Stability and stabilization with applications

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FMSz

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Outline

- Stability analysis & inequalities
  - Multiple integral inequalities
  - Linear systems with constant discrete and distributed delays
  - Multiple summation inequalities
- Connections of different estimation approaches
  - Comparison with free matrix based approaches
  - Combination of basic inequalities with convexifying inequalities
- Control design
  - Synchronization of networks
  - Finite frequency H_∞ dynamic output feedback
  - Case studies

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Motivation

Linear continuous-time delay system

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + A_d x(t-\tau), \qquad t \in \mathbf{R}_{\geq 0}, && (1)\\
x_0(t) &= \varphi(t),\ t \in [-\tau, 0] \subset \mathbf{R}, && (2)
\end{aligned}
$$

where

- $x(t) \in \mathbf{R}^{n_x}$ is the state,
- A and A_d are constant matrices,
- τ is a constant time delay and
- $x_0(.)$ is the initial function.

Discrete-time analogue

$$
\begin{aligned}
x(t+1) &= Ax(t) + A_d x(t-\tau), \qquad t \in \mathbf{Z}_{\geq 0}, && (3)\\
x_0(t) &= \varphi(t),\ t \in [-\tau, 0] \subset \mathbf{Z}, , && (4)
\end{aligned}
$$

Motivation

Stability analysis:

- via the characteristic equation or the Razumikhin theorem,
- based on Lyapunov - Krasovskii functional.

A Lyapunov-Krasovskii functional: let $x_t(s) = x(t+s)$, and

$$\begin{aligned} V(x_t, \dot{x}_t) &= x(t)^T P x(t) + \int_{-\tau}^{0} x(s)^T Q x(s) ds \\ &+ \int_{-\tau}^{0} \int_{t+s}^{t} \dot{x}(\sigma)^T R \dot{x}(\sigma) d\sigma ds, \quad P, Q, R \in \mathbf{S}_{n_x}^{+}, \end{aligned}$$

The time derivative contains the term

$$-\int_{t-\tau}^{t} \dot{x}(\sigma)^T R \dot{x}(\sigma) d\sigma$$

How can it be estimated? $\Rightarrow$ Integral inequalities

Integral inequalities

(Join work with T. Takács, SCL, 96(2016)72-80)

Let $W \in \mathbf{S}_n^+$, and $\ell \in \mathbf{Z}_{\geq 0}$ be given. Consider

$$\begin{aligned} J_{W,\ell,a,b}(f) &= \frac{\ell!}{(b-a)^\ell} \int_a^b \int_{v_1}^b \dots \int_{v_\ell}^b f^T(s) W f(s) ds dv_\ell \dots dv_1 \\ &= \int_a^b \left(\frac{s-a}{b-a} \right)^\ell f^T(s) W f(s) ds, \qquad f \in C([a,b], \mathbf{R}^{\overline{n}}). \end{aligned}$$

Aim: derive a lower estimation of this functional.

For $g_1, g_2 \in L_2[a,b]$ define a scalar product by

$$\langle g_1, g_2 \rangle_{\ell,[a,b]} = \int_a^b \left(\frac{s-a}{b-a} \right)^\ell g_1(s) g_2(s) ds.$$

Then

$$J_{W,\ell,a,b}(f) = \langle f, Wf \rangle_{\ell,[a,b]}.$$

Integral inequalities

Orthogonal polynomials with respect to the previous scalar product:

$$p_{\ell,n}(t) = P_{\ell,n}\left(\frac{t-a}{b-a}\right) \qquad t \in [a,b],$$

where $P_{\ell,n}$ can be given by the *generalized Rodrigues-formula*:

$$\begin{aligned} P_{\ell,0}(x) &\equiv 1, & x \in [0,1], \\ P_{\ell,n}(x) &= \frac{1}{n!}\frac{1}{x^\ell}\frac{d^n}{dx^n}\left(x^\ell (x^2-x)^n\right), & n = 1,2,\ldots \end{aligned}$$

Properties:

$$\begin{aligned} p_{\ell,n}(b) &= 1, \qquad p_{\ell,n}(a) = (-1)^n \frac{\ell+n}{n}, \\ \|p_{\ell,n}\|^2_{\ell,[a,b]} &= \frac{b-a}{\ell+2n+1}. \end{aligned}$$

