Step-size coefficients for boundedness of linear multistep methods

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Step-size coefficients for boundedness of LMMs

Context

- Presentation is based on the paper L. L., *Exact optimal values of step-size coefficients for boundedness of linear multistep methods*, Numerical Algorithms, 75 (2017)
- Numerical experiments and symbolic proofs have been carried out by using the Wolfram Language (*Mathematica*)
- Consider an initial-value problem $u'(t) = F(u(t))$ for $t \geq 0$ with $u(0) = u_0$
- Approximate its solution by a linear multistep method (LMM):
  $u_n = \sum_{j=1}^{k} a_j\, u_{n-j} + \Delta\mathrm{t} \sum_{j=0}^{k} b_j\, F(u_{n-j})$ (for $n \geq k$)
- Basic assumptions on the LMM: consistency, zero-stability, irreducibility, $b_0 \geq 0$
- Monotonicity or boundedness properties play an important role: $\exists\, ?\, \mu \geq 1$ such that $\|u_n\| \leq \mu \max_{0 \leq j \leq k-1} \|u_j\|$ (for $n \geq k$)
- How to guarantee the monotonicity or boundedness property?
- One possibility: impose some restrictions on the step size $\Delta\mathrm{t}$

Step-size coefficients for boundedness of LMMs

Restriction on the step size: SCB or SSP coefficients

Definition 1.2. *Suppose that some method coefficients $a_j \in \mathbb{R}$ $(1 \leq j \leq k)$ and $b_j \in \mathbb{R}$ $(0 \leq j \leq k)$ satisfying (5) are given. We say that $\gamma > 0$ is a step-size coefficient for boundedness (SCB) of the corresponding LMM, if $\exists\, \mu \geq 1$ such that*

- *for any vector space with seminorm* $(\mathbb{V}, \|\cdot\|)$,
- *for any function* $F : \mathbb{V} \to \mathbb{V}$ *satisfying*

$$\exists \tau > 0 \quad \forall v \in \mathbb{V} \ : \ \|v + \tau F(v)\| \leq \|v\|,$$

- *for any* $\Delta t \in (0, \gamma\, \tau]$,
- *and for any starting vectors* $u_j \in \mathbb{V}$ $(0 \leq j \leq k-1)$,

the sequence u_n *generated by* (3) *has the property* $\|u_n\| \leq \mu \cdot \max_{0 \leq j \leq k-1} \|u_j\|$ *for all* $n \geq k$.

- If $\mu = 1$, the method is SSP (strong-stability preserving), and $\gamma > 0$ is the SSP coefficient.
- Clearly: larger $\gamma > 0 \Longrightarrow$ larger step sizes $\Longrightarrow$ more efficient numerical method

Step-size coefficients for boundedness of LMMs

Fundamental questions

- Decide if $\exists\, \gamma>0$ SCB or SSP coefficient
- Decide if a given $\gamma>0$ is an SCB or SSP coefficient
- Find the maximum $\gamma>0$ SCB or SSP coefficient
- Clearly: $\exists\, \gamma>0$ SSP coefficient $\Longrightarrow \gamma$ is an SCB as well
- It is easy to answer the above questions for SSP coefficients

Step-size coefficients for boundedness of LMMs

Existence and computation of SSP coefficients

- There are *simple* (necessary and sufficient) conditions to check whether ∃ an SSP coefficient or to find the largest SSP coefficient for a given LMM:
- ∃ SSP coefficient $\Longleftrightarrow$

$$b_0 \geqslant 0, \qquad a_j \geqslant 0, \qquad b_j \geqslant 0 \quad (\text{for } 1 \leqslant j \leqslant k), \quad \text{and} \quad a_i > 0 \quad \text{for all } i \in \{1, 2, \ldots, k\} \text{ with } b_i > 0.$$

- For a given $\gamma > 0$ to be an SSP coefficient, it is necessary and sufficient:

$$b_0 \geqslant 0, \quad \text{and} \quad a_j \geqslant 0, \qquad b_j \geqslant 0, \qquad \gamma b_j \leqslant a_j \quad (\text{for } 1 \leqslant j \leqslant k).$$

- However, for many practically relevant methods: ∄ positive SSP coefficient, but ∃ positive SCB

