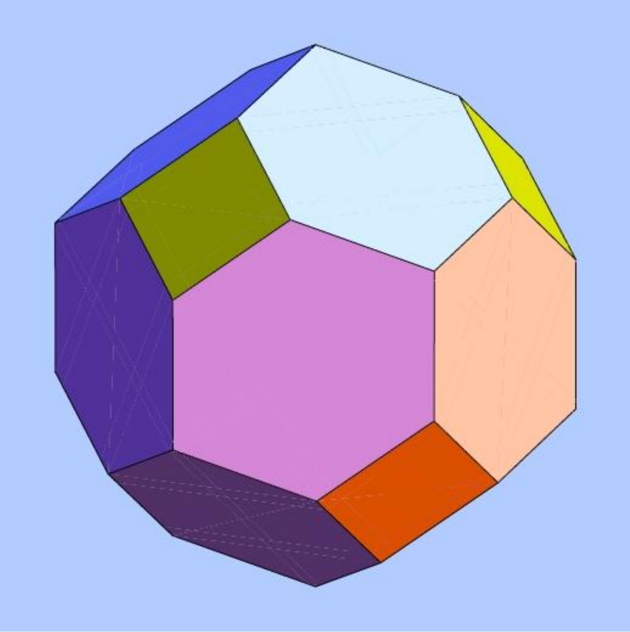
Nested truncated pentagonal dipyramid

German, 17th century, Museo degli Argenti, Florence

Rhombic dodecahedron



Rhombic dodecahedron



Truncated rhombic dodecahedron

F=18

Contrefait sphere

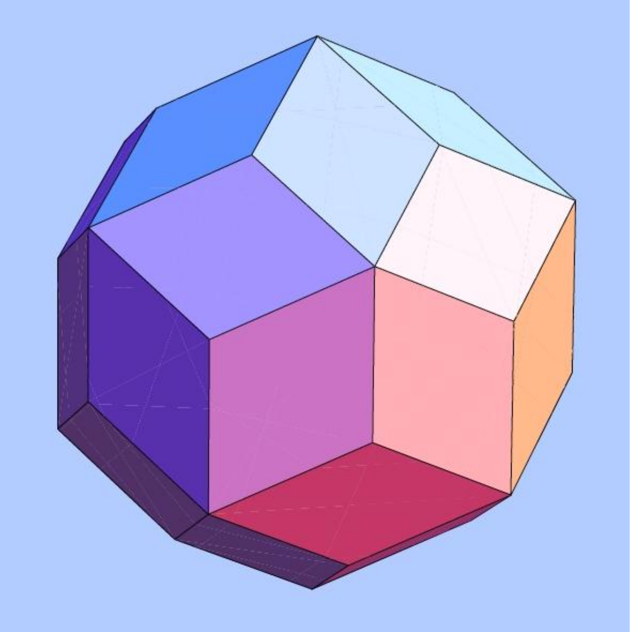
with 18 circular openings,

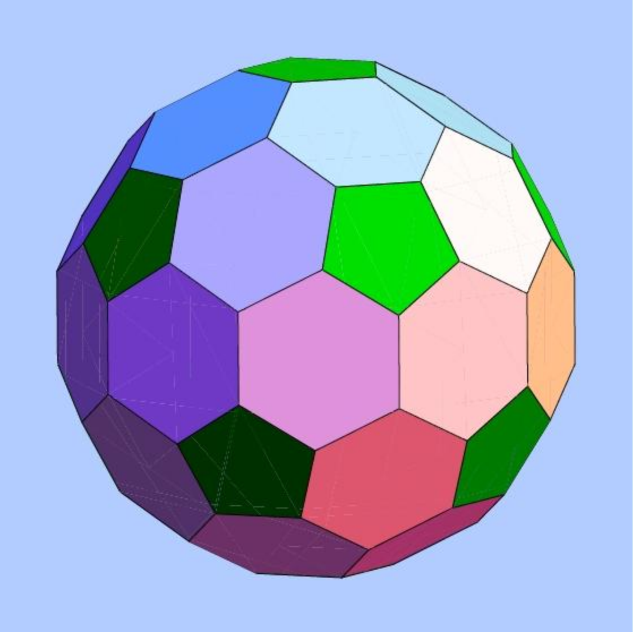
C. Zick, Nuremberg, 1685?

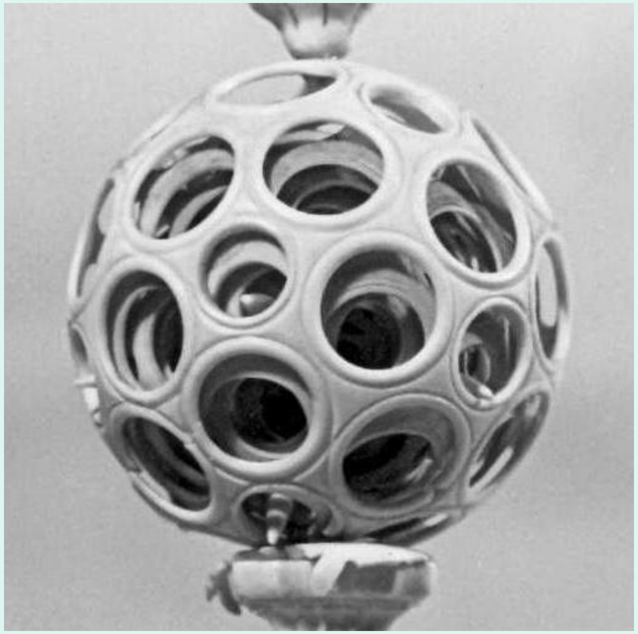
Danish Royal Collections,

Copenhagen

Rhombic triacontahedron







Rhombic triacontahedron

Truncated rhombic triacontahedron
F=42

Contrefait sphere
with 42 circular openings, 17th century, Museo degli Argenti, Florence

COVERING THE SPHERE WITH EQUAL CIRCLES

The covering problem

How must the sphere be covered by *n* equal circles (spherical caps) without interstices so that the angular radius of the circles will be as small as possible?

Circle covering of nested ivory spheres

Sideway cuts
(wooden model)
(From the Internet)

14 equal holes with O_h symmetry
(only the external shell)

Circle covering of a layer of a nested ivory sphere

Sideway cuts form a covering of the sphere by 14 equal circles (with octahedral symmetry constraint)

Proven solutions

(circle centres at the vertices of)

$n = 2$

$n = 3$ equilateral triangle,

$n = 4$ regular tetrahedron,

$n = 5$ trigonal bipyramid,

$n = 6$ regular octahedron,

$n = 7$ pentagonal bipyramid,

$n = 10$ bicapped square antiprism,

$n = 12$ regular icosahedron,

$n = 14$ bicapped hexagonal antiprism

Conjectural solutions

Up to $n = 20$ and some sporadic values of n:

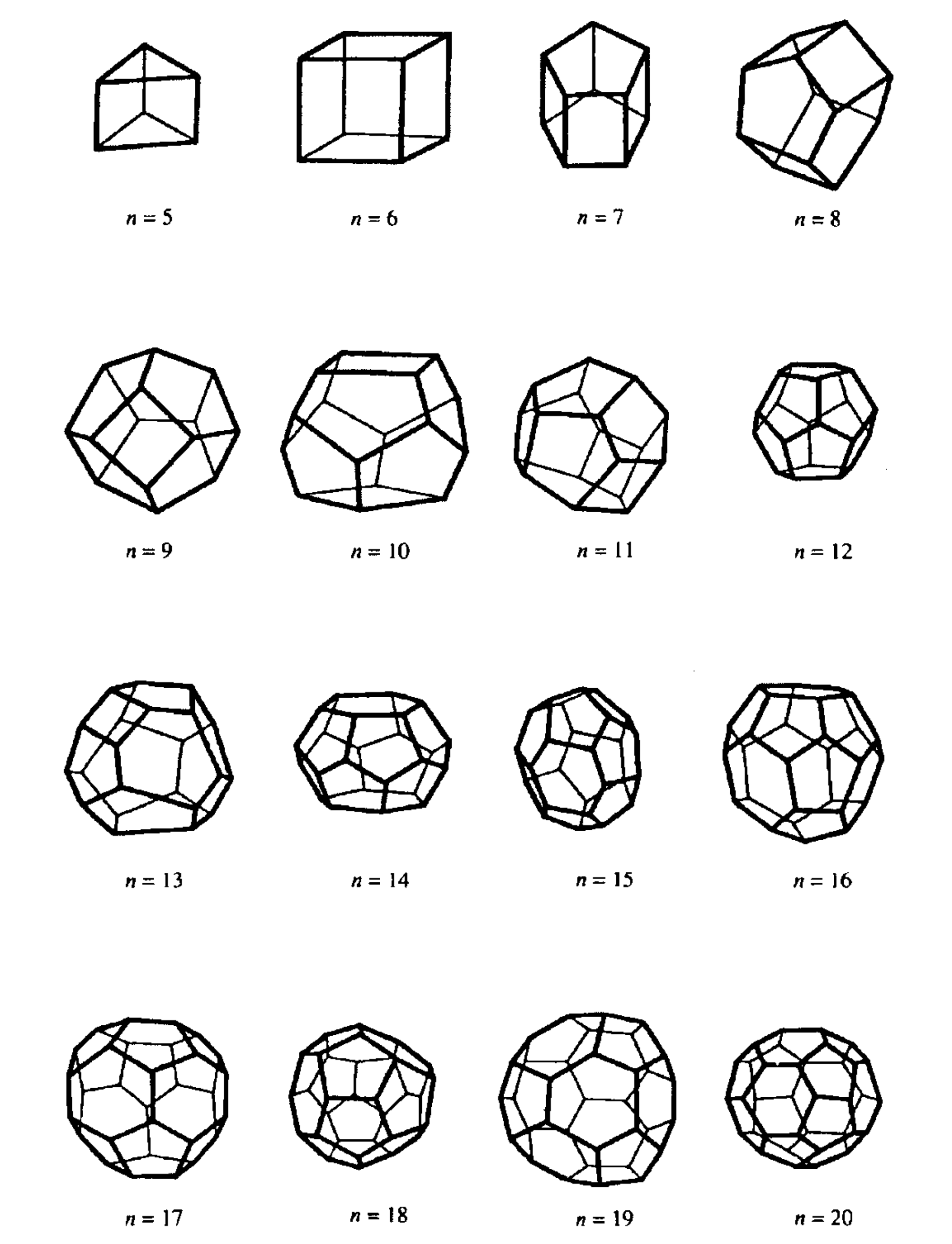
T.T., Gáspár, Zs., *Math. Proc. Cambridge Philos. Soc.* **110**, 71-89 (1991).

Up to $n = 130$:

Hardin, R.H., Sloane, N.J.A., Smith, W.D., Tables of spherical coverings (~2000), http://neilsloane.com/coverings/index.html

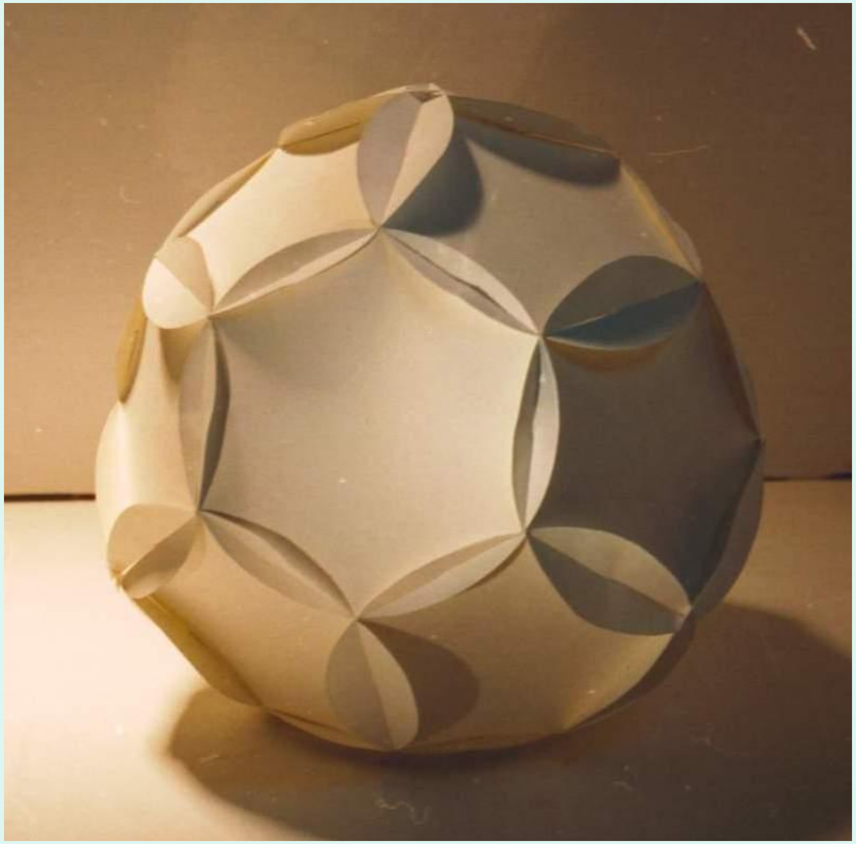
Conjectural solutions for up to n = 20

Polyhedra formed by the Dirichlet cells of circles of the best coverings

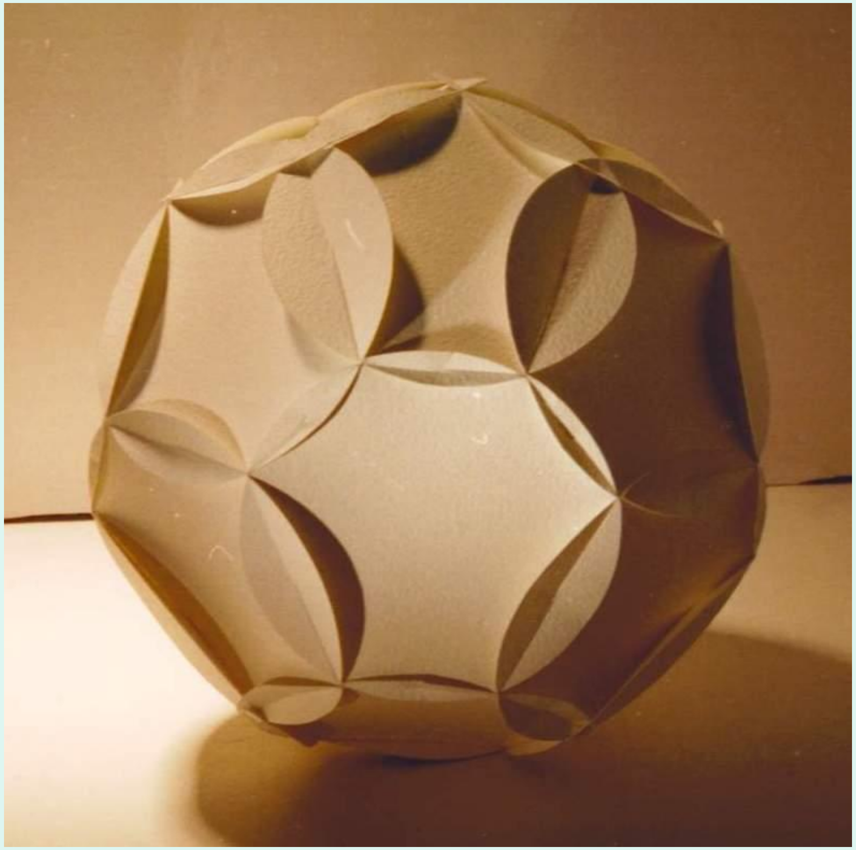


T.T., Gáspár, Zs., *Math. Proc. Cambridge Philos. Soc.* **110**, 71-89 (1991).