Integral inequalities

Lemma

Let $M > 0$, $\ell \geq 0$ and $\nu_\ell \geq 0$ be given integers satisfying the condition $\ell + \nu_\ell \leq M - 1$. Then

$$J_{W,\ell,a,b}(f) \geq \frac{1}{b-a} \Phi_M^T \left(\Xi_\ell \otimes I\right)^T \mathcal{W}_\ell \left(\Xi_\ell \otimes I\right) \Phi_M, \tag{5}$$

where

$$\begin{aligned} \mathcal{W}_\ell &= diag\{(\ell+1), (\ell+3), \ldots, (\ell+2\nu_\ell+1)\} \otimes W, \\ \Phi_M^T &= \left[\phi_0^T, \quad \ldots, \quad \phi_{M-1}^T\right] \text{ with } \phi_j = \int_a^b p_{0j}(s) f(s) ds, \end{aligned}$$

matrix $\Xi_\ell = \Xi_\ell(\nu_\ell, M-1)$ is connected with a basis transformation.

(The explicit formula is given in Gyurkovics-Takács, SCL, 96(2016)72-80)

Integral inequalities

Some special cases.

1. Case $\ell = 0, \nu_\ell = M-1, M \geq 1$:

$$J_{W,\ell,a,b}(f) \geq \frac{1}{b-a}\sum_{j=0}^{M-1}(2j+1)\phi_j^T W \phi_j,$$

   which is identical with

   - the Jensen inequality, if $M = 1$; (e.g. Briat, "Linear parameter-varying and time-delay systems." (2014).
   - the Wirtinger inequality, if $M = 2$; (Seuret-Gouaisbaut, Automatica, 49(2013)2860-2866.)
   - the Bessel-Legendre inequalities, if $M \geq 1$; (Seuret-Gouaisbaut, SCL, 81(2015))

Integral inequalities

2. Case $\ell > 0$,
   - for $\ell = 1, \nu_\ell = 0, M = 1$, it is the double integral Jensen inequality, (e.g. Sun et al. Int. J. Robust Nonlin. Control 19 (2009) 1364-1375)
   - for $\ell = 1, \nu_\ell = 1, M = 2$, it is the improvement of the double integral Wirtinger inequality, (Park et al. Automatica 55 (2015) 204-208)
   - for $\ell \geq 1, \nu_\ell = 1, M \geq 2$, it is the improvement of the multiple integral inequality of (Lee et al. JFI, 352 (2015) 5627-5645)
   - for the general case equivalent inequalities in (Park et al. Appl.Math. Lett. 77 (2018) 6-12)

Integral inequalities

Application to $J_{W,\ell,a,b}(f') = \langle f', Wf' \rangle_{\ell,[a,b]}$.

Lemma

Let $M > 0$, $\ell \geq 0$ and $\nu_\ell \geq 0$ be given integers satisfying the condition $\ell + \nu_\ell \leq \max\{0, M-1\}$. Then

$$J_{W,\ell,a,b}(f') \geq \frac{1}{b-a} \widetilde{\Phi}_M^T \left(\mathcal{Z}_\ell \otimes I\right)^T \mathcal{W}_\ell \left(\mathcal{Z}_\ell \otimes I\right) \widetilde{\Phi}_M, \qquad (6)$$

where $\mathcal{W}_\ell$ is the same as before , $\widetilde{\Phi}_0 = col\{f(b),\ f(a)\}$,
$\widetilde{\Phi}_M = col\left\{f(b),\ f(a),\ \frac{1}{b-a}\phi_0, \ldots, \frac{1}{b-a}\phi_{M-1}\right\}$, if $M > 0$,
$\mathcal{Z}_0 = \begin{bmatrix} \underline{\ell}_{\nu_0}^{(1)} & \underline{\ell}_{\nu_0}^{(2)} & -Z_0 \end{bmatrix}$, $\mathcal{Z}_\ell = \begin{bmatrix} \underline{\ell}_{\nu_\ell}^{(1)} & \underline{0}_{\nu_\ell} & -Z_\ell \end{bmatrix}$, if $\ell > 0$,
$\underline{\ell}_k^{(1)} = (1, \ldots, 1)^T$, $\underline{0}_k = (0, \ldots, 0)^T$, $\underline{\ell}_k^{(2)} = (-1, 1, \ldots, \pm 1)^T$ and $Z_\ell = Z_\ell(\nu_\ell, M-1)$ is connected with a basis transformation.

