On the parabolic Cauchy problem for quantum graphs with vertex noise

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Outline

Continuity in vertex v

Berkolaiko, Kuchment (2013): quantum graphs

$G = (V, E)$ a simple graph with finite vertex set V and edge set E

$$e \cong [0, \ell_e], \quad e \in E,\ \ell_e > 0$$

u is a function on G if $u = (u_e)_{e \in E}$, $u_e \colon [0, \ell_e] \to \mathbb{R}$

E_v: the set of edges incident to v, let

$$u_e(v) = u_e(0) \text{ or } u_e(v) = u_e(\ell_e) \text{ for } e \in E_v$$

Definition

u is continuous in vertex v if $u_e(v)$ is the same value for each $e \in E_v$, notation: $u(v)$.
u is continuous on G, if it is continuous in each vertex.

Kirchhoff–Neumann-condition in vertex v

If the one-sided derivatives of u_{e} exist at the endpoints of $\mathrm{e} \cong [0, \ell_{\mathrm{e}}]$, let

$$u_{\mathrm{e}}'(\mathrm{v}) = u_{\mathrm{e}}'(0) \text{ or } u_{\mathrm{e}}'(\mathrm{v}) = -u_{\mathrm{e}}'(\ell_{\mathrm{e}}) \text{ for } \mathrm{e} \in \mathrm{E}_{\mathrm{v}}.$$

For some $(c_{\mathrm{e}}(\mathrm{v}))_{\mathrm{e} \in \mathrm{E}_{\mathrm{v}}}$

Definition

Kirchhoff(–Neumann)-condition is satisfied in v *if*

$$\sum_{\mathrm{e} \in \mathrm{E}_{\mathrm{v}}} c_{\mathrm{e}}(\mathrm{v}) \cdot u_{\mathrm{e}}'(\mathrm{v}) = 0.$$

Diffusion problem on G

$$\begin{cases} \dot{u}_{\mathrm{e}}(t,x) = (c_{\mathrm{e}} u_{\mathrm{e}}')'(t,x) - p_{\mathrm{e}}(x) u_{\mathrm{e}}(t,x), & x \in (0,\ell_{\mathrm{e}}),\ t>0,\ \mathrm{e} \in \mathrm{E}, \quad (a) \\ u(t,\cdot) \text{ is continuous on G}, & t>0, \quad (b) \\ 0 = \sum_{\mathrm{e} \in \mathrm{E_v}} c_{\mathrm{e}}(\mathrm{v}) \cdot u_{\mathrm{e}}'(t,\mathrm{v}), & t>0,\ \mathrm{v} \in \mathrm{V}, \quad (c) \\ u_{\mathrm{e}}(0,x) = u_{0,\mathrm{e}}(x), & x \in [0,\ell_{\mathrm{e}}],\ \mathrm{e} \in \mathrm{E}, \quad (d) \end{cases}$$

$$c_{\mathrm{e}} \in C[0,\ell_{\mathrm{e}}], \quad 0 < c_0 \le c_{\mathrm{e}}(x), \quad x \in [0,\ell_{\mathrm{e}}],\ \mathrm{e} \in \mathrm{E}$$

$$0 \le p_{\mathrm{e}} \in L^\infty(0,\ell_{\mathrm{e}}), \quad \mathrm{e} \in \mathrm{E}$$

notice: $2 \cdot |\mathrm{E}|$ boundary conditions!

Spaces and operators

state space of the edges:

$$\mathcal{H} := \prod_{\mathsf{e}\in\mathsf{E}} L^2(0,\ell_{\mathsf{e}}),\ \mathcal{H}^2 := \prod_{\mathsf{e}\in\mathsf{E}} H^2(0,\ell_{\mathsf{e}})$$

boundary space of the vertices: $\mathbb{R}^n$ with $|\mathsf{V}| = n$
"feedback" operator $C\colon \mathcal{D}(C) \subset \mathcal{H} \to \mathbb{R}^n$,

$$\mathcal{D}(C) = \left\{u \in \mathcal{H}^2 \colon u \text{ is continuous on } \mathsf{G}\right\};$$

$$Cu = \left(\sum_{\mathsf{e}\in\mathsf{E}_{\mathsf{v}}} c_{\mathsf{e}}(\mathsf{v}) \cdot u'_{\mathsf{e}}(\mathsf{v})\right)_{\mathsf{v}\in\mathsf{V}} \in \mathbb{R}^n$$

system operator $A\colon \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$,

$$\mathcal{D}(A) := \{u \in \mathcal{H}^2 \colon u \text{ is continuous on } \mathsf{G} \text{ and } Cu = 0_{\mathbb{R}^n}\};$$

$$A := \operatorname{diag}\left(\frac{d}{dx}\left(c_{\mathsf{e}}\frac{d}{dx}\right) - p_{\mathsf{e}}\right)_{\mathsf{e}\in\mathsf{E}}$$