Examples of the best coverings



$n = 16$



$n = 20$

THE ROUNDEST POLYHEDRA

The isoperimetric problem for polyhedra

Among the polyhedra with a given surface area and given number of faces n, which has the maximum volume?

Isoperimetric quotient (Pólya)

Solutions to this problem are those polyhedra which have the maximum value of the isoperimetric quotient $0 \le IQ \le 1$

$$IQ = 36\pi \frac{V^2}{A^3}$$

V and A are the volume and the surface area of the polyhedron with *n* faces.

The solutions are the *roundest polyhedra.*

The Lindelöf theorem (1869)

A necessary condition for a polyhedron to maximize the volume is that the faces of the polyhedron are tangent to a sphere at the centroid of the faces.

For polyhedra satisfying Lindelöf’s necessary condition and where the radius of the sphere is the unity:

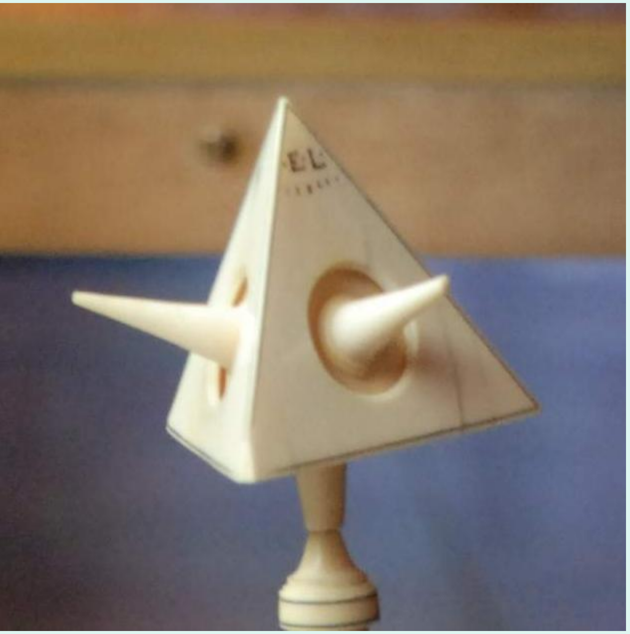
$$IQ = \frac{4\pi}{3}\frac{1}{V} = 4\pi\frac{1}{A}$$

Thus, IQ is a maximum if A or V is a minimum.

Equivalent formulation of the isoperimetric problem

Determine those polyhedra of minimum surface area (or volume) that can be circumscribed about the unit sphere and have *n* faces.

Roundest turned ivory polyhedra



$n = 4$
Regular tetrahedron
Egidius Lobenigk, 1588
Grünes Gewölbe Dresden

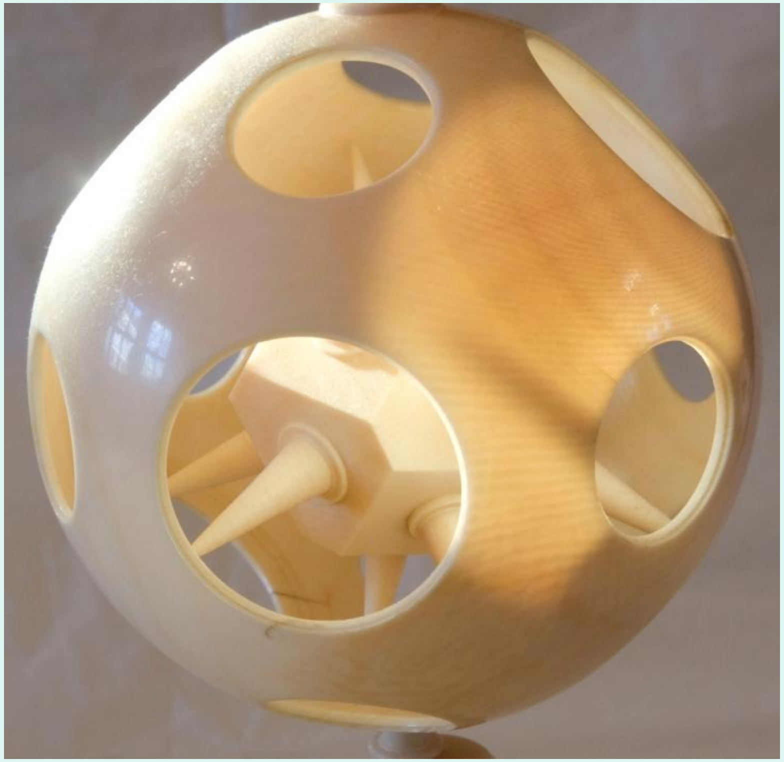
$n = 6$
Regular hexahedron
Around 1600
Grünes Gewölbe Dresden

$n = 12$
Regular dodecahedron
Around 1600,
Grünes Gewölbe Dresden

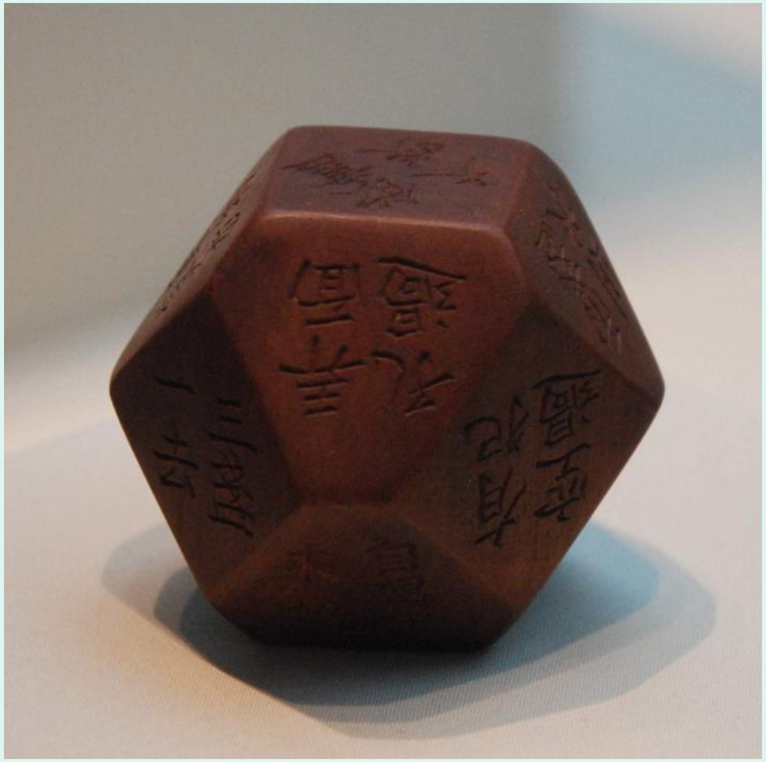
Solution under octahedral symmetry constraint, $n = 14$