Stability analysis

Consider

$$\begin{aligned}\dot{x}(t) &= Ax(t) + A_{d_1}x(t-\tau) + A_{d_2}\int_{t-\tau}^{t} x(s)ds,\ t \geq 0,\\ x_0(t) &= \varphi(t),\ t \in [-\tau, 0],\end{aligned}$$

Let $M > 0,\ m_1 \geq 0,\ m_2 \geq 1$ be given integers,

$x_t(s) = x(t+s)$ be the solution,

$\Phi_M(t) = \text{col}\left\{\phi_0(t), ..., \phi_{M-1}(t)\right\}$ with

$\phi_j(t) = \int_{-\tau}^{0} p_{0j}(s)x_t(s)ds$, ($p_{0j}$ is the Legendre polynomial)

and introduce the exteded and augmented state variables

$$\begin{aligned}\widetilde{x}(t) &= \text{col}\left\{x(t), \Phi_M(t)\right\},\\ \widetilde{\Phi}_M(t) &= col\left\{x(t), x(t-\tau), \frac{1}{\tau}\Phi_M(t)\right\}.\end{aligned}$$

Stability analysis

Consider the LKF candidate

$$V(x_t, \dot{x}_t) = V_1(x_t) + V_2(x_t) + V_3(\dot{x}_t),$$

where

$$\begin{aligned}
V_1(x_t) &= \widetilde{x}(t)^T P \widetilde{x}(t), \\
&\qquad\qquad\qquad P \in \mathbf{S}_{n_x(M+1)}, \\
V_2(x_t) &= \sum_{j=0}^{m_1} \int_{-\tau}^{0} \left(\frac{s+\tau}{\tau}\right)^j x_t(s)^T Q_j x_t(s) ds, \\
&\qquad\qquad\qquad Q_j \in \mathbf{S}^+_{n_x}, \quad j = 0, \ldots, m_1, \\
V_3(\dot{x}_t) &= \tau \sum_{j=1}^{m_2} \int_{-\tau}^{0} \left(\frac{s+\tau}{\tau}\right)^j \dot{x}_t(s)^T R_j \dot{x}_t(s) ds, \\
&\qquad\qquad\qquad R_j \in \mathbf{S}^+_{n_x}, \quad j = 1, \ldots, m_2.
\end{aligned}$$

Stability analysis

Theorem

Let $M > 0$, $m_1 \geq 0$, $m_2 \geq 1$ *and* $\nu_{1,j} \geq 0$, $(j = 0, \ldots, m_1)$, $\nu_{2,k} \geq 0$, $(k = 0, \ldots, m_2 - 1)$ *satisfy* $j + \nu_{1,j} < M$, $k + \nu_{2,k} < M$, $\forall j, k$. *The system is asymptotically stable, if*

$$\exists P \in \mathbf{S}_{n_x(M+1)}, Q_j \in \mathbf{S}^+_{n_x}, R_k \in \mathbf{S}^+_{n_x},$$

$j = 0, ..., m_1$ *and* $k = 1, ..., m_2$ *such that*

$$(\mathcal{L}_{m,M}) \quad \begin{cases} \Psi^0_{m_1,M}(\tau) > 0, \\ \Psi^1_M(\tau) + \Psi^2_{m_1,M} + \Psi^{3,1}_{m_2,M}(\tau) - \Psi^{3,2}_{m_2,M}(\tau) < 0 \end{cases}$$

hold true, where

Stability analysis

(continuation)

