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On axiomatizations of the Shapley value for assignment games

René van den Brink and Miklós Pintér

Corvinus University of Budapest, Department of Mathematics and MTA-BCE "Lendület" Strategic Interactions Research Group

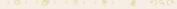
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If an axiomatization works on class of games A, and

• $A \subseteq B$, then it does not mean the axiomatization works on class of games B,





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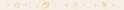
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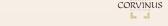


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- it implies the axiomatization works on $A \cup B$ (!).
- Conclusion: we have to check (almost) all classes of games one by one.



Shapley value

The Shapley value (Shapley, 1953) of Player i in game v is as follows:

$$\phi_i(v) \stackrel{\circ}{=} \sum_{T \subseteq N \setminus \{i\}} \frac{|T|!(|N \setminus T|-1)!}{|N|!} v_i'(T) ,$$

where $v_i'(T) \stackrel{\circ}{=} v(T \cup \{i\}) - v(T)$ is Player *i*'s marginal contribution to coalition *T* in game *v*.





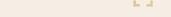
Solution ϕ on class of games $A \subseteq \mathcal{G}^N$

• is Pareto optimal (or efficient), if $\sum_{i \in N} \phi_i(v) = v(N)$ for all $v \in A$;



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- is anonymous, if $\phi(v) \circ \pi = \phi(v \circ \pi)$, for all $v \in A$ and permutation π on N such that $v \circ \pi \in A$;





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- is anonymous, if $\phi(v) \circ \pi = \phi(v \circ \pi)$, for all $v \in A$ and permutation π on N such that $v \circ \pi \in A$;
- satisfies the equal treatment property, if $\phi_i(v) = \phi_j(v)$ for all $v \in A$ and symmetric $(v_i'(S) = v_j'(S)$, if $S \subseteq N \setminus \{i, j\})$ players i, j in v;



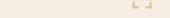
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- is covariant under strategic equivalence, if $\phi(\alpha v \oplus \beta) = \alpha \phi(v) + \beta$, for all $v \in A$, $\alpha > 0$ and $\beta \in \mathbb{R}^N$ convinue such that $\alpha v \oplus \beta \in A$;

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- satisfies coalitional strategic equivalence, if $\phi_i(v) = \phi_i(v + \alpha u_T)$, for all $v \in A$, $i \in N$, $T \subseteq N \setminus \{i\}$ and $\alpha > 0$ such that $v + \alpha u_T \in A$;





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- satisfies fairness, if $\phi_i(v+w) \phi_i(v) = \phi_j(v+w) \phi_j(v)$ for all $v, w \in A$ and $i, j \in N$ such that i, j are symmetric in w and $v+w \in A$.

Further axioms

Let $A \subseteq \mathcal{G}_N = \cup_{T \subseteq N} \mathcal{G}^T$ be a class of games, ϕ be a solution on A, and for all $v \in A$, $T \subseteq N$, $T \neq \emptyset$, such that $v^{S \cup (N \setminus T)} \in A$, let

$$v_{T,\phi}(S) \stackrel{\circ}{=} v(S \cup (N \setminus T)) - \sum_{i \in N \setminus T} \phi_i(v^{S \cup (N \setminus T)}) \text{ for all } S \subseteq T, \ S \neq \emptyset,$$

and $v_{T,\phi}(\emptyset) = 0$. Then $v_{T,\phi} \in \mathcal{G}^T$ is called the ϕ -reduced game of v on coalition T. Solution ϕ defined on $A \subseteq \mathcal{G}$

• is $\mathit{HM-consistent}$ (Hart and Mas-Colell, 1989), briefly consistent, if for all $T \subseteq N$, $v \in A \cap \mathcal{G}^T$ and $S \subseteq T$, $S \neq \emptyset$, such that $v_{S,\phi} \in A$, it holds that $\phi_i(v_{S,\phi}) = \phi_i(v)$ for all $i \in S_{\text{ORVINUS}}^{\text{TOTALLS}}$



From the literature it follows that the Shapley value is the unique solution on \mathcal{G}^N that satisfies the following sets of axioms:

 Pareto optimality, the null player property, the equal treatment property and additivity (Shapley, 1953);





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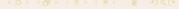
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- Pareto optimality, the null player property and fairness (van den Brink, 2001).



