On an Invasive Species Model with Harvesting

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Relations between the people, trees and rats in Easter Island

$$\left.\begin{aligned}
\dot{P} &= aP\left(1-\frac{P}{T}\right), \\
\dot{R} &= cR\left(1-\frac{R}{T}\right)-gR, \\
\dot{T} &= \frac{b}{1+fR}T\left(1-\frac{T}{M}\right)-hP
\end{aligned}\right\} \qquad (1)$$

The biological feasibility of the model

Proposition: All solutions of (1) with positive initial conditions $P(0) > 0$, $R(0) > 0$ and $T(0) > 0$ remain positive for all $t \geq 0$ in their domain of existence.

Proposition: The system (1) is dissipative.

Proposition: There is no (nontrivial) periodic solution

1. in the positive quadrant of the $[P, T]$ plane, provided the harvesting rate of the trees by the people is less than the growth rate of the human population, i.e. $h \leq a$ holds.
2. in the positive quadrant of the $[R, T]$ plane.

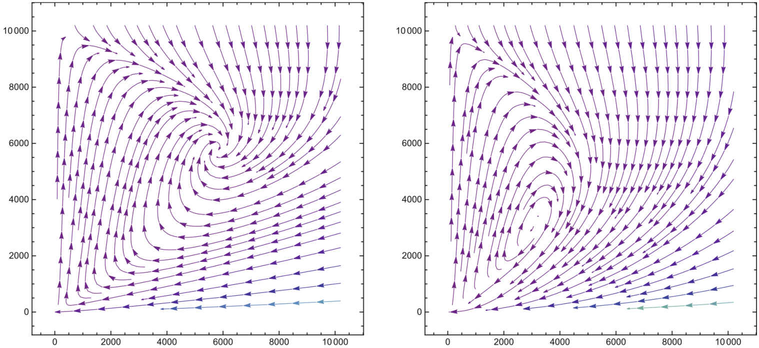


Fig. 1 A phase portrait of the $[P, T]$ plane of system for $h \leq a$ and $h > a$

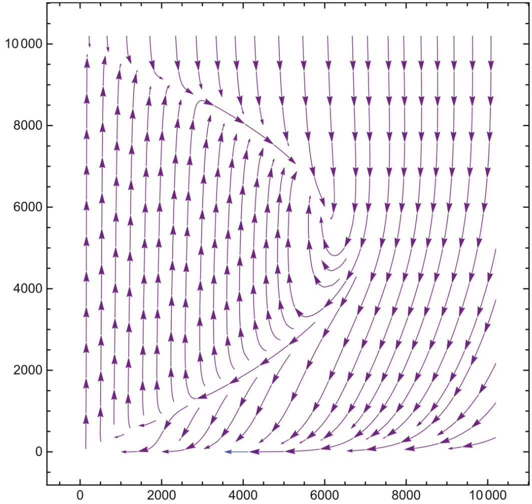


Fig. 2 A phase portrait of the $[R, T]$ plane of system

Equilibria of the system (1)

$$E_1(0, 0, M)$$

$$E_2\left(0, M\left(1-\frac{g}{c}\right), M\right)$$

$$E_3\left(M\left(1-\frac{h}{b}\right), 0, M\left(1-\frac{h}{b}\right)\right)$$

$$E_4\left(\frac{c(b-h)M}{bc+f(c-g)hM}, \frac{c(b-h)M}{bc+f(c-g)hM}\left(1-\frac{g}{c}\right), \frac{c(b-h)M}{bc+f(c-g)hM}\right)$$

Proposition: The equilibrium E_1 is unstable

Proposition: If $g < c$ holds, then the equilibrium E_2 is unstable

Proposition: If $g < c$, $h < b$ hold, then the equilibrium E_3 is unstable

Proposition: If $g > c$, $h < b$, $a < b$ hold, then the equilibrium E_3 is asymtotically stable

Proposition: If $g > c$, $h < b$, $a > b$ hold, then the equilibrium E_3 is asymptotically stable iff $h < (a + b)/2$ holds.

Proposition: If $g < c$, $h < \min\{a, b, c-g\}$ hold, then the equilibrium E_4 is asymptotically stable.