$$
\begin{aligned}
\Psi^{0}_{m_1,M}(\tau) &= \tau P + \sum_{j=0}^{m_1} diag\left\{0, (\Xi_j \otimes I)^T \mathcal{Q}_j^{(j)} (\Xi_j \otimes I)\right\}, \\
\Psi^{1}_{M}(\tau) &= \Gamma_M^T P \Lambda_M + \Lambda_M^T P \Gamma_M, \\
\Psi^{2}_{m_1,M} &= diag\left\{\sum_{j=0}^{m_1} Q_j,\ -Q_0,\ -\sum_{j=1}^{m_1} j\,(\Xi_{j-1} \otimes I)^T \mathcal{Q}_{j-1}^{(j)} (\Xi_{j-1} \otimes I)\right\}, \\
\Psi^{3,1}_{m_2,M}(\tau) &= \tau \mathcal{A}^T \sum_{j=1}^{m_2} R_j \mathcal{A}, \\
\Psi^{3,2}_{m_2,M}(\tau) &= \frac{1}{\tau} \sum_{j=1}^{m_2} j\,(\mathcal{Z}_{j-1} \otimes I)^T \mathcal{R}_{j-1}^{(j)} (\mathcal{Z}_{j-1} \otimes I),
\end{aligned}
$$

matrices Ξ_k, $\mathcal{Z}_k$ *are given previously with* $\nu_k = \nu_{1,k}$, *and* $\nu_k = \nu_{2,k}$,

(continuation)

$$
\begin{aligned}
\mathcal{Q}_j^{(k)} &= diag\{(j+1),\ (j+3),\ \ldots\ ,\ (j+(2M-1))\} \otimes Q_k, \\
\mathcal{A} &= (A,\ A_{d_1},\ \tau A_{d_2}, 0, \ldots, 0) \in \mathbf{R}^{n_x \times n_x(M+2)}, \\
\Lambda_M &= \begin{bmatrix} \mathcal{A} \\ \widetilde{\mathcal{L}}_0 \otimes I \end{bmatrix},\ \Gamma_M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \tau I_M \end{bmatrix} \otimes I, \\
\widetilde{\mathcal{L}}_0 &= \begin{bmatrix} \underline{\ell}_{M-1}^{(1)} & \underline{\ell}_{M-1}^{(2)} & -Z_0(M-1, M-1) \end{bmatrix}, \\
\mathcal{R}_{j-1}^{(j)} &= diag\{jR_j,\ (j+2)R_j,\ \ldots\ ,\ (j+2\nu_{j-1})R_j\},
\end{aligned}
$$

where $\underline{\ell}_{M-1}^{(1)}$, $\underline{\ell}_{M-1}^{(2)}$ $Z_0(M-1, M-1)$ *are given previously.*

Hierarchy of the LMI stability conditions

Aim: Comparison of the stability conditions for different M, m.
Assume that

$$\begin{array}{lll} M \geq 1, & m_1 = m - 1, & m_2 = m, \\ & \nu_{1,j} = M - j - 1, & \nu_{2,k} = M - k, \\ & j = 0, 1, \ldots, m-1, & k = 0, \ldots, m-1. \end{array}$$

$\mathcal{L}_{m,M}$ depends on τ : write $\mathcal{L}_{m,M}(\tau)$ when considering $\mathcal{L}_{m,M}$ for a given value of τ.

Definition

Let the pairs (m, M) and $(\hat{m}, \hat{M})$ be given. We will say that $\mathcal{L}_{\hat{m},\hat{M}}$ outperforms $\mathcal{L}_{m,M}$, if, for every τ for which $\mathcal{L}_{m,M}(\tau)$ has a feasible solution, $\mathcal{L}_{\hat{m},\hat{M}}(\tau)$ has a feasible solution, too. This is denoted by $\mathcal{L}_{m,M} \prec \mathcal{L}_{\hat{m},\hat{M}}$.

Hierarchy of the LMI stability conditions

Theorem

Let the integer parameters satisfy the previous assumption. Then

$$\begin{aligned}\mathcal{L}_{m,M} &\prec \mathcal{L}_{m,M+1},\\ \mathcal{L}_{m,M} &\prec \mathcal{L}_{m+1,M}.\end{aligned}$$