Diffusion problem – (ACP)

diffusion problem on G $\Longleftrightarrow$

$$\begin{cases} \dot{u}(t) = A\,u(t), & t > 0, \\ u(0) = u_0, \end{cases} \tag{ACP}$$

$u_0 = (u_{0,\mathrm{e}})_{\mathrm{e}\in\mathrm{E}} \in \mathcal{H}$

Operator semigroups

Engel, Nagel (2000); Arendt, Batty, Hieber, Neubrander (2011)

Let E be a Banach space, $\mathcal{L}(E)$ the bounded linear operators on E.

Definition

The function $S\colon [0,\infty) \to \mathcal{L}(E)$ is called strongly continuous semigroup or C_0-semigroup on E if

1. $S(t+s) = S(t)S(s), \quad s,t \geq 0, \quad S(0) = \mathrm{Id}_E$;
2. $[0,\infty) \ni t \mapsto S(t)x \in E$ *is continuous for all* $x \in E$.

Notation: $(S(t))_{t\geq 0}$.

Operator semigroups

Definition

It can be seen that each C_0-semigroup S admits a unique generator $(A, \mathcal{D}(A))$ that is a densely defined and closed linear operator with

$$Ax = \lim_{t \downarrow 0} \frac{S(t)x - x}{t}, \quad \mathcal{D}(A) = \left\{ x \in E : \exists \lim_{t \downarrow 0} \frac{S(t)x - x}{t} \right\}.$$

Proposition

For each C_0-semigroup $(S(t))_{t \geq 0}$ $\exists\, M \geq 1$ and $\exists\, w \in \mathbb{R}$ s.t.

$$\|S(t)\| \leq M \cdot e^{wt}, \quad t \geq 0$$

and

$$\{\lambda : \operatorname{Re} \lambda > w\} \subset \rho(A).$$

Operator semigroups

Let A be a closed linear operator, define the abstract Cauchy problem

$$\begin{cases} \dot{u}(t) = A\,u(t), t > 0, \\ u(0) = x \in E. \end{cases} \tag{ACP}$$

A mild solution of (ACP) is a function $u \in C(\mathbb{R}_+, E)$ with

$$\int_0^t u(s)\,ds \in \mathcal{D}(A) \text{ and } A\int_0^t u(s)\,ds = u(t) - x,\ \ t > 0.$$

Theorem

T.f.a.e.

- (i) *For all $x \in E$ there exists a unique mild solution to (ACP);*
- (ii) *The operator $(A, \mathcal{D}(A))$ generates a C_0-semigroup $(S(t))_{t\geq 0}$ on E.*

In this case, the mild solution is given by $u(t) = S(t)x$, $t \geq 0$.

Operator semigroups

Definition

The semigroup S is called bounded analytic if $\exists\, \theta \in (0, \frac{\pi}{2}]$ s.t. S has a holomorphic extension to the sector

$$\Sigma_\theta := \{z \in \mathbb{C} \setminus \{0\} \colon |\arg z| < \theta\}$$

which is bounded on $\Sigma_{\theta'}$ for each $\theta' \in (0, \theta)$. An analytic semigroup is contractive if $S(t)$ is a contraction for each $t > 0$.

