What is the difference between weakly and strongly stable linear multistep methods?

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in the memory of I. Mezei

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Linear multistep methods

IVP

$$\begin{cases} u(0) = u^0 \\ u'(t) = f(u(t)) \end{cases}$$

f Lipschitz

LMM

$$\begin{cases} u_i = c^i\,, & i = 0, \ldots, k-1 \\ \dfrac{1}{h}\displaystyle\sum_{j=0}^{k} \alpha_j u_{i-j} = \sum_{j=0}^{k} \beta_j f(u_{i-j})\,, & i = k, \ldots, n+k-1 \end{cases}$$

Characteristic polynomials

Characteristic polynomials

$$\varrho(x) = \sum_{j=0}^{k} \alpha_j x^{k-j}\,, \qquad \sigma(x) = \sum_{j=0}^{k} \beta_j x^{k-j}\,.$$

Consistency

$$\varrho(1) = 0 \qquad \text{and} \qquad \varrho'(1) = \sigma(1)$$

Root-conditions

Definition

- An LMM is *strongly stable*: for every root $\xi_i \in \mathbb{C}$ of the first characteristic polynomial $|\xi_i| < 1$ holds except $\xi_1 = 1$, which is a simple root.
- A not strongly stable method is *weakly stable*: for every root $\xi_i \in \mathbb{C}$ of the first characteristic polynomial $|\xi_i| \leq 1$ holds and if $|\xi_i| = 1$ then it is a simple root, moreover $\xi_1 = 1$.

Why should we distinguish them?

Motivational example

Test-equation

$$\begin{cases} \dot{y}(t) = \lambda y(t) \\ y(0) = 1 \end{cases}$$

solution: $y(t) = e^{\lambda t}$

Midpoint vs. AB2

$$y_n = y_{n-2} + 2h\, f_{n-1} \qquad \text{(Midpoint method)}$$

$$y_n = y_{n-1} + h\left(\frac{3}{2} f_{n-1} - \frac{1}{2} f_{n-2}\right) \qquad \text{(Adams–Bashforth 2)}$$

$\lambda < 0$, $|h\lambda| \ll 1$

Error

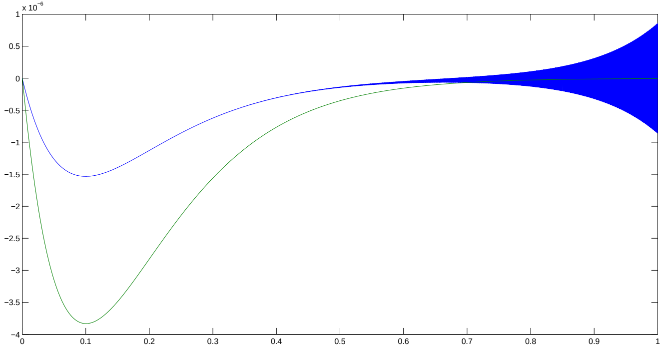

Figure 1 : $h = 5 \cdot 10^{-4}$, $T = 1$, $\lambda = -10$. Errors: Midpoint method (blue), Adams-Bashforth 2 (green).

Error

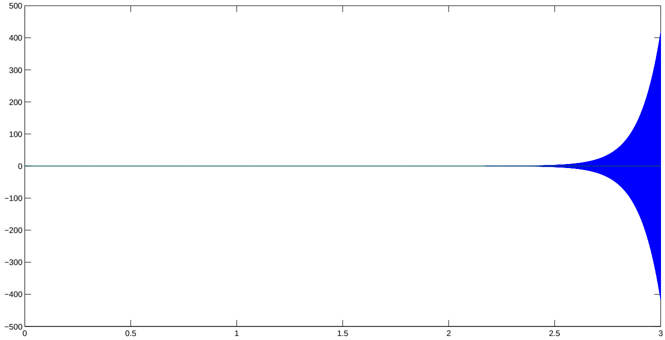

Figure 2 : $h = 5 \cdot 10^{-4}$, $T = 3$, $\lambda = -10$. Errors: Midpoint method (blue), Adams-Bashforth 2 (green).

Why?