The LMIs can be arranged into a bidirectional hierarchy:

$$\begin{array}{ccccccccccc}
\mathcal{L}_{0,1} & \prec & \mathcal{L}_{0,2} & \prec & \mathcal{L}_{0,3} & \prec & \dots & \prec & \mathcal{L}_{0,M} & & \dots \\
 & & \curlywedge & & \curlywedge & & & & \curlywedge & & \\
 & & \mathcal{L}_{1,2} & \prec & \mathcal{L}_{1,3} & \prec & \dots & \prec & \mathcal{L}_{1,M} & & \dots \\
 & & & & \curlywedge & & & & \curlywedge & & \\
 & & & & \mathcal{L}_{2,3} & \prec & \dots & \prec & \mathcal{L}_{2,M} & & \dots \\
 & & & & & & & & \curlywedge & & \\
 & & & & & & \ddots & & \vdots & & \\
 & & & & & & & & \mathcal{L}_{M-1,M} & & \dots \\
 & & & & & & & & & & \ddots
\end{array}$$

Summation inequalities

(Join work with K. Kiss, I. Nagy, T. Takács, JFI, 354(2017)123-144.)
Let $m, N \in \mathbf{Z}_{\geq 0}$, $s_i = i$, if $(i = 0, 1, \ldots, N-1)$. For $f, g : \mathbf{Z} \to \mathbf{R}$, define a scalar product by

$$\ll f, g \gg_m \;=\; \sum_{i_1=0}^{N-1} \sum_{i_2=0}^{i_1} \cdots \sum_{i_m=0}^{i_{m-1}} f(i_m) g(i_m), \tag{7}$$

and denote the corresponding norm by $|||f|||_m$. Equivalently

$$\ll f, g \gg_m = \frac{1}{(m-1)!} \sum_{i=0}^{N-1} r_{N,m-1}(i) f(i) g(i),$$

where
$r_{N,m-1}(x) = (N-1-x+m-1)(N-1-x+m-2)\ldots(N-1-x+1)$.
Further,

$$< f, g >_m = \sum_{i=0}^{N-1} r_{N,m-1}(i) f(i) g(i), \quad \text{with } \|f\|_m^2 = < f, f >_m .$$

Summation inequalities

Let $\{p_{mj}\}$ be a set of orthogonal polynomials w.r.t. $< .,. >_m$.

$$J_{R,m,0,N}(f) \;=\; \sum_{i_1=0}^{N-1} \sum_{i_2=0}^{i_1} \dots \sum_{i_m=0}^{i_{m-1}} f^T(i_m) R f(i_m) = \ll f, Rf \gg_m .$$

Lemma

Let m, ν_1, ν_m, N be given integers satisfying conditions $m \geq 1$ and $\nu_m < \nu_1 < N$. Let $R \in \mathbf{S}_n^+$ and $f : \mathbf{Z} \to \mathbf{R}^n$. Then

$$J_{R,m,0,N}(f) \geq \frac{1}{(m-1)!} \, \Phi^T \left(\Xi_m \otimes I\right)^T \mathcal{R}_m \left(\Xi_m \otimes I\right) \Phi.$$

where $\mathcal{R}_m = diag\left\{ \frac{1}{\|p_{m0}\|_m^2} R, \; \dots, \; \frac{1}{\|p_{m\nu_m}\|_m^2} R \right\},$

$\Phi = col\{\phi_0, \; \dots, \; \phi_{\nu_1}\}, \; \phi_j = < f, p_{1j} >_1$

and Ξ_m *is connected with a basis transformation.*

Summation inequalities

Remark

With special choices of m and ν_m, one can obtain from the previous estimation

- *the single and double summation Jensen inequality*
- *the single and double summation Wirtinger inequality*
- *and some recently published higher "order" inequalities.*

Summation inequalities for differences

Let ρ be given by function $f : \mathbf{Z} \to \mathbf{R}^n$ as

$$\rho : \mathbf{Z} \to \mathbf{R}^n,\ \rho(i) = f(i+1) - f(i),\ i = 0, 1, ..., N-1.$$

Summation inequalities

Lemma

Let m, ν_1, ν_m, N satisfy condition $\nu_m + m - 1 \leq \nu_1 < N$, and let $R \in \mathbf{S}_n^+$. Then

$$J_{R,m,0,N}(\rho) \geq \frac{1}{(m-1)!} \widetilde{\Phi}^T \left(\mathcal{Z}_m \otimes I\right)^T \mathcal{R}_m \left(\mathcal{Z}_m \otimes I\right) \widetilde{\Phi},$$

where $\widetilde{\Phi} = col\left\{f(N), f(0), \phi_0, \ldots, \phi_{\nu_1 - 1}\right\}, \quad if\ \nu_1 > 0,$
$\widetilde{\Phi} = col\left\{f(N), f(0)\right\}, \quad if\ \nu_1 = 0,$
$\mathcal{R}_m$ is defined previously and
$\mathcal{Z}_m$ is connected with a basis transformation.