Step-size coefficients for boundedness of LMMs

Existence and computation of SCB

- It is more difficult to check whether ∃ an SCB,
- or to determine if a given positive number is a SCB,
- or to compute the maximum SCB–even for a single LMM.
- In W. Hundsdorfer (1954–2017), A. Mozartova, M. N. Spijker, *Stepsize restrictions for boundedness and monotonicity of multistep methods*, J. Sci. Comput. 50 (2012), 265–286, they define

$$\mu_n(\gamma) := \begin{cases} 0 & \text{for } n < 0, \\ b_n - \gamma\, b_0 \mu_n(\gamma) + \displaystyle\sum_{j=1}^{k} (a_j - \gamma\, b_j) \mu_{n-j}(\gamma) & \text{for } 0 \le n \le k, \\ -\gamma\, b_0 \mu_n(\gamma) + \displaystyle\sum_{j=1}^{k} (a_j - \gamma\, b_j) \mu_{n-j}(\gamma) & \text{for } n > k. \end{cases}$$

- Notice that $\mu_n(\gamma)$ is determined *only by the coefficients* of the LMM

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Determining if a given positive number is a SCB

- W. Hundsdorfer, A. Mozartova, M. N. Spijker:

> **Theorem 1.6** *Suppose the LMM satisfies* (5) *and let* $\gamma > 0$ *be given. Then* γ *is a SCB if and only if*
>
> $$-\gamma \in \operatorname{int}(\mathcal{S}), \text{ and } \mu_n(\gamma) \geq 0 \text{ for all } n \in \mathbb{N}^+. \tag{8}$$

- We need to check ∞ *many* sign conditions
- They typically checked these conditions for $1 \leq n \leq 1000$–if $\exists$ positive SCB
- But what if $\nexists$ positive SCB?

Step-size coefficients for boundedness of LMMs

Deciding whether ∃ positive SCB

- M. N. Spijker, *The existence of stepsize-coefficients for boundedness of linear multistep methods,* Appl. Numer. Math. 63 (2013) 45–57:

$$\tau_n = \begin{cases} 0 & \text{for } n < 0, \\ b_n + \sum_{j=1}^{k} a_j \tau_{n-j} & \text{for } 0 \le n \le k, \\ \sum_{j=1}^{k} a_j \tau_{n-j} & \text{for } n > k. \end{cases}$$

- An *almost* necessary and sufficient condition for ∃ positive SCB: the strict positivity of τ_n
- The above sequence is easier to study: no dependence on a parameter (γ)
- The author analyzes the LMM families: Adams–Moulton (or implicit Adams), Adams–Bashforth (or explicit Adams), BDF, extrapolated BDF (EBDF), Milne–Simpson, Nyström

Step-size coefficients for boundedness of LMMs

Example (M. N. Spijker)

- Abbreviation here: SCM = step-size coefficient for monotonicity = SSP coefficient

Corollary 2.2 *In the EBDF family*

- $\exists$ SCM $>$ 0 *for the 1-step EBDF method;*
- $\nexists$ SCM $>$ 0 *but* $\exists$ SCB $>$ 0 *for the k-step EBDF method with* $k \in \{2, 3, 4, 5\}$*;*
- $\nexists$ SCB $>$ 0 *for the 6-step EBDF method.*

Step-size coefficients for boundedness of LMMs

Exact optimal value $\gamma_{\sup}$ of SCB in the Adams–Bashforth family

- We effectively use the criterion with the parametric sequence: $\mu_n(\gamma) \geq 0$ for all $n \in \mathbb{N}^+$
- In the AB family, $\mu_n(\gamma)$ is a polynomial in γ for each n

Theorem 2.4 *The optimal values of the step-size coefficients for boundedness in the Adams–Bashforth family are given by the rational numbers below:*

- $\gamma_{\sup,1} = 1$;
- $\gamma_{\sup,2} = 4/9 \approx 0.44444$;
- $\gamma_{\sup,3} = 84/529 \approx 0.15879$;
- *for* $k = 4$, $\nexists$ SCB > 0.