Midpoint method

$$y_n = c_1\xi_1^n + c_2\xi_2^n$$

where ξ_1, ξ_2 are the roots of $\xi^2 = 1 + 2h\lambda\xi$.

$$\xi_1 = e^{h\lambda} + \mathcal{O}\left(h^3\lambda^3\right) \quad \text{- principal root}$$
$$\xi_2 = -e^{-h\lambda} + \mathcal{O}\left(h^3\lambda^3\right) \quad \text{- parasitic root}$$

Error

$$|\xi_1| < 1 < |\xi_2|, \qquad \xi_2 < 0$$

Endpoint error:

$$c_2 e^{-\lambda T}$$

Fortunately $c_1 = 1 + \delta$, $c_2 = -\delta$, with $\delta = \mathcal{O}\left(h^3\lambda^3\right)$

Why?

AB2

$$\xi_1 = e^{h\lambda} + \mathcal{O}\left(h^3\lambda^3\right)$$
$$\xi_2 = \frac{1}{2}\left(h\lambda - h^2\lambda^2\right) + \mathcal{O}\left(h^3\lambda^3\right)$$

Conclusion and questions

Midpoint method: growing oscillations

- Extension: general weakly stable case?!
- Extension: general (nonlinear) IVP?!

Lax-Stetter framework

Problem

$$F(u) = 0$$

$\mathcal{X}$ and $\mathcal{Y}$ are normed spaces, $\mathcal{D} \subset \mathcal{X}$ and $F : \mathcal{D} \to \mathcal{Y}$ is a (nonlinear) operator. It is assumed that there exists a unique solution $\bar{u}$.

Numerical method

Sequence $(\mathcal{X}_n, \mathcal{Y}_n, F_n)_{n\in\mathbb{N}}$ which generates a sequence of problems

$$F_n(u_n) = 0\,, \quad n = 1, 2, \dots\,,$$

where $\mathcal{X}_n, \mathcal{Y}_n$ are normed spaces, $\mathcal{D}_n \subset \mathcal{X}_n$ and $F_n : \mathcal{D}_n \to \mathcal{Y}_n$. If there exists a unique solution of the (approximating) problems, it will be denoted by $\bar{u}_n$.

Numerical methods

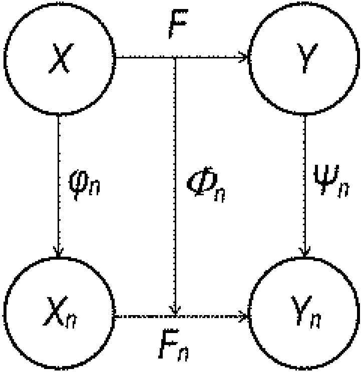

Figure 3 : The general scheme of numerical methods.

Definition

A numerical method is

- *convergent* if
$$\lim \|\varphi_n(\bar{u}) - \bar{u}_n\|_{\mathcal{X}_n} = 0$$
- *consistent* if
$$\lim \|F_n(\varphi_n(\bar{u}))\|_{\mathcal{Y}_n} = 0$$
- *stable* if there exist $S \in \mathbb{R}$, $R \in (0, \infty]$ such that $\forall (u_n)_{n\in\mathbb{N}}, (v_n)_{n\in\mathbb{N}}$ which satisfy $u_n, v_n \in B_R(\varphi_n(\bar{u}))$
$$\|u_n - v_n\|_{\mathcal{X}_n} \leq S \, \|F_n(u_n) - F_n(v_n)\|_{\mathcal{Y}_n}$$

Results

- $+$ some natural assumption $\Rightarrow \exists \bar{u}_n \in B_R(\varphi_n(\bar{u}))$
- consistency $+$ stability implies convergence:

$$\|\varphi_n(\bar{u}) - \bar{u}_n\|_{\mathcal{X}_n} \leq S\,\|F_n(\varphi_n(\bar{u})) - F_n(\bar{u}_n)\|_{\mathcal{Y}_n} =$$
$$S\,\|F_n(\varphi_n(\bar{u}))\|_{\mathcal{Y}_n} \to 0$$

LMMs in the Lax–Stetter framwork

$$F_n(\mathbf{u}_n) = \mathbf{A}_n\mathbf{u}_n - \mathbf{B}_n f(\mathbf{u}_n) - \mathbf{c}_n \,,$$

$$\mathbf{A}_n = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{A}_{n,\partial} & \mathbf{A}_{n,0} \end{pmatrix}, \qquad \mathbf{B}_n = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{n,\partial} & \mathbf{B}_{n,0} \end{pmatrix}$$

Example: Midpoint-method

$$(F_n(\mathbf{u}_n))_i = \begin{cases} u_0 - c_0 \,, & i = 0 \\ u_1 - c_1 \,, & i = 1 \\ \dfrac{u_i - u_{i-2}}{2h} - f_{i-1} \,, & i = 2, \dots, n \end{cases}$$