Royal Danish Collection

Further examples for *n* = 14

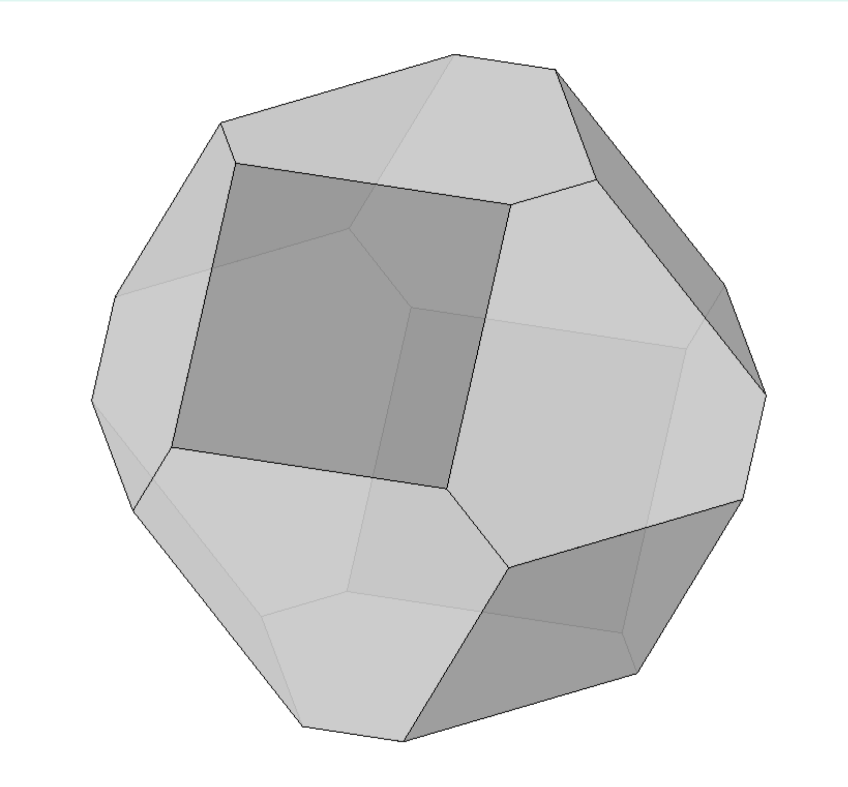


Spiked core of a sphere
1st half of the 17th century
Inv. No. II 284
Grünes Gewölbe Dresden

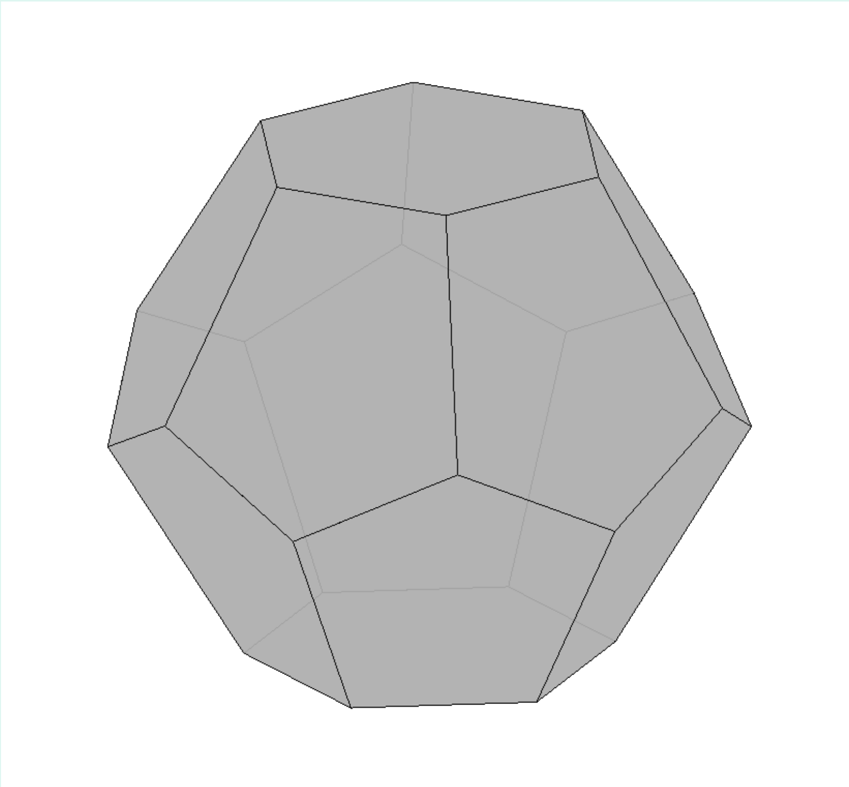


Wooden die
Unified Silla, 7th-9th century,
National Museum Gyeongju
Korea

Solutions for $n = 14$



With symmetry constraint
$IQ = 0.78163\ldots$



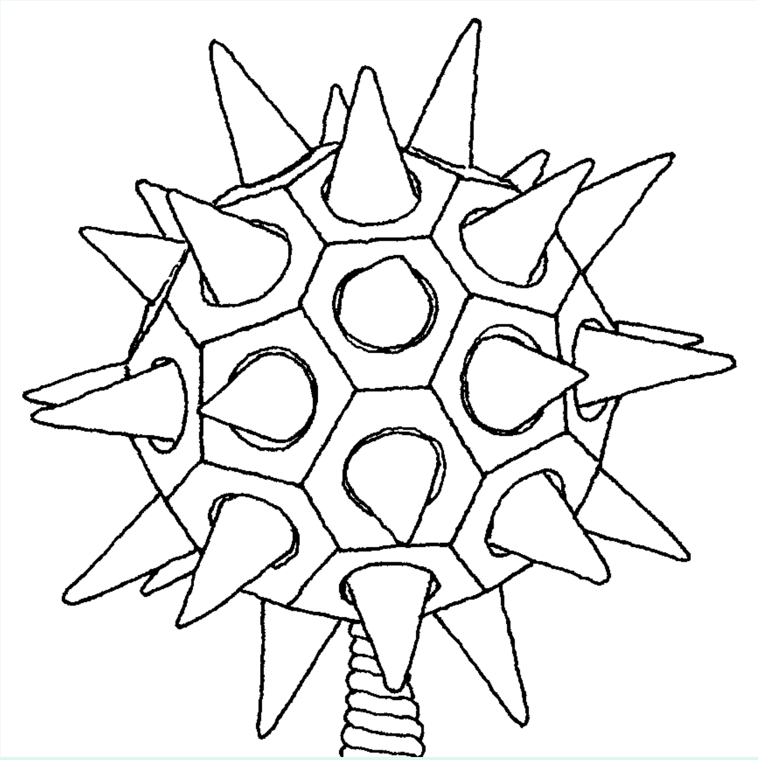
With no symmetry constraint
$IQ = 0.78569\ldots$