Step-size coefficients for boundedness of LMMs

Exact optimal values $\gamma_{\sup}$ of SCB in the BDF family for $k = 1$ and $k = 2$:

- We again use the criterion with the parametric sequence: $\mu_n(\gamma) \geq 0$ for all $n \in \mathbb{N}^+$
- In the BDF family, $\mu_n(\gamma)$ is a rational function in γ for each n, and it turns out that
  - $\gamma_{\sup,1} = +\infty$;
  - $\gamma_{\sup,2} = 1/2$;
- For $k = 1$ (implicit Euler method), the corresponding recursion is

$$(\gamma + 1)\mu_n(\gamma) - \mu_{n-1}(\gamma) = 0 \qquad (n \geq 1)$$

with

$$\mu_0(\gamma) = \frac{1}{\gamma + 1}.$$

The explicit solution is $\mu_n(\gamma) = 1/(\gamma + 1)^{n+1} > 0$, so, due to Theorem 1.6, we have that γ is a SCB for any $\gamma > 0$.

- For $k = 2$, the corresponding recursion is

$$(2\gamma + 3)\mu_n(\gamma) - 4\mu_{n-1}(\gamma) + \mu_{n-2}(\gamma) = 0 \qquad (n \geq 2)$$

with

$$\mu_0(\gamma) = \frac{2}{2\gamma + 3}, \quad \mu_1(\gamma) = \frac{8}{(2\gamma + 3)^2}.$$

Its characteristic polynomial $\mathcal{P}_2(\cdot, \gamma)$ is quadratic for $\gamma > 0$. This polynomial has

- two distinct real roots for $0 < \gamma < 1/2$;
- a double real root for $\gamma = 1/2$;
- a pair of complex conjugate roots for $\gamma > 1/2$.

For any fixed $\gamma > 1/2$ we thus have

$$\mu_n(\gamma) = |\varrho_1(\gamma)|^n \left[c_1(\gamma) \left(\frac{\varrho_1(\gamma)}{|\varrho_1(\gamma)|} \right)^n + \overline{c_1(\gamma)} \left(\frac{\overline{\varrho_1(\gamma)}}{|\varrho_1(\gamma)|} \right)^n \right]$$

- Since the polynomial is quadratic, one can directly handle these expressions

Step-size coefficients for boundedness of LMMs

Computational details in the BDF family for $k = 3$: an upper bound

- For $k = 3$, the corresponding recursion is

$$(6\gamma + 11)\mu_n(\gamma) - 18\mu_{n-1}(\gamma) + 9\mu_{n-2}(\gamma) - 2\mu_{n-3}(\gamma) = 0 \qquad (n \geq 3)$$

with

$$\mu_0(\gamma) = \frac{6}{6\gamma + 11}, \quad \mu_1(\gamma) = \frac{108}{(6\gamma + 11)^2}, \quad \mu_2(\gamma) = \frac{54(-6\gamma + 25)}{(6\gamma + 11)^3}.$$

- A useful lemma involving a simple root of the function $\mu_n(\cdot)$:

Lemma 3.1 *Suppose there exist some $n \in \mathbb{N}^+$ and $\gamma^* > 0$ such that $\mu_n(\gamma^*) = 0$ and $\mu_n'(\gamma^*) \in \mathbb{R} \setminus \{0\}$. Then $\gamma_{\text{sup}} \leq \gamma^*$.*

- Let us consider the 6th term

$$\mu_6(\gamma) = \frac{6\left(5184\gamma^4 - 539352\gamma^3 + 4277340\gamma^2 - 7093698\gamma + 3248425\right)}{(6\gamma + 11)^7}.$$

The polynomial $\{5184, -539352, 4277340, -7093698, 3248425\}$ in the numerator has 4 real roots; let $\gamma^* \approx 0.831264$ denote its smallest root (the other three zeros are located at ≈ 1.22747, ≈ 6.42689, and ≈ 95.556). Then, due to Lemma 3.1, we have $\gamma_{\text{sup},3} \leq \gamma^*$.