Example: Midpoint-method

$$F_n(\mathbf{u}_n) = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ -\frac{1}{2h} & 0 & \frac{1}{2h} & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & -\frac{1}{2h} & 0 & \frac{1}{2h} \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ \vdots \\ u_N \end{pmatrix} -$$

$$\begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{n+1} \end{pmatrix} - \begin{pmatrix} c_0 \\ c_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Norms

$k \in \mathbb{N}$ fixed, $\mathbf{u}_n \in \mathbb{R}^{k+n}$

- $k\infty$ norm:

$$\|\mathbf{u}_n\|_{k\infty} = \max_{0 \le i \le k-1} |u_i| + \max_{k \le i \le k+n-1} |u_i|$$

- $k1$ norm:

$$\|\mathbf{u}_n\|_{k1} = \max_{0 \le i \le k-1} |u_i| + h \sum_{i=k}^{k+n-1} |u_i|$$

- k–Spijker norm:

$$\|\mathbf{u}_n\|_{k\$} = \max_{0 \le i \le k-1} |u_i| + h \max_{k \le l \le k+n-1} \left| \sum_{i=k}^{l} u_i \right|$$

Norms

If $\mathbf{A}$ is a regular matrix and $\|\cdot\|_\star$ is a norm then $\|\mathbf{u}\|_{\mathbf{A},\star} = \|\mathbf{Au}\|_\star$ defines a norm.

Norm pairs

Emphasizing the importance of the norms in the stability estimate: a numerical method is stable in the norm pair $\left(\|\cdot\|_{\mathcal{X}_n}, \|\cdot\|_{\mathcal{Y}_n}\right)$ if

$$\|\mathbf{u}_n - \mathbf{v}_n\|_{\mathcal{X}_n} \leq S \, \|F_n(\mathbf{u}_n) - F_n(\mathbf{v}_n)\|_{\mathcal{Y}_n}$$

Stability results

Results

- Weakly stable methods are stable in the following norm pairs: $(\|\cdot\|_{k\infty}, \|\cdot\|_{k\infty})$, $(\|\cdot\|_{k1}, \|\cdot\|_{k1})$ and $(\|\cdot\|_{k\infty}, \|\cdot\|_{k1})$.
- Strongly stable methods are stable in the $(\|\cdot\|_{k\infty}, \|\cdot\|_{k\$})$ norm pair, as well.

Spijker's example

The midpoint-method is not stable in the $\left(\|\cdot\|_{k\infty}, \|\cdot\|_{k\$}\right)$ norm pair:

$$\|\mathbf{u}_n - \mathbf{v}_n\|_{k\infty} \leq S \|F_n(\mathbf{u}_n) - F_n(\mathbf{v}_n)\|_{k\$}$$

does not hold.

Idea

Growing oscillation

$$\begin{cases} u'(t) = 0 \\ u(0) = 0 \end{cases}$$

$\mathbf{u}_n = (0, 0, 1, -2, 3, -4, \dots)^T$, $\mathbf{v}_n = \mathbf{0}$

Spijker's example

$\mathbf{u}_n = (0, 0, 1, -2, 3, -4, \dots)^T$, $\mathbf{v}_n = \mathbf{0}$

$$\|F_n(\mathbf{u}_n) - F_n(\mathbf{v}_n)\|_{k\$} = \|\mathbf{A}_{n,0}\mathbf{u}_{n,0}\|_{\$}$$

$$\|\mathbf{u}_n - \mathbf{v}_n\|_{k\infty} = \|\mathbf{u}_{n,0}\|_{\infty} \, .$$

We can calculate

$$\mathbf{A}_{n,0}\mathbf{u}_{n,0} = \left(\frac{1}{2h}, -\frac{1}{h}, \frac{1}{h}, -\frac{1}{h}, \frac{1}{h}, \dots\right)^T ,$$

thus

$$\|\mathbf{A}_{n,0}\mathbf{u}_{n,0}\|_{\$} = \frac{1}{2} \quad \text{while} \quad \|\mathbf{u}_{n,0}\|_{\infty} = n \, .$$

Extention

Theorem

Weakly stable methods are not stable in the $\left(\|\cdot\|_{k\infty}, \|\cdot\|_{k\$}\right)$ *norm pair.*

Idea of the proof

A different growing oscillation: using the root at the boundary.