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Further, it is the unique solution on G_N that satisfies

• Pareto optimality, covariance, the equal treatment property and consistency (Hart and Mas-Colell, 1989).





A trivial remark

Theorem

If $|B \cup S| = 2$, then solution ϕ on $\mathcal{G}^{B,S}$ satisfies Pareto optimality and the equal treatment property if and only if it is the Shapley value.





Let OR(N) be the set of all (linear) orderings on set N. Consider the following two solutions for assignment games with buyer and seller sets B and S. First, let

$$OR_B \stackrel{\circ}{=} \{ \tau \in OR(B \cup S) \mid \tau(i) \leq |B| \Rightarrow i \in B \}$$
,

be the orders where the buyers come first, and let

$$OR_S \stackrel{\circ}{=} \{ \tau \in OR(B \cup S) \mid \tau(i) \leq |S| \Rightarrow i \in S \}$$
,

be the orders where the sellers come first.

Now, for all $v \in \mathcal{G}^{B,S}$ and $i \in B \cup S$, let

$$\phi_i^B(v) \stackrel{\circ}{=} \frac{1}{|OR_B|} \left(\sum_{\tau \in OR_B} (v(\{j \in B \cup S \mid \tau(j) \leq \tau(i)\}) \right)$$

$$-v(\{j\in B\cup S\mid \tau(j)<\tau(i)\}))\right),$$



be the average marginal contribution of buyer or seller i over all orders where the buyers come first, and

$$\phi_i^{S}(v) \stackrel{\circ}{=} \frac{1}{|OR_S|} \left(\sum_{\tau \in OR_S} \left(v(\{j \in B \cup S \mid \tau(j) \le \tau(i)\} \right) - v(\{j \in B \cup S \mid \tau(j) < \tau(i)\}) \right)$$

be the average marginal contribution of buyer or seller i over all orders where the sellers come first. Then, for all $v \in \mathcal{G}^{B,S}$, let

$$\phi^{B,S}(v) \stackrel{\circ}{=} \frac{\phi^B(v) + \phi^S(v)}{2}$$

be the average of these two solutions.



Results on $\phi^{B,S}(v)$

Theorem

Solution $\phi^{B,S}$ is a convex combination of random order values and satisfies anonymity, the equal treatment property, covariance, additivity, and strong monotonicity on $\mathcal{G}^{B,S}$.





An example

Consider
$$B=\{1,2\},\ S=\{3\},\ \text{and}\ v\in\mathcal{G}^{B,S}$$
 determined by $a_{1,3}=1$ and $a_{2,3}=2$, that is, $v(\{1,3\})=1,\ v(\{2,3\})=v(\{1,2,3\})=2$ and $v(T)=0$ otherwise. Then $\phi^B(v)=(0,0,2),\ \phi^S(v)=(\frac{1}{2},\frac{3}{2},0),\ \text{and thus}$ $\phi^{B,S}(v)=(\frac{1}{4},\frac{3}{4},1).$ However, $\phi^{Sh}(v)=(\frac{1}{6},\frac{2}{3},\frac{7}{6}).$



Corollary

On the class $\mathcal{G}^{B,S}$ of assignment games, the following axiomatizations of the Shapley value do not work:

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- Chun's axiomatization (Pareto optimality, the equal treatment property and coalitional strategic equivalence),
- van den Brink's axiomatization (Pareto optimality, the null player property and fairness).



Potential

Definition

Let $A \subseteq \mathcal{G}_N$. For every function $P : A \to \mathbb{R}$, $T \subseteq N$, $T \neq \emptyset$, and for all $v \in \mathcal{G}^T \cap A$ and $i \in T$ such that |T| = 1 or $v^{T \setminus \{i\}} \in A$, let

$$P_i'(v) \stackrel{\circ}{=} \begin{cases} P(v), & \text{if } |T| = 1\\ P(v) - P(v^{T \setminus \{i\}}) & \text{otherwise.} \end{cases}$$
 (1)

lf

$$\sum_{i\in T} P_i'(v) = v(T),$$

for all $v \in \mathcal{G}^T \cap A$ such that either |T| = 1 or $v^{T \setminus \{i\}} \in A$ for all $i \in T$, then P is called a potential on A.