Step-size coefficients for boundedness of LMMs

Computational details in the BDF family for $k = 3$: a lower bound

- It is easily seen from the definition that

$$\exists\, \gamma_0 > 0 : \mu_n(\gamma_0) \geq 0 \; (\forall\, n \in \mathbb{N}^+) \implies \gamma_{\sup} \geq \gamma_0.$$

- So to finish the proof of $\gamma_{\sup} = \gamma^* \approx 0.831264$ (an algebraic number of degree 4), we verify that $\mu_n(\gamma^*) \geq 0$ for each $n \in \mathbb{N}$
- In this case, the explicit form of $\mu_n(\gamma^*)$ is

$$\mu_n(\gamma^*) = c_1 \varrho_1^n + c_2 \varrho_2^n + \overline{c_2} (\overline{\varrho_2})^n \qquad (n \geq 0),$$

where

- $\varrho_1 \approx 0.500518$ is the largest real root of the polynomial

$$P_{\text{BDF31}} = \{34012224, -85030560, 108650160, -91171656, 55033668, -25076142, 8777889, -2366334, 486000, -75816, 10080, -1152, 64\};$$

- $\varrho_2 \approx 0.312678 + 0.390087 i$ is the root of P_{BDF31} with the largest real part;

Step-size coefficients for boundedness of LMMs

Computational details in the BDF family for $k = 3$: final steps of the proof of $\gamma_{\text{sup}} = \gamma^* \approx 0.831264$ (an algebraic number of degree 4)

- We take into account that $c_1 \approx 0.50155509$, and $\rho_1 \approx 0.500518$, and $|\rho_2| \approx 0.499935$ (exact algebraic numbers)
- ρ_1 and ρ_2 are close to each other

Now, clearly, $\mu_n(\gamma^*) = \varrho_1^n \left[c_1 + c_2 (\varrho_2/\varrho_1)^n + \overline{c_2}\, (\overline{\varrho_2}/\varrho_1)^n\right]$, and we have

$$\left| c_2 \left(\frac{\varrho_2}{\varrho_1}\right)^n + \overline{c_2} \left(\frac{\overline{\varrho_2}}{\varrho_1}\right)^n \right| \leq 2|c_2| \left|\frac{\varrho_2}{\varrho_1}\right|^n < 2 \cdot \frac{2777}{10000} \left(\frac{9989}{10000}\right)^n .$$

On the other hand,

$$2 \cdot \frac{2777}{10000} \left(\frac{9989}{10000}\right)^n < \frac{50155}{100\,000} < c_1$$

for $n \geq 93$, therefore $\mu_n(\gamma^*) > 0$ for $n \geq 93$.

Finally, one checks that $\mu_n(\gamma^*) > 0$ for $n \in \{0, 1, \ldots, 92\} \setminus \{6\}$ (recall that $\mu_6(\gamma^*) = 0$), so the proof is complete.

Remark 3.9 *We have* $\mu_{92}(\gamma^*) \approx 1.585176 \cdot 10^{-28}$.

Step-size coefficients for boundedness of LMMs

Computational details in the BDF family for $k = 4$

- For $k = 4$, the recursion is

$$(12\gamma + 25)\mu_n(\gamma) - 48\mu_{n-1}(\gamma) + 36\mu_{n-2}(\gamma) - 16\mu_{n-3}(\gamma) + 3\mu_{n-4}(\gamma) = 0 \qquad (n \geq 4)$$

with

$$\mu_0(\gamma) = \frac{12}{12\gamma + 25}, \quad \mu_1(\gamma) = \frac{576}{(12\gamma + 25)^2}, \quad \mu_2(\gamma) = \frac{1296(-4\gamma + 13)}{(12\gamma + 25)^3},$$

$$\mu_3(\gamma) = \frac{192\left(144\gamma^2 - 1992\gamma + 2137\right)}{(12\gamma + 25)^4}.$$

- This time it turns out that

$\gamma_{\text{sup},4} \approx 0.486220284043$ *is the unique real root of the fifth-degree polynomial*

{147456, −4065024, 97751296, −178921248, 146499984, −39945535};

Step-size coefficients for boundedness of LMMs

Computational details in the BDF family for $k = 4$: harder to analyze

- Recall Lemma 3.1: a simple root of the function $\mu_n(\cdot)$ for some n is an upper bound on γ_{sup}