Proposition:
If $c < g$, $a < h < b$ hold, then the equilibrium E_3 is losing its stability by a Hopf bifurcation at

$$h_H := \frac{a+b}{2}$$

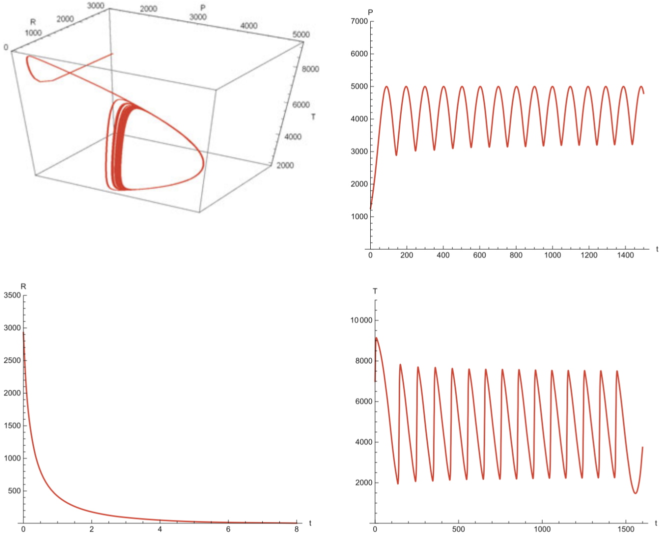
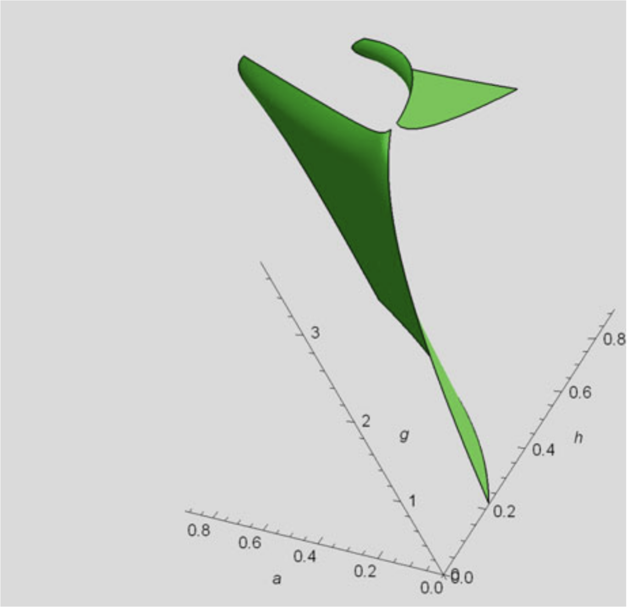
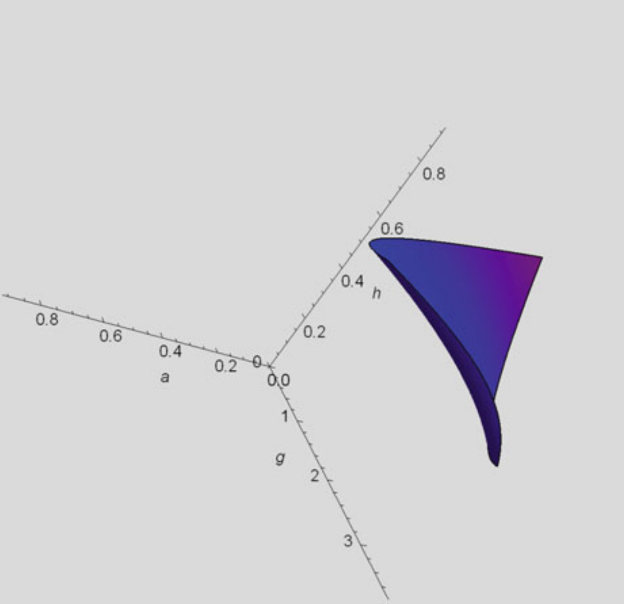
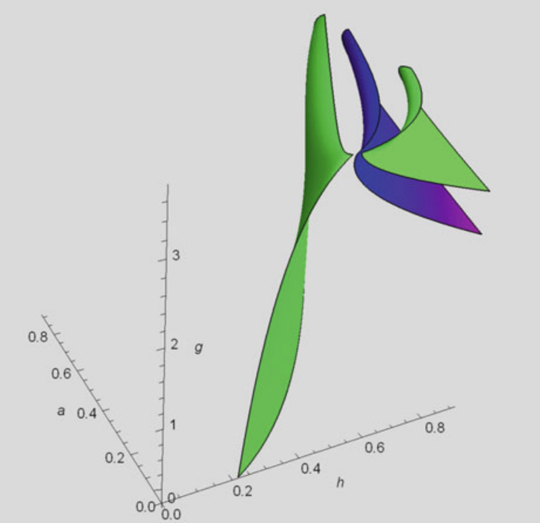