Sufficient LMI stability condition can be derived for discrete-time systems, as well.

Generalized free-matrix-based approach

(Join work with T. Takács SCL 123(2019)40-46)
Let $D_i \subset \mathbf{R}$ or $D_i \subset \mathbf{Z}$, $D_0 = [a, b]$, $D_1 = [a, c)$, $D_2 = [c, b]$. (If $D_i \subset \mathbf{Z}$, then, e.g. $[a, b] := \{l \in \mathbf{Z} : a \leq l \leq b\}$.

Let $\mathcal{V}_i$ $(i = 0, 1, 2)$ be the inner product space of φ from $D_i \to \mathbf{R}$ with the scalar product $\langle ., . \rangle_i$ containing the elements $\pi_{0i}(t) \equiv 1$, $t \in D_i$, having the following properties:

(P1) If $\varphi, \psi \in \mathcal{V}_i$, then $\varphi\psi \in \mathcal{V}_i$ and $\langle \varphi, \psi \rangle_i = \langle \pi_{0i}, \varphi\psi \rangle_i$;

(P2) If for $\varphi \in \mathcal{V}_i$ $\varphi(t) \geq 0$ for all $t \in D_i$, then $\langle \pi_{0i}, \varphi \rangle_i \geq 0$.

(P3) $\langle \varphi, \psi \rangle_0 = \langle \varphi_1, \psi_1 \rangle_1 + \langle \varphi_2, \psi_2 \rangle_2$, where φ_i and ψ_i are the restrictions of φ and ψ respectively to D_i.

Typically, $\mathcal{V}_i = L_2(D_i)$ or $\mathcal{V}_i = l_2(D_i)$ with the scalar product $\langle \varphi, \psi \rangle_i = \int_{D_i} \varphi(t)\psi(t)dt$ and $\langle \varphi, \psi \rangle_i = \sum_{t \in D_i} \varphi(t)\psi(t)$, respectively.

Generalized free-matrix-based approach

Let $\pi_{0i}, \pi_{1i}, \ldots, \pi_{\nu i}$ be an orthogonal system in $\mathcal{V}_i$ for $\nu \in \mathbf{N}$, let $\mathcal{V}_0^n = \left\{\phi = (\phi^1, \ldots, \phi^n)^T : \phi^i \in \mathcal{V}_0\right\}$ and consider $f \in \mathcal{V}_0^n$. Set $M_1 = (\nu + 1)n$, $M_2 = 2M_1$ and define $w \in \mathbf{R}^{M_2}$, with

$$w = \begin{bmatrix} w^1 \\ w^2 \end{bmatrix} = \begin{bmatrix} \left[\langle f_1, \pi_{01}\rangle_1^T, \quad \ldots, \quad \langle f_1, \pi_{\nu 1}\rangle_1^T\right]^T \\ \left[\langle f_2, \pi_{02}\rangle_2^T, \quad \ldots, \quad \langle f_2, \pi_{\nu 2}\rangle_2^T\right]^T \end{bmatrix},$$

where f_i is the restriction of f to D_i. Set Ψ^i $(i = 1, 2)$ as

$$\Psi^i = \begin{bmatrix} Z_{00}^i & \ldots & Z_{0\nu}^i & N_0^i \\ \vdots & \ddots & \vdots & \vdots \\ {Z_{0\nu}^i}^T & \ldots & Z_{\nu\nu}^i & N_\nu^i \\ {N_0^i}^T & \ldots & {N_\nu^i}^T & W \end{bmatrix},$$

where $Z_{kl}^i \in \mathbf{R}^{M_2 \times M_2}$, $N_k^i \in \mathbf{R}^{M_2 \times n}$, $k, l = 0. \ldots, \nu$.