Remark 3.4 *Obtaining the exact value of* $\gamma_{\text{sup},4} \approx 0.48622$ *proved to be significantly harder than determining that of* $\gamma_{\text{sup},3}$*, because we could not apply Lemma 3.1 to bound* $\gamma_{\text{sup},4}$ *from above. The value of* $\gamma_{\text{sup},4}$ *was found via a series of numerical experiments. For example, to see* $\gamma_{\text{sup},4} < 0.48625$*, one checks that the sequence* μ_n *in Theorem 1.6 for* $1 \leq n \leq 27000$ *satisfies*

$$\mu_n(48625/100000) < 0 \Longleftrightarrow n \in \{26814, 26875, 26886, 26936, 26947, 26997\}.$$

To find all these six indices, we used 16000 digits of precision to evaluate the terms of the recursion $\mu_n(48625/100000)$*—15000 digits would be insufficient. In fact, these experiments led to the formulation of Lemma 3.2.*

Step-size coefficients for boundedness of LMMs

Computational details in the BDF family for $k = 4$

- A dominant root of the characteristic polynomial = a root with maximum modulus
- A new observation: if a positive dominant root loses its dominant property at a certain value of the parameter γ^*, then $\gamma_{\sup} \leq \gamma^*$. More precisely we have the following *elementary* lemma.

The following lemma will be applied to bound $\gamma_{\sup}$ from above when the characteristic polynomial has a unique pair of complex conjugate roots that are dominant.

Lemma 3.2 *Suppose that* $z \in \mathbb{C} \setminus \mathbb{R}$ *with* $|z| = 1$, $w \in \mathbb{C} \setminus \{0\}$, *and a real sequence* $\nu_n \to 0$ $(n \to +\infty)$ *are given. Then* $wz^n + \bar{w}(\bar{z})^n + \nu_n < 0$ *for infinitely many* $n \in \mathbb{N}$.

- The situation becomes much harder to analyze, if there are e.g. 4 dominant complex roots. Many unsolved questions in this area, related to deep theorems in Diophantine approximation. Some recent progress:

11. Ouaknine, J., Worrell, J.: Decision problems for linear recurrence sequences. In: reachability problems, 6th International Workshop, RP 2012, Bordeaux, France, September 17–19 (2012). doi:10.1007/978-3-642-33512-9_3
12. Ouaknine, J., Worrell, J.: Positivity problems for low-order linear recurrence sequences. In: proceedings of the 25th Annual ACM-SIAM Symposium on Discrete Algorithms (2014). doi:10.1137/1.9781611973402.27
13. Ouaknine, J., Worrell, J.: Automata, languages, and programming. Part II: on the positivity problem for simple linear recurrence Sequences, pp. 318–329. Springer, Heidelberg (2014)
14. Ouaknine, J., Worrell, J.: Ultimate positivity is decidable for simple linear recurrence sequences, pp. 330-341. Springer, Heidelberg (2014)

Step-size coefficients for boundedness of LMMs

Computational details in the BDF family for $k = 4$: final steps of $\gamma_{\text{sup}} = \gamma^* \approx 0.48622$ (a degree 5 algebraic number)

- For $k = 4$ and $0 < \gamma < \frac{7}{12} \approx 0.5833$, the characteristic polynomial has 2 positive real ($\rho_{1,2}$) and 2 conjugate complex roots ($\rho_{3,4}$), and $0 < \rho_2 < \rho_1$
- Moreover we have
  - $|\varrho_3(\gamma)| = |\varrho_4(\gamma)| < \varrho_1(\gamma)$ for $0 < \gamma < \gamma^*$;
  - $|\varrho_3(\gamma^*)| = |\varrho_4(\gamma^*)| = \varrho_1(\gamma^*)$ for $\gamma = \gamma^*$;
  - $\varrho_1(\gamma) < |\varrho_3(\gamma)| = |\varrho_4(\gamma)|$ for $\gamma^* < \gamma < 7/12$.
- So Lemma 3.2 becomes applicable, and the rest of the proof is similar to the final steps of the proof in the previous $k = 3$ case