Proposition

The C_0-semigroup $(S(t))_{t \geq 0}$ is bounded analytic iff its generator $(A, \mathcal{D}(A))$ is sectorial, that is, $\exists\, \theta \in (0, \frac{\pi}{2}]$ s.t.

$$\Sigma_{\frac{\pi}{2}+\theta} \subset \rho(A) \quad \text{and} \quad \sup_{\lambda \in \Sigma_{\frac{\pi}{2}+\theta-\varepsilon}} \|\lambda R(\lambda, A)\| < \infty \text{ for all } \varepsilon > 0.$$

Back to the network problem

$$A := \operatorname{diag}\left(\frac{d}{dx}\left(c_{\mathrm{e}}\frac{d}{dx}\right) - p_{\mathrm{e}}\right)_{\mathrm{e}\in\mathrm{E}}$$

$$\mathcal{D}(A) := \{u \in \mathcal{H}^2 \colon u \text{ is continuous on G and } Cu = 0_{\mathbb{R}^n}\}$$

on $\mathcal{H} = \prod_{\mathrm{e}\in\mathrm{E}} L^2(0,\ell_{\mathrm{e}})$ with $C\colon \mathcal{H}^2 \to \mathbb{R}^n$, $Cu = \left(\sum_{\mathrm{e}\in\mathrm{E}_{\mathrm{v}}} c_{\mathrm{e}}(\mathrm{v}) \cdot u_{\mathrm{e}}'(\mathrm{v})\right)_{\mathrm{v}\in\mathrm{V}}$.

Proposition (Mugnolo '07, Kovács, S. '21)

1. *$(A,\mathcal{D}(A))$ generates a C_0-semigroup on $\mathcal{H}$, that is, for all $u_0 \in \mathcal{H}$ there exists a unique mild solution to (ACP) – hence, for the diffusion problem on G;*
2. *Moreover, $(A,\mathcal{D}(A))$ is dissipative, sectorial and self-adjoint, $(0,+\infty) \subset \rho(A)$. Thus the C_0-semigroup $(S(t))_{t\geq 0}$ generated by $(A,\mathcal{D}(A))$ is analytic, positive and contractive.*

Noise in the Kirchhoff–Neumann-condition

$$\begin{cases} \dot u_\mathrm{e}(t,x) = (c_\mathrm{e} u_\mathrm{e}')'(t,x) - p_\mathrm{e}(x) u_\mathrm{e}(t,x), & x \in (0,\ell_\mathrm{e}),\ t \in (0,T],\ \mathrm{e} \in \mathrm{E}, \quad (a) \\ u(t,\cdot) \text{ is continuous on } \mathrm{G}, & t \in (0,T], \quad (b) \\ \dot\beta_\mathrm{v}(t) = \sum_{\mathrm{e} \in \mathrm{E}_\mathrm{v}} c_\mathrm{e}(\mathrm{v}) \cdot u_\mathrm{e}'(t,\mathrm{v}), & t \in (0,T],\ \mathrm{v} \in \mathrm{V}, \quad (c) \\ u_\mathrm{e}(0,x) = u_{0,\mathrm{e}}(x), & x \in [0,\ell_\mathrm{e}],\ \mathrm{e} \in \mathrm{E}, \quad (d) \end{cases}$$

$(\Omega, \mathscr{F}, \mathbb{P})$ complete probability space, $(\mathscr{F}_t)_{t\in[0,T]}$ right continuous filtration,

$$(\beta(t))_{t\in[0,T]} = \left((\beta_\mathrm{v}(t))_{t\in[0,T]} \right)_{\mathrm{v}\in\mathrm{V}},$$

$\mathbb{R}^n$-valued Brownian-motion (Wiener-process) with covariance matrix

$$Q \in \mathbb{R}^{n\times n}$$

Mild solution

$X(\cdot) \in C\left([0,T], L^2(\Omega,\mathcal{H})\right)$ is a mild solution to the problem if $\{X(t)\}_{t\in[0,T]}$ is $(\mathscr{F}_t)_{t\in[0,T]}$-adapted and for all $t\in[0,T]$

$$X(t) = X(t,u_0) = S(t)u_0 + \int_0^t (1-A)S(t-s)D\,d\beta(s), \quad t\in[0,T]$$

$\mathbb{P}$-a.e. where $D := D_{C,1}$ is the Dirichlet-operator

Greiner (87) $\Rightarrow$ for all $\lambda\in\rho(A)$ exists the Dirichlet-operator

$$D_{C,\lambda} = \left(C\mid_{\mathrm{Ker}(\lambda - A_{\max})}\right)^{-1} \in \mathcal{L}(\mathbb{R}^n;\mathcal{H}) \text{ with}$$

$$A_{\max} := \mathrm{diag}\left(\frac{d}{dx}\left(c_e\frac{d}{dx}\right) - p_e\right)_{e\in E}$$

$$\mathcal{D}(A_{\max}) := \left\{u\in\mathcal{H}^2 \colon u \text{ is continuous on G}\right\}$$