Solution under icosahedral symmetry constraint, $n = 32$

Minimum circle covering

Around 1600, Dresden,
Grünes Gewölbe Dresden

Application for n = 32



Grünes Gewölbe Dresden

„Hyperball” the roundest ball prior to inflation

Proven solutions

$n = 4$ regular tetrahedron,

$n = 5$ trigonal prism,

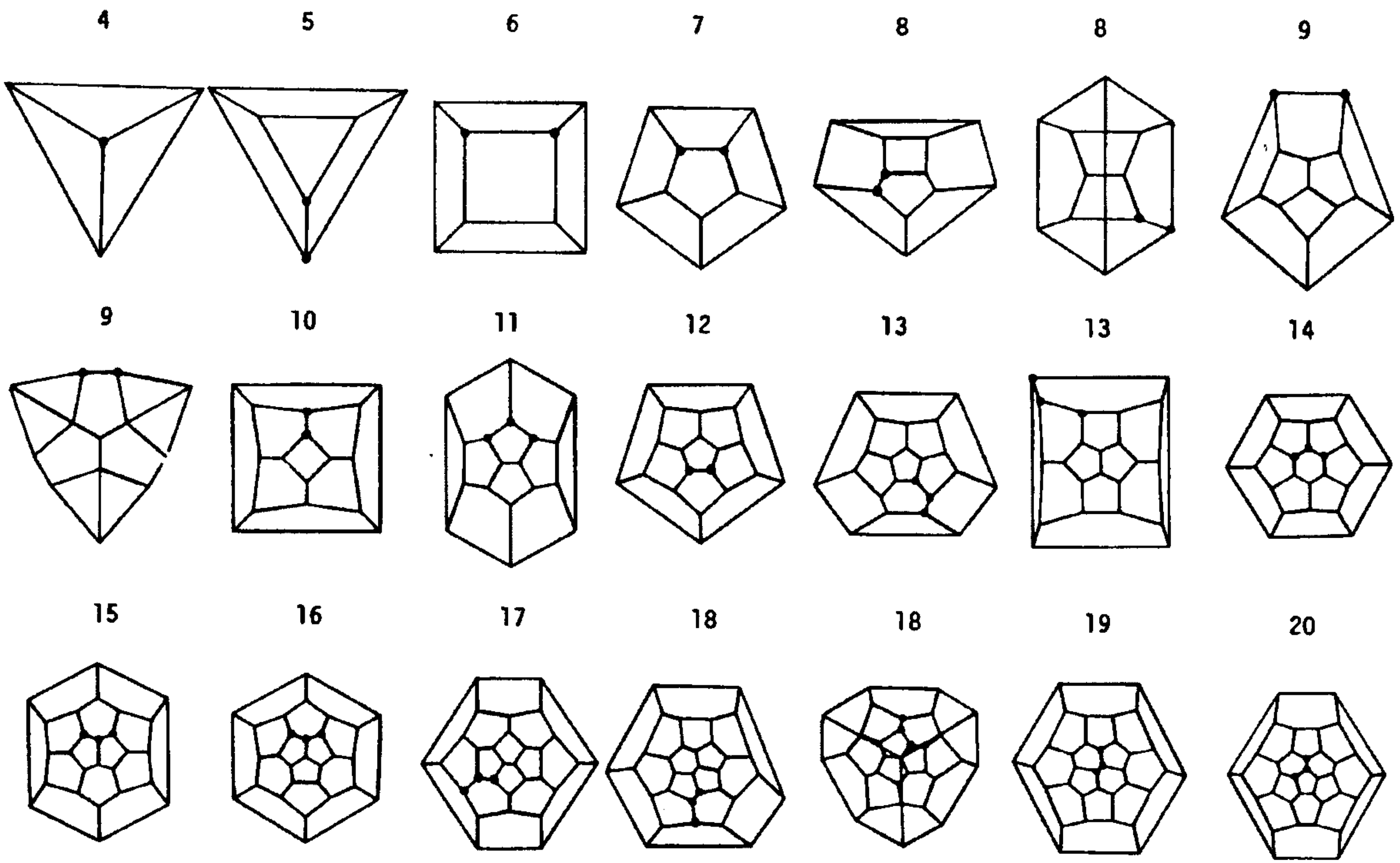
$n = 6$ cube,

$n = 7$ pentagonal prism (proven only „tentatively”),

$n = 12$ regular dodecahedron.

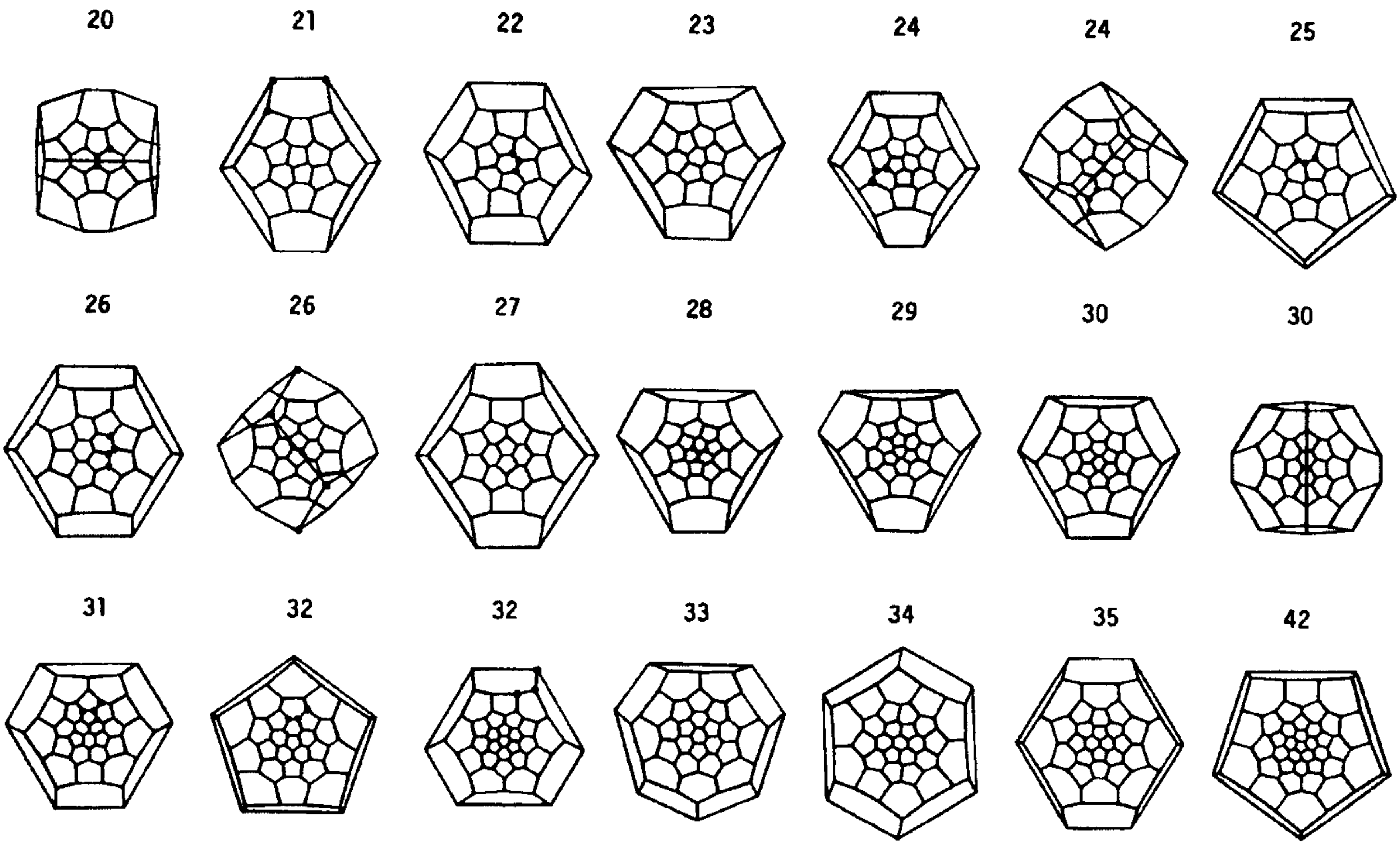
All these solutions are identical to the duals of the respective minimum circle coverings.