Why should we upgrade it?

Problems with Spijker's approach: stability in $\left(\|\cdot\|_{k\infty}, \|\cdot\|_{k\$}\right)$ also leads to convergence in the norm $\|\cdot\|_{k\infty}$.

Tricky example

$$(F_n(\mathbf{u}_n))_i = \begin{cases} u_0 - c_0 & \text{, if} \quad i = 0\,, \\ \dfrac{u_i - u_{i-1}}{h} - f_{i-1} & \text{, if} \quad 1 \leq i \leq n \quad \text{odd}\,, \\ \dfrac{u_i - u_{i-1}}{h} - f_i & \text{, if} \quad 2 \leq i \leq n \quad \text{even}\,. \end{cases}$$

Consistent of order 2 with respect to the $k\$$ norm, while seemingly it is consistent of order 1 with respect to the $k\infty$ or $k1$ norms.

Upgrading

$$F_n(\mathbf{u}_n) - F_n(\mathbf{v}_n) = \mathbf{A}_n(\mathbf{u}_n - \mathbf{v}_n) - \mathbf{B}_n(f(\mathbf{u}_n) - f(\mathbf{v}_n))$$

taking absolute value

$$|F_n(\mathbf{u}_n) - F_n(\mathbf{v}_n)| = |\mathbf{A}_n(\mathbf{u}_n - \mathbf{v}_n) - \mathbf{B}_n(f(\mathbf{u}_n) - f(\mathbf{v}_n))| \geq$$
$$|\mathbf{A}_n(\mathbf{u}_n - \mathbf{v}_n)| - |\mathbf{B}_n||f(\mathbf{u}_n) - f(\mathbf{v}_n)| \geq |\mathbf{A}_n(\mathbf{u}_n - \mathbf{v}_n)| - L|\mathbf{B}_n||\mathbf{u}_n - \mathbf{v}_n|$$

taking norms

$$\|F_n(\mathbf{u}_n) - F_n(\mathbf{v}_n)\|_{k1} \geq \|\mathbf{A}_n(\mathbf{u}_n - \mathbf{v}_n)\|_{k1} - L\,\||\mathbf{B}_n|\|_{k1}\,\|\mathbf{u}_n - \mathbf{v}_n\|_{k1}$$

using the stability estimate

$$S\,\|F_n(\mathbf{u}_n) - F_n(\mathbf{v}_n)\|_{k1} \geq \|\mathbf{u}_n - \mathbf{v}_n\|_{k1}$$

we get

$$C_1\,\|F_n(\mathbf{u}_n) - F_n(\mathbf{v}_n)\|_{k1} \geq \|\mathbf{u}_n - \mathbf{v}_n\|_{\mathbf{A}_n,k1}$$

$$C_1 \left\| F_n(\mathbf{u}_n) - F_n(\mathbf{v}_n) \right\|_{k1} \geq \left\| \mathbf{u}_n - \mathbf{v}_n \right\|_{\mathbf{A}_n, k1}$$

or similarly

$$C_2 \left\| F_n(\mathbf{u}_n) - F_n(\mathbf{v}_n) \right\|_{k\infty} \geq \left\| \mathbf{u}_n - \mathbf{v}_n \right\|_{\mathbf{A}_n, k\infty} .$$

Meaning:...

What is the difference between strongly and weakly stable methods?

Idea

Avoiding each type of growing oscillations.

Definition

An LMM is discrete C^1 stable if it is stable in every $\left(\|\cdot\|_{\mathbf{L}_n,k\infty}, \|\cdot\|_{k\infty}\right)$ norm pair, where $\mathbf{L}_n$ represents a k-step differentiation formula.

Theorem

An LMM is discrete C^1 stable if and only if it is stable in the $\left(\|\cdot\|_{\mathbf{E}_n,k\infty}, \|\cdot\|_{k\infty}\right)$ norm pair, where $\mathbf{E}_n$ represents the explicit Euler method

$$\mathbf{E}_n = \begin{pmatrix} 1 & 0 & 0 & \dots & \dots & 0 \\ 0 & \ddots & 0 & \dots & \dots & 0 \\ 0 & \dots & 1 & 0 & \dots & 0 \\ \dots & 0 & -\frac{1}{h} & \frac{1}{h} & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & -\frac{1}{h} & \frac{1}{h} \end{pmatrix}$$

Theorem

Strongly stable LMMs are discrete C^1 stable, while weakly stable LMMs are not.

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Thank you for your attention!