Generalized free-matrix-based approach

Lemma (GFMB inequality)

For $i = 1, 2$, $W \in \mathbf{S}_n^+$, *and*

$$\Psi^i \geq 0,$$

the following generalized free-matrix-based inequality holds true for all $\rho_k^i \geq \|\pi_{ki}\|_i^2$*:*

$$\langle f_i, Wf_i \rangle_i \geq -\chi_i^T \left(\sum_{k=0}^{\nu} \rho_k^i Z_{kk}^i \right) \chi_i - He\left(\chi_i^T N^i w \right),$$

where $\chi_i \in \mathbf{R}^{M_2}$ $(i = 1, 2)$ *is arbitrary,*
$N^1 = \left(\widehat{N}^1, 0 \right) = \left(N_0^1, \ldots, N_\nu^1, 0, \ldots, 0 \right) \in \mathbf{R}^{M_2 \times M_2}$
$N^2 = \left(0, \widehat{N}^2 \right) = \left(0, \ldots, 0, N_0^2, \ldots, N_\nu^2 \right) \in \mathbf{R}^{M_2 \times M_2}$.

Generalized free-matrix-based approach

Independent functions based GFMB inequality:

Lemma (I-GFMB inequality)

Let $\{p_{ki}\}_{k=0}^{\nu}$ *be a system of linearly independent functions in* $\mathcal{V}_i$, $\widetilde{w}_k^i = \langle f_i, p_{ki} \rangle_i$ *and* $\gamma_{kl} = \langle p_{ki}, p_{li} \rangle_i$. *If* $W \in \mathbf{S}_n^+$, *and* $\Psi^i \geq 0$, , *then*

$$\langle f_i, Wf_i \rangle_i \geq -\sum_{k=0}^{\nu} He\left(\chi_i^T N_k^i \widetilde{w}_k^i\right) - \chi_i^T \left(\sum_{k=0}^{\nu} \gamma_{kk} Z_{kk}^i + \sum_{k=0}^{\nu} \sum_{l=k+1}^{\nu} He\left(\gamma_{kl} Z_{kl}^i\right)\right) \chi_i.$$

where $\chi_i \in \mathbf{R}^{M_2}$ *is arbitrary.*

Generalized free-matrix-based approach

Notations: Set $\mathcal{W}_i = \operatorname{diag}\left\{\frac{1}{\rho_0^i}, \dots, \frac{1}{\rho_\nu^i}\right\} \otimes W$, $\widehat{\mathcal{W}}_1 = \operatorname{diag}(\mathcal{W}_1, 0)$, $\widehat{\mathcal{W}}_2 = \operatorname{diag}(0, \mathcal{W}_2)$, and $\widehat{\mathcal{W}}_i^-$ is defined analogously with $\mathcal{W}_i^{-1}$.

Corollary (Simplified GFMB inequality)

GFMB inequality lemma implies that

$$
\begin{array}{lll}
\textbf{\textit{S-GFMB}} & \langle f_i, Wf_i \rangle \geq & -He\left(\chi_i^T N^i w\right) - \chi_i^T N^i \widehat{\mathcal{W}}_i^- {N^i}^T \chi_i \\
\textbf{\textit{S-FMB}} & \langle f_i, Wf_i \rangle \geq & -w^T \left(He\left(N^i\right) + N^i \widehat{\mathcal{W}}_i^- {N^i}^T \right) w \\
\textbf{\textit{BBI}} & \langle f_i, Wf_i \rangle \geq & {w^i}^T \mathcal{W}_i w^i
\end{array}
$$

Moreover, the right hand side of the last inequality is always greater than or equal to the right hand side of the others.

Generalized free-matrix-based approach

Theorem

Suppose that $\{p_{ki}\}_{k=0}^{\nu}$ *and* $\{\pi_{ki}\}_{k=0}^{\nu}$ *span the same subspace of* $\mathcal{V}_i$. *Let* $\rho_k^i = \|\pi_{ki}\|_i^2$.
Then the GFMB inequality, and the I-GFMB inequality are equivalent.
Moreover, the GFMB-, S-GFMB-, S-FMB- and BBI inequalities are equivalent.