Conjectural solutions



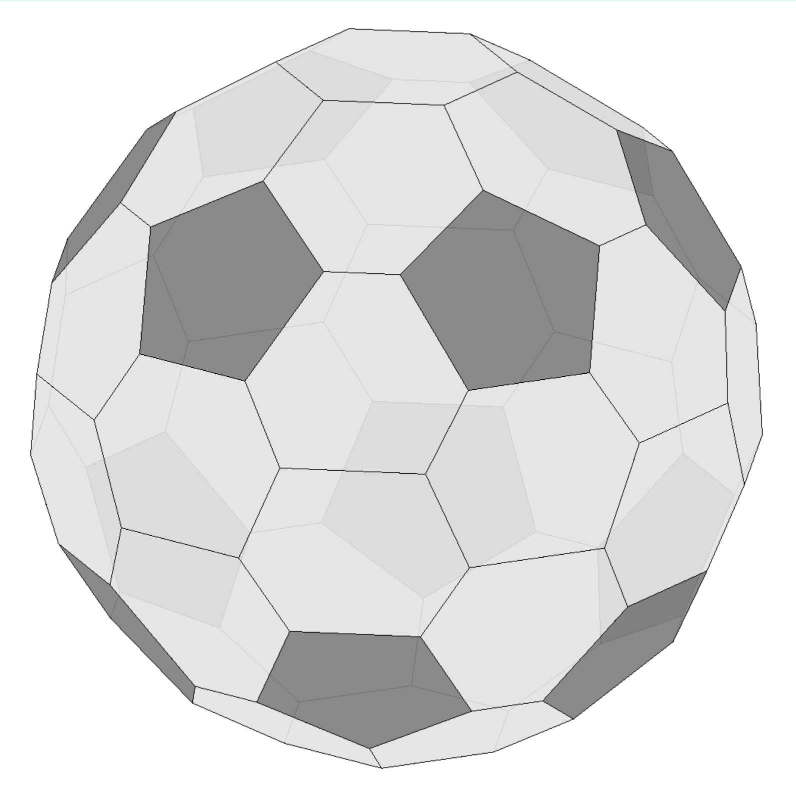
Schoen, A., *Proc. 2nd ACM Symp. Comp. Geometry*, 1986, pp. 159-168.

Conjectural solutions

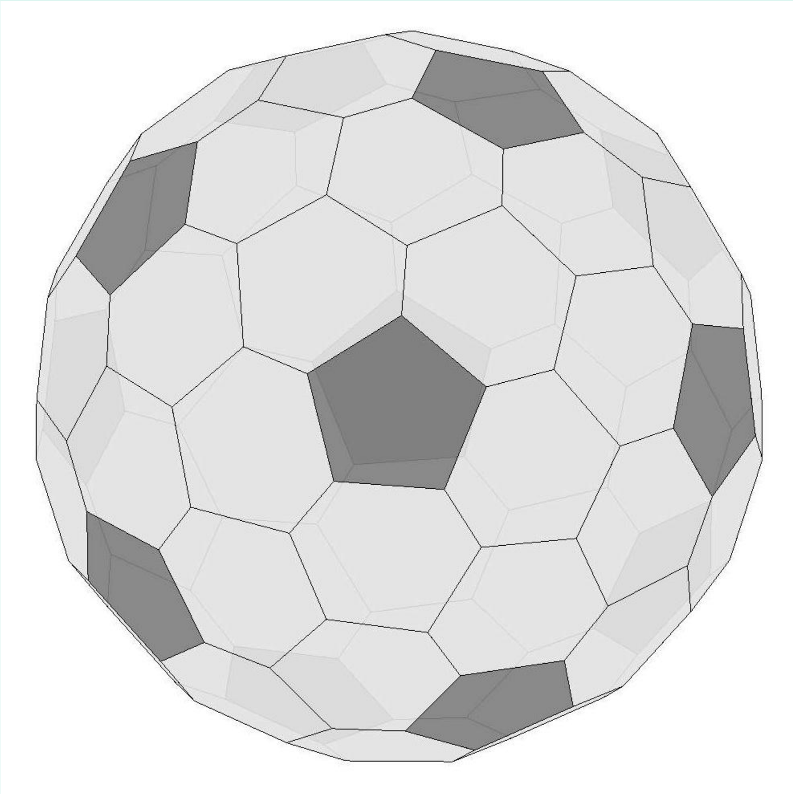


Schoen, A., *Proc. 2nd ACM Symp. Comp. Geometry*, 1986, pp. 159-168.