Step-size coefficients for boundedness of LMMs

Computational details for $k = 5$

$$(60\gamma + 137)\mu_n(\gamma) - 300\mu_{n-1}(\gamma) + 300\mu_{n-2}(\gamma) - 200\mu_{n-3}(\gamma) + 75\mu_{n-4}(\gamma) - 12\mu_{n-5}(\gamma) = 0 \qquad (n \geq 5)$$

with

$$\mu_0(\gamma) = \frac{60}{60\gamma + 137}, \quad \mu_1(\gamma) = \frac{18000}{(60\gamma + 137)^2}, \quad \mu_2(\gamma) = \frac{18000(-60\gamma + 163)}{(60\gamma + 137)^3},$$

$$\mu_3(\gamma) = \frac{12000\left(3600\gamma^2 - 37560\gamma + 30469\right)}{(60\gamma + 137)^4},$$

$$\mu_4(\gamma) = \frac{4500\left(-216000\gamma^3 + 8600400\gamma^2 - 22146420\gamma + 10021847\right)}{(60\gamma + 137)^5};$$

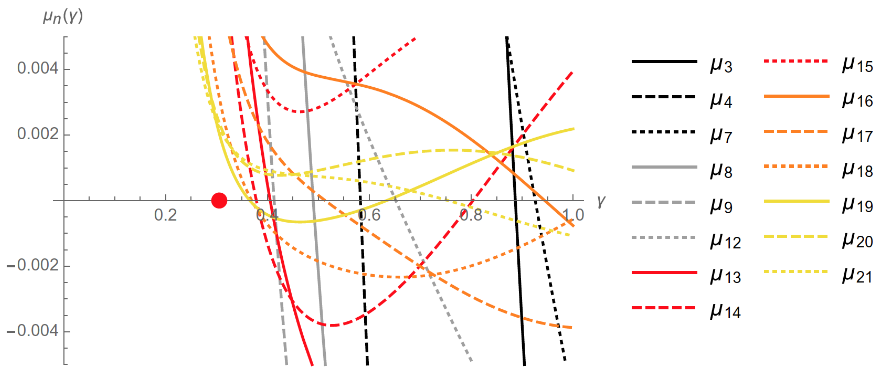


Fig. 1 The functions $\gamma \mapsto \mu_n(\gamma)$ for $1 \leq n \leq 21$ corresponding to the BDF5 method are shown (the curves with indices $n \in \{1, 2, 5, 6, 10, 11\}$ are not visible in this plot window). The *red dot* is placed at $\gamma = \gamma_{\mathrm{sup},5} \approx 0.30421$

Step-size coefficients for boundedness of LMMs

Computational details for $k = 5$

- $\gamma_{\text{sup},5} \approx 0.304213712525$ *is the smaller real root of the tenth-degree polynomial*

$$
\begin{array}{rrr}
\{9183300480000000000, & 8581284115200000000, & 119228009560272000000, \\
-1582364597979312000000, & 13003728314556711240000, & -34695982088244754164000, \\
52222192306393709117100000, & -49383429122661370894800000, & 28296029023568096013528000, \\
-8971403601204733655413800, & 1134065322004973267201570\};
\end{array}
$$

Step-size coefficients for boundedness of LMMs

Computational details for $k = 6$

$$3(20\gamma+49)\mu_n(\gamma) - 360\mu_{n-1}(\gamma) + 450\mu_{n-2}(\gamma) - 400\mu_{n-3}(\gamma) + 225\mu_{n-4}(\gamma) -$$

$$72\mu_{n-5}(\gamma) + 10\mu_{n-6}(\gamma) = 0 \qquad (n \geq 6)$$

with

$$\mu_0(\gamma) = \frac{20}{20\gamma+49}, \quad \mu_1(\gamma) = \frac{2400}{(20\gamma+49)^2},$$

$$\mu_2(\gamma) = \frac{3000(-20\gamma+47)}{(20\gamma+49)^3}, \quad \mu_3(\gamma) = \frac{8000\left(400\gamma^2 - 3440\gamma + 2131\right)}{3(20\gamma+49)^4},$$

$$\mu_4(\gamma) = \frac{500\left(-24000\gamma^3 + 695600\gamma^2 - 1343380\gamma + 474833\right)}{(20\gamma+49)^5},$$

$$\mu_5(\gamma) = \frac{160\left(480000\gamma^4 - 53296000\gamma^3 + 283987200\gamma^2 - 212499240\gamma + 84071653\right)}{(20\gamma+49)^6}.$$