Definition

A collection $A \subseteq \mathcal{G}$ is subgame closed, if $v^{T \setminus \{i\}} \in A$ for all $T \subseteq N$ with |T| > 1, $i \in T$ and $v \in \mathcal{G}^T$ such that $v \in A$.

Theorem

Let $A \subseteq \mathcal{G}_N$ be a subgame closed set of games. Then function P on \mathcal{G} is a potential, if and only if $P_i'(v) = \phi_i(v)$ for all $T \subseteq N$, $T \neq \emptyset$, $v \in \mathcal{G}^T \cap A$ and $i \in T$.





(semi-)Negative result on the potential

Corollary

On the class of assignment games there is a potential P such that there exists an assignment game v and a player i such that $P'_i(v) \neq \phi_i(v)$.



HM-consistency

First, let \mathcal{G}_a be the collection of all assignment games, that is, $\mathcal{G}_a = \{v \in \mathcal{G}_N \mid \text{there exist } B, S \subset N, \\ B \neq \emptyset, \ S \neq \emptyset, \ B \cap S = \emptyset, \ B \cup S = N \text{ such that } v \in \mathcal{G}^{B,S}\}.$ Moreover, let solution $\overline{\phi}$ on \mathcal{G}_a for each $v \in \mathcal{G}_a \cap \mathcal{G}^{B,S}$ be given by

$$\overline{\phi}_i(v) \stackrel{\circ}{=} \left\{ \begin{array}{l} \frac{v(B \cup S)}{|(B \cup S) \setminus NP(v)|}, & \text{if } i \notin NP(v) \\ 0 & \text{otherwise.} \end{array} \right.$$

It is worth noticing that for any assignment game v and player $i \notin NP(v)$, $\overline{\phi}_i(v) > 0$.



Negative result on HM-consistency

Theorem

Solution $\overline{\phi}$ satisfies Pareto optimality, anonymity, the equal treatment property, covariance and consistency on \mathcal{G}_a .





Submarket

Definition

Let $A \in \mathcal{A}^{B,S}$ be an assignment situation. Then $B' \cup S'$, $B' \subseteq B$, $S' \subseteq S$, $B' \cup S' \neq \emptyset$, is a submarket of A if $a_{i,j} = 0$ for all $(i,j) \in (B' \times (S \setminus S')) \cup ((B \setminus B') \times S')$.





Axioms

Definition

Solution f on $A^{B,S}$ satisfies

- submarket efficiency, if for all $A \in \mathcal{A}^{B,S}$ and for all submarkets (B',S') of A, it holds that $\sum_{i\in B'\cup S'}f_i(v)=v_A(B'\cup S')$,
- valuation fairness, if for every buyer $i \in B$, every seller $j \in S$ and every pair of assignment situations $A, \overline{A} \in \mathcal{A}^{B,S}$ such that $\overline{a}_{i,j} = 0$ and $a_{g,h} = \overline{a}_{g,h}$ for all $(g,h) \in ((B \setminus \{i\}) \times S) \cup (B \times (S \setminus \{j\}))$, it holds that $f_i(A) f_i(\overline{A}) = f_i(A) f_i(\overline{A})$.





The positive result (á la Myerson (1977))

Theorem

The Shapley value ϕ is the unique solution for assignment situations that satisfies submarket efficiency and valuation fairness.

and a further result (compare it to Solymosi, Brugueras and Raghavan (201?))

$\mathsf{Theorem}$

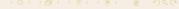
Consider assignment situations $A, \overline{A} \in \mathcal{A}^{B,S}$ such that for some $i \in B, j \in S$ it holds that $\overline{a}_{i,j} \geq a_{i,j}$, and $a_{g,h} = \overline{a}_{g,h}$ for all $(g,h) \in ((B \setminus \{i\}) \times S) \cup (B \times (S \setminus \{j\}))$. Then $\phi_i(\overline{A}) \geq \phi_i(A)$ and $\phi_j(\overline{A}) \geq \phi_j(A)$.





On the class of assignment games





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