New conjectural solutions



$n = 50$



$n = 72$

T. T., Gáspár, Zs., Lengyel, A., *Phil. Mag.* **93** Nos. 31-33, 3970-3982 (2013)

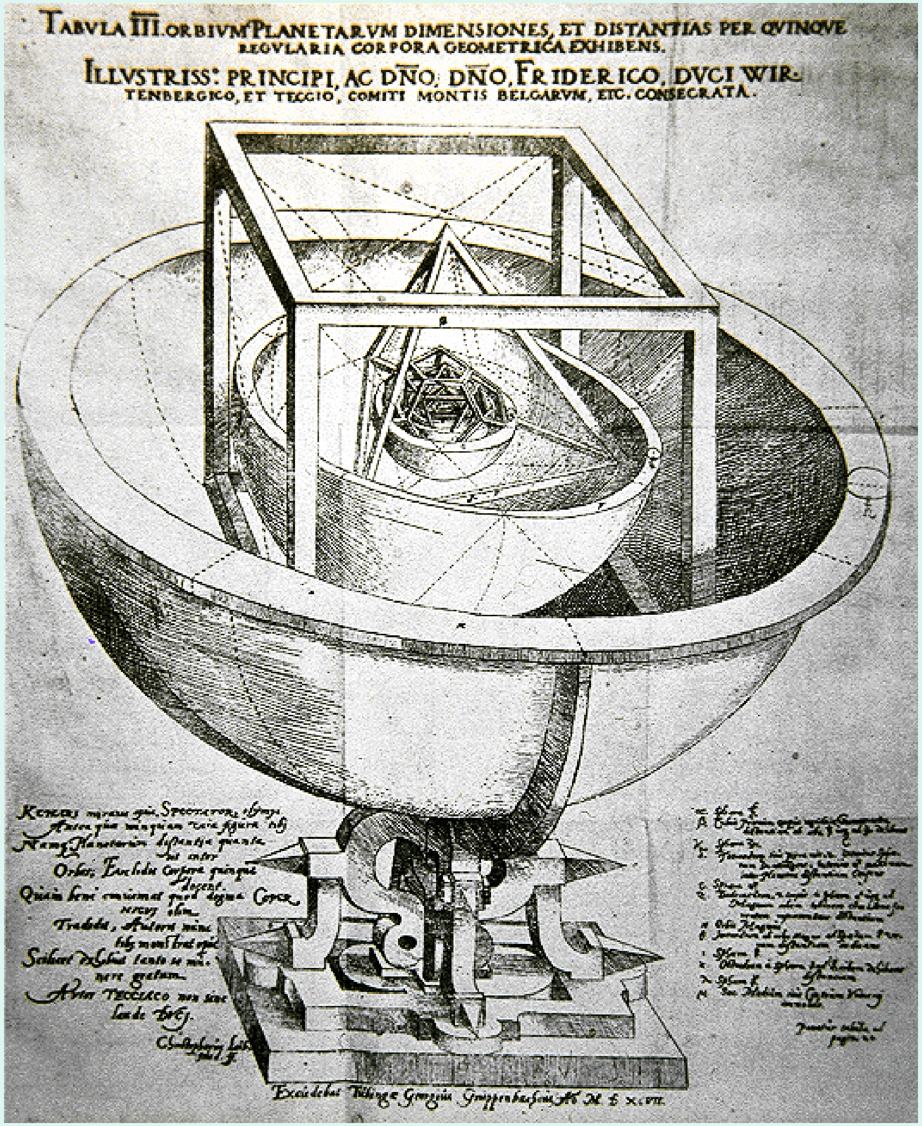
PROBABLE DIRECT SCIENTIFIC INFLUENCE

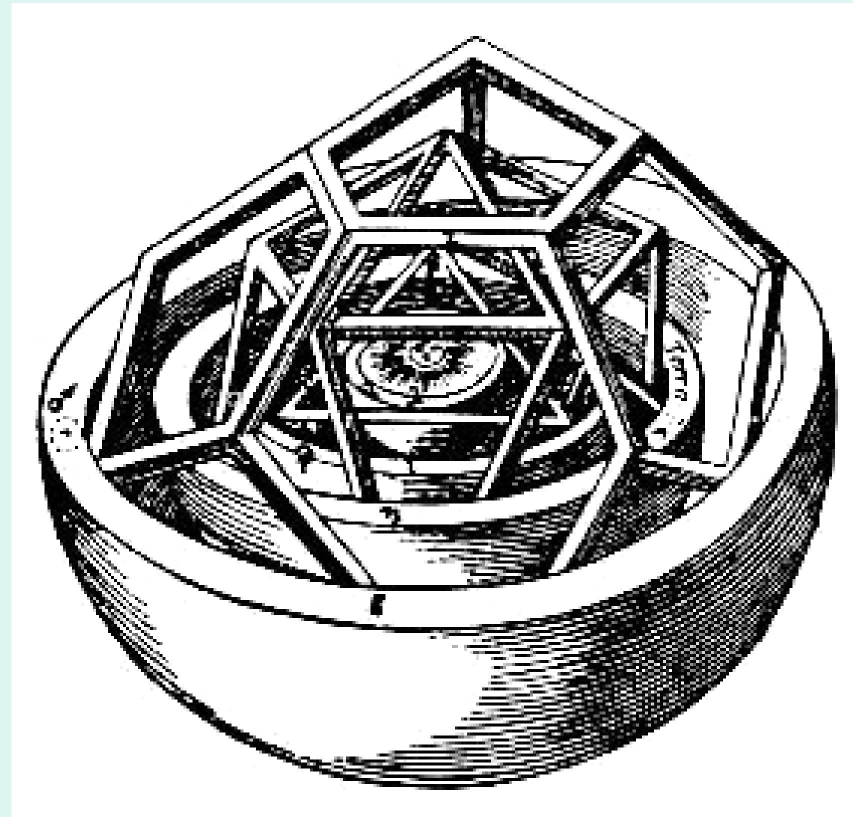
Early nested polyhedra

Georg Wecker, 1581

(SKD Online)

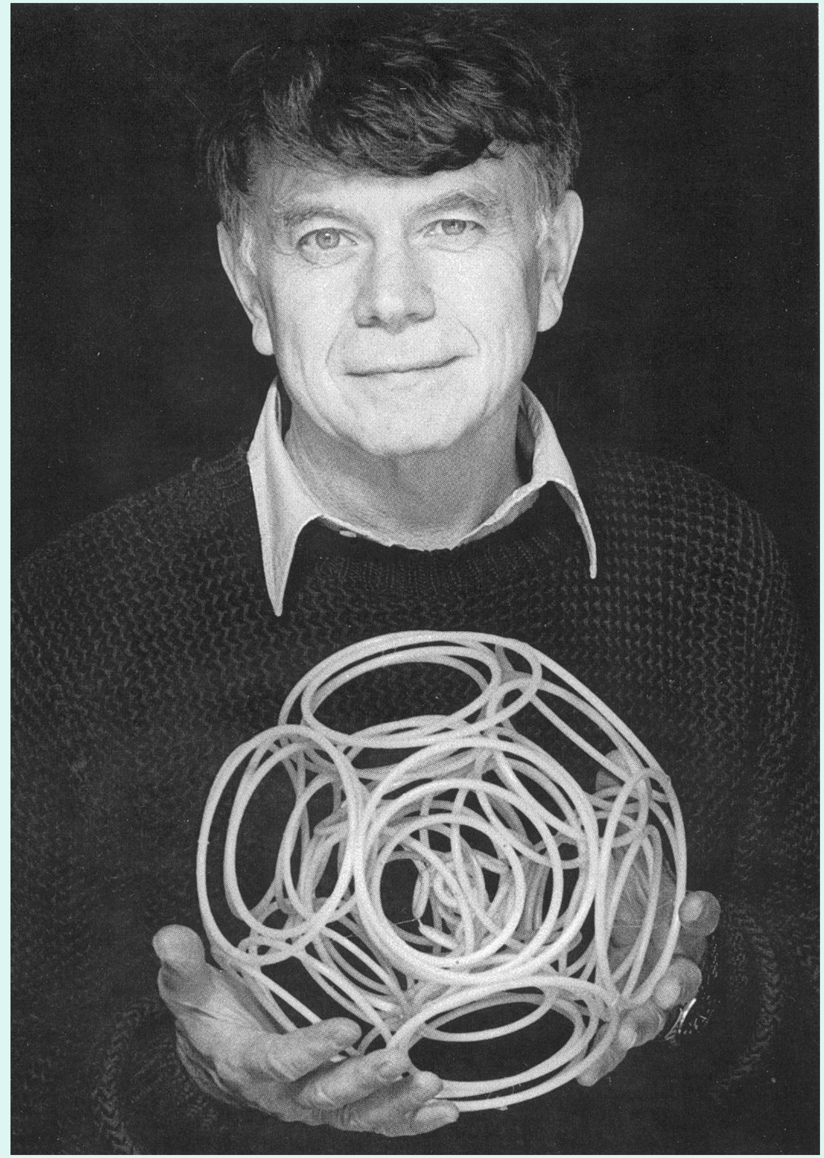
J. Kepler, *Mysterium cosmographicum*, 1596





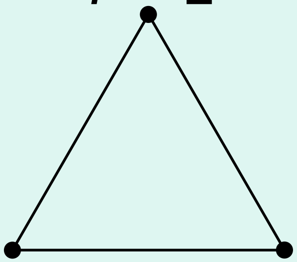
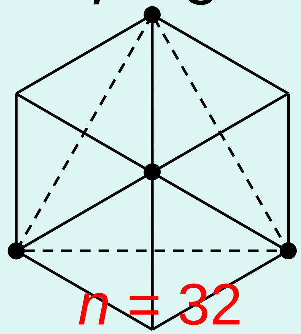
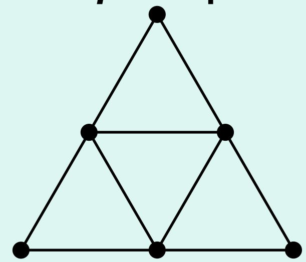
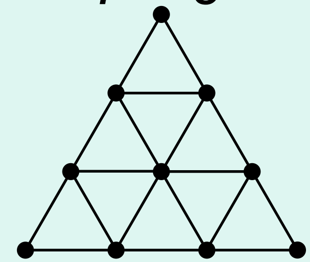
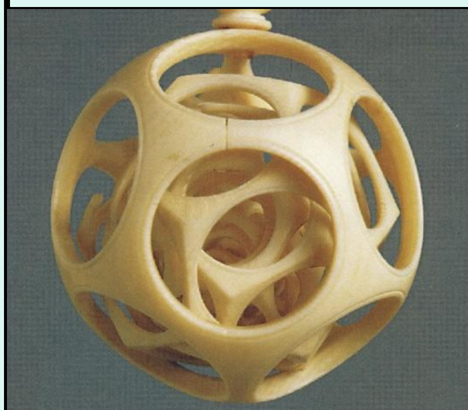
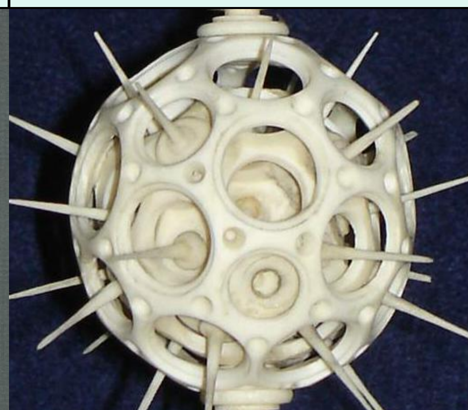
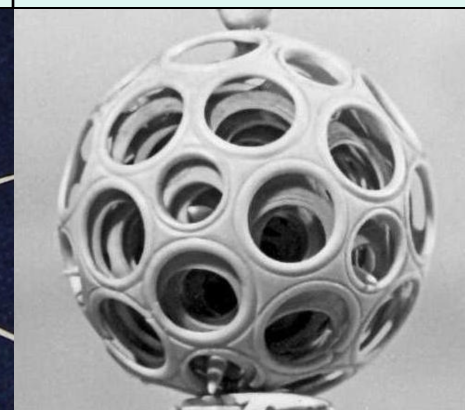
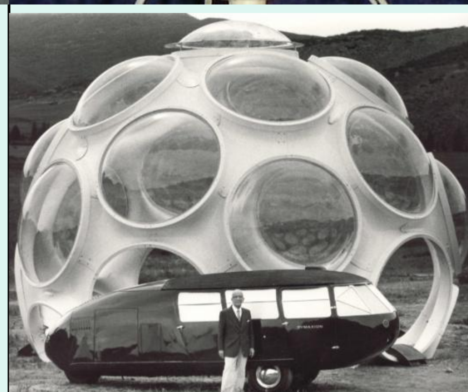
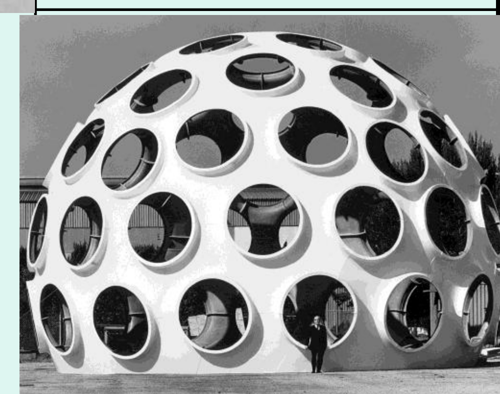
Nested Platonic polyhedra for explanation of the solar system