Existence and uniqueness of the mild solution

A generates a contractive, analytic semigroup $\Rightarrow$ for $\alpha \in (0,1)$ we can define the fractional domain spaces of A as

$$\mathcal{H}_\alpha := \mathcal{D}((1-A)^\alpha), \quad \|u\|_\alpha := \|(1-A)^\alpha u\|, \quad u \in \mathcal{D}((1-A)^\alpha)$$

Theorem (Kovács, S. '24)

For $\alpha < \frac{1}{4}$ and $u_0 \in \mathcal{H}_\alpha$, there exists a unique mild solution to the vertex noise problem, and it has a continuous version in $\mathcal{H}_\alpha$.

Proof: Da Prato, Zabczyk (93) $\Rightarrow$ we have to show that for $\alpha < \frac{1}{4}$ there exists $\gamma > 0$ s. t.

$$\int_0^T t^{-\gamma} \left\|(1-A)S(t)D_{C,1}Q^{\frac{1}{2}}\right\|^2_{HS(\mathbb{R}^n,\mathcal{H}_\alpha)} dt < +\infty.$$

By a recent result of Bolin, Kovács, Kumar and Simas (2024) for $\alpha < \frac{1}{4}$ and $\varepsilon > 0$ small enough, $(1-A)^{\frac{1}{2}+\alpha+\varepsilon}D_{C,1}$ is a bounded operator from $\mathbb{R}^n$ to $\mathcal{H}$.

Existence and uniqueness of the mild solution

Proof (continued):

$$
\begin{aligned}
&\int_0^T t^{-\gamma} \left\|(1-A)S(t)D_{C,1}Q^{\frac{1}{2}}\right\|^2_{\mathrm{HS}(\mathbb{R}^n,\mathcal{H}_\alpha)} dt = \\
&\int_0^T t^{-\gamma} \left\|(1-A)^{\frac{1}{2}-\varepsilon}S(t)(1-A)^{\alpha+\varepsilon-\frac{1}{2}}(1-A)D_{C,1}Q^{\frac{1}{2}}\right\|^2_{\mathrm{HS}(\mathbb{R}^n,\mathcal{H})} dt \\
&\leq \int_0^T t^{-\gamma} \left\|(1-A)^{\frac{1}{2}-\varepsilon}S(t)\right\|^2 dt \cdot \left\|(1-A)^{\alpha+\varepsilon+\frac{1}{2}}D_{C,1}\right\|^2_{\mathrm{HS}(\mathbb{R}^n,\mathcal{H})} \cdot \mathrm{Tr}(Q) \\
&\leq c_T \cdot \int_0^T t^{-\gamma} \cdot \frac{1}{t^{1-2\varepsilon}} dt,
\end{aligned}
$$

$\gamma < 2\varepsilon \implies$ the last integral is finite.

Strong Feller property of the transition semigroup

Definition

The transition semigroup of the solutions $X(t) = X(t, u_0)$ is defined as

$$(\mathcal{P}_t\phi)(u_0) := \mathbb{E}\left(\phi(X(t, u_0))\right), \qquad t \in [0, T], \ \ \phi \in \mathcal{B}_b(\mathcal{H}).$$

$(\mathcal{P}_t)_{t\geq 0}$ *is said to be strong Feller at time* $T > 0$ *if*

$$\text{for any } \phi \in \mathcal{B}_b(\mathcal{H}), \quad \mathcal{P}_T\phi \in C_b(\mathcal{H}) \text{ holds.}$$

Theorem (Fkirine, Kovács, S. '25)

Let G *be a tree,* $c_\mathsf{e} = 1$, $p_\mathsf{e} = 0$, $\mathsf{e} \in \mathsf{E}$; *that is,* A *is the Laplacian on* G. *If the covariance matrix* $Q = \mathrm{diag}(q_\mathsf{v})_{\mathsf{v}\in\mathsf{V}}$ *of the Kirchhoff noise is diagonal, and* $q_\mathsf{v} \neq 0$ *for all terminal vertices except for, possibly, one of them, then the transition semigroup* $\mathcal{P}_t$ *is strong Feller at any time* $T > 0$.

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Thank you for your attention!