Step-size coefficients for boundedness of LMMs

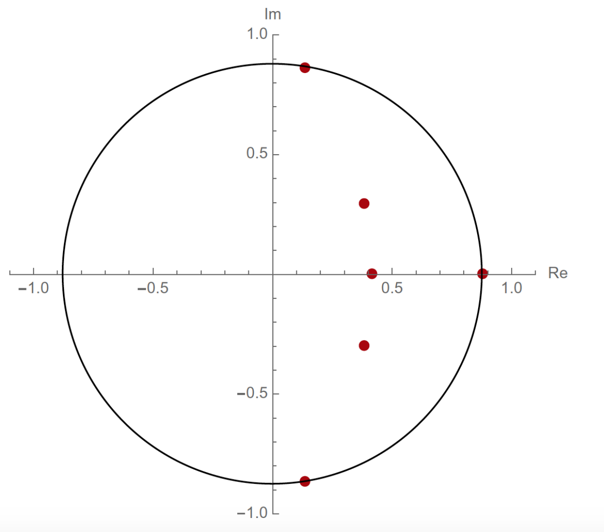
Computational details for $k = 6$

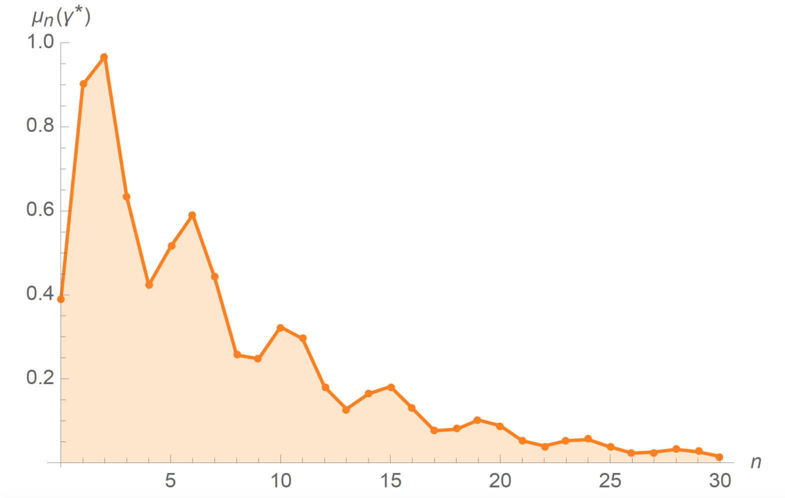
- $\gamma_{\text{sup},6} \approx 0.131359487166$ *is the smaller real root of the 18th-degree polynomial*

$$
\begin{array}{rr}
\{301499153838045275528311603200000000, & 1226395855345048398189452014387200000000, \\
384963168041618344234237602954215424000000, & 2754957003308188522312802320744458485760000, \\
6883218301719049493344792020881093689344000000, & -384146941872396676115776998321179378948505600, \\
114843588487750902323103668249803599786305126400, & -1006269459507863531788997342497299304467812843520, \\
5587246198359348966734174906666273788289332150272, & -17429944795858965010882996868073155329514839408640, \\
35959114141443095864886240750517884787497897431040, & -53357827225132542443145327442029250536098863687680, \\
58779078470720235677143648519968524504336318905600, & -48117131040654192740877887801688549303578668712064, \\
28809153195856173726312967696976168633917662024240, & -12158530101520566099221248226347019432756062262240, \\
3383327891741061214240426918034255832010259451480, & -541370800878125712591610585145194659522378896880, \\
333280926411862545507602476611681487682629370 67\}. &
\end{array}
$$

Step-size coefficients for boundedness of LMMs

Computational details: $k = 6$





Step-size coefficients for boundedness of LMMs

Summary for the BDF and AB families

We have identified two types of conditions that characterize γ_{sup} in these multistep families:

(i) a positive real dominant root of the characteristic polynomial corresponding to the recursion $\mu_n(\gamma)$ loses its dominant property at $\gamma = \gamma_{\text{sup}}$, or

(ii) there is an index $n_0 \in \mathbb{N}$ such that γ_{sup} is a simple root of the function $\mu_{n_0}(\cdot)$.

It turns out that γ_{sup} is determined

- by condition (i) for the BDF methods with $k \in \{2, 4, 5, 6\}$ steps;
- by condition (ii) with $n_0 = 6$ for the 3-step BDF method;
- by condition (ii) with $n_0 = 2$ for the Adams–Bashforth methods with $k \in \{1, 2, 3\}$ steps.