Snelson’s atom model

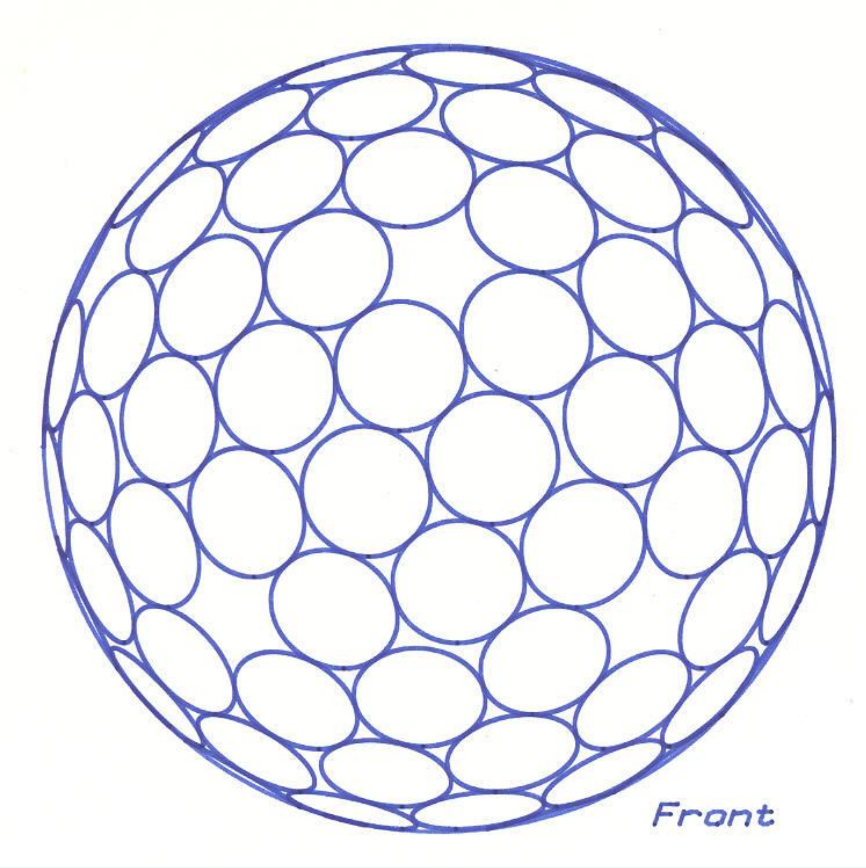


„Fly’s eye” dome is based on K. Snelson’s idea of atom in the 1960’s.

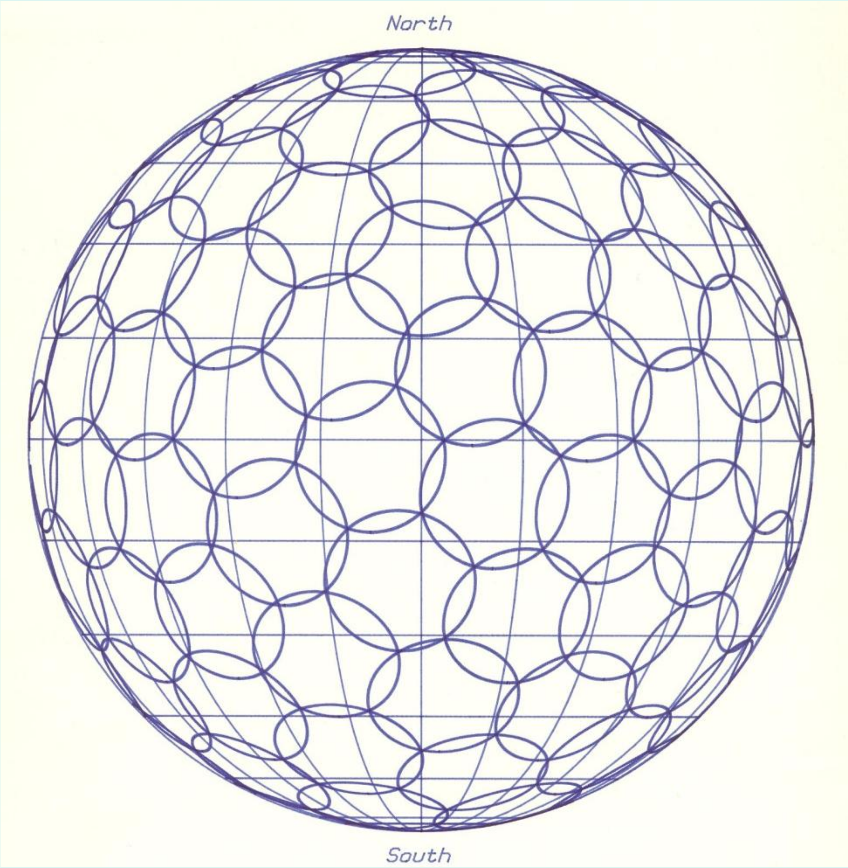
Geodesic spheres (triangulation)

$T = 1$	$T = 3$	$T = 4$	$T = 9$
			
$n = 12$	$n = 32$	$n = 42$	$n = 92$
			
			

Inspiration for discrete geometry



Circle packing



Circle covering

CONCLUSIONS

- A survey of turned single- and multi-layer ivory spheres and polyhedra has been given.
- The 5 Platonic solids, 7 Archimedean solids, 4 Catalan solids have been identified.
- Turned ivory spheres and polyhedra have given inspiration to study problems of discrete geometry and to apply their results in practice.

Acknowledgements

I thank Dr. H. Almegaard,
Prof. A. Florian,
Dr. K. Hincz,
Mr. S. Kabai,
Ms. B. Kresling,
Dr. A. Lengyel,
Ms. M. Tupputi
Mr. D. Weber
Dr. T. Wehli,
Dr. Zhong You
for repeated help in preparation of this paper.
Supported by OTKA grant K81146.