Fig. 4 A solution of system (1) with parameters $c < g$, $a < b$ and $h = (a + b)/2$


3
2
1
g
0.8
0.6
0.4
0.2
h
0.8
0.6
0.4
0.2
0.0
a
0.0
0.0


0.8
0.6
0.4
h
0.2
0.8
0.6
0.4
0.2
0
a
0.0
1
g
2
3

Bifurcation from the Interior Equilibrium E_4



Discretization with Explicit Euler method

$$
\left.
\begin{aligned}
P_{n+1} &= P_n + saP_n\left(1 - \frac{P_n}{T_n}\right), \\
R_{n+1} &= R_n + s\left(cR_n\left(1 - \frac{R_n}{T_n}\right) - gR_n\right), \\
T_{n+1} &= T_n + s\left(\frac{b}{1 + fR_n}T_n\left(1 - \frac{T_n}{M}\right) - hP_n\right)
\end{aligned}
\right\} \qquad (2)
$$

Proposition:
Supppose that $d \in \mathbb{N}$, $A \in \mathbb{R}^{d \times d}$, $B := I_d + hA$, furthermore conditions

$$\lambda \in \sigma(A) \qquad \text{and} \qquad h < \frac{-\max^2(\Im(\lambda)) - 2s(A)}{s^2(A)}$$

hold. Then $s(A) < 0$ implies $\rho(B) < 1$.

Proposition:
Suppose that $d \in \mathbb{N}$, $A \in \mathbb{R}^{d \times d}$,

$$B := I_d + hA.$$

Then $s(A) > 0$ implies $\rho(B) > 1$ independent of h.

Discretization with a non-standard Mickens method

$$\left.\begin{aligned} P_{n+1} &= P_n + \varphi(s) a P_{n+1}\left(1 - \frac{P_n}{T_n}\right), \\ R_{n+1} &= R_n + \varphi(s)\left(cR_n\left(1 - \frac{R_n}{T_n}\right) - gR_n\right), \\ T_{n+1} &= T_n + \varphi(s)\left(\frac{b}{1 + fR_n} T_n\left(1 - \frac{T_n}{M}\right) - hP_n\right) \end{aligned}\right\} \tag{3}$$

Neimark-Sacker bifurcation

$\alpha(h) \pm \beta(h) i \quad \in \sigma(\text{Jacobian of the RHS at a FP}), \quad h$ is the bifurcation parameter

1. $| \alpha(h_{NS}) \pm \beta(h_{NS}) i | = 1$ and $\neq 1$ for other eugenvalues

2. $\partial_h \left(\alpha^2(h_{NS}) + \beta^2(h_{NS}) \right) \neq 0$

Preservation of bifurcation after discretization

$$\alpha(h) \pm \beta(h)i \quad \in \sigma(\text{Jacobian of the RHS at any EP in the CS})$$

$$\Rightarrow$$

$$1 + \varphi(s)\alpha(h) \pm \varphi(s)\beta(h)i \ \in \sigma(\text{Jacobian of the RHS at the same EP in the DS})$$

$$\Rightarrow$$

Proposition:
If Hopf bifurcation occurs from an EP with h_H critical value, then Neimark-Sacker bifurcation cannot occur from the same EP with the same critical value.

Neimark-Sacker bifurcation from E_3 on model (1)

$$h_{NS} := \frac{\left(\sqrt{-3a^2\varphi(s)^2 + 4a\varphi(s)(b\varphi(s)+2)+16} + a\varphi(s) + 2b\varphi(s) - 4\right)}{4\varphi(s)}$$

Neimark-Sacker bifurcation after discretization

$$\alpha(h) \pm \beta(h)i \quad \in \sigma(\text{Jacobian of the RHS at any EP in the CS})$$

$$\Rightarrow$$

$$\varphi(s) = \frac{-2\alpha(h)}{\alpha^2(h) + \beta^2(h)} \iff |\, 1 + \varphi(s)\alpha(h) \pm \varphi(s)\beta(h)i \,| = 1$$

Thank you very much for your attention!