Comparison of complexity characterized by the number of parameters involved:

$$\textit{BBI} \prec \textit{S-FMB} \prec \textit{S-GFMB} \prec\prec \textit{GFMB} \prec\prec \textit{I-GFMB}$$

Estimations for two connected intervals

Let $\mathcal{V}_i = L_2(D_i)$ or $\mathcal{V}_i = l_2(D_i)$ with the scalar product given previously, and let $W \in \mathbf{S}_n^+$. Define

$$\mathcal{W} = \operatorname{diag}\{1, 3, ..., 2\nu + 1\} \otimes W.$$

Then

$$\langle f, Wf \rangle_0 = \langle f_1, Wf_1 \rangle_1 + \langle f_2, Wf_2 \rangle_2.$$

Suppose that $a < c < b$, and introduce the notations $h = b - a$, $\alpha = \frac{c-a}{h}$. Then S-FMB inequality yields

$$\langle f, Wf \rangle_0 \geq \frac{1}{h} w^T \Omega_F(\alpha, W, \widehat{N}^1, \widehat{N}^2) w,$$

where

$$\begin{aligned}\Omega_F(\alpha, W, \widehat{N}^1, \widehat{N}^2) = -h\,\mathrm{He}\left(\begin{bmatrix}\widehat{N}^1 & 0\end{bmatrix} + \begin{bmatrix}0 & \widehat{N}^2\end{bmatrix}\right)\\ -(-h\widehat{N}^1)(\alpha\mathcal{W}^{-1})(-h\widehat{N}^1)^T\\ -(-h\widehat{N}^2)((1-\alpha)\mathcal{W}^{-1})(-h\widehat{N}^2)^T.\end{aligned}$$

Estimations for two connected intervals

On the other hand, the BBI inequality is

$$\langle f, Wf \rangle_0 \geq \frac{1}{h} w^T \Omega_B(\alpha, W) w,$$

where

$$\Omega_B(\alpha, W) = \begin{bmatrix} \frac{1}{\alpha}\mathcal{W} & 0 \\ 0 & \frac{1}{(1-\alpha)}\mathcal{W} \end{bmatrix}.$$

Advantage of BBI: tighter lower bound and less complexity.
Disadvantage: it is non-convex in the lengths of the intervals, if it is applied in case of time-varying delays. To avoid the non-convexity, *further lower estimation is needed.*
*Possibilities:*the application of

- the classical reciprocally convex combination (RCC) lemma of Park et al. (2015),
- the extended RCC (E-RCC) lemma of Seuret et al. (2016),
- the modified Moon lemma of Liu et al. (2016),
- a simplified and a modified version of the E- RCC lemma.

Estimations for two connected intervals

Theorem

Let $W \in \mathbf{S}_n^+$ be given.

(A) *S-FMB* $\Leftrightarrow$ *(BBI & modified Moon lemma)*
The modified Moon lemma: for all $U_1, U_2 \in \mathbf{R}^{M_2 \times M_1}$ *and for all* $\alpha \in (0,1)$

$$\Omega_B(\alpha, W) \geq \Omega_1(\alpha, W, U_1, U_2) = He\left(\begin{bmatrix} U_1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & U_2 \end{bmatrix}\right) - \alpha U_1 \mathcal{W}^{-1} U_1^T - (1-\alpha) U_2 \mathcal{W}^{-1} U_2^T.$$

Estimations for two connected intervals

(continuation)

(B) *(BBI & modified Moon lemma)* $\Rightarrow$
(BBI & simplified E-RCC), but not conversely.
The simplified E-RCC:
If $Y_1, Y_2 \in \mathbf{R}^{M_1 \times M_1}$ *are arbitrary matrices and*
$\widehat{X}_1 = \mathcal{W} - Y_1 \mathcal{W}^{-1} Y_1^T$, $\widehat{X}_2 = \mathcal{W} - Y_2^T \mathcal{W}^{-1} Y_2$,
then

$$\Omega_B(\alpha, W) \geq \Omega_2(\alpha, W, Y_1, Y_2)$$
$$= \begin{bmatrix} \mathcal{W} + (1-\alpha)\widehat{X}_1 & \alpha Y_1 + (1-\alpha) Y_2 \\ * & \mathcal{W} + \alpha \widehat{X}_2 \end{bmatrix} \quad \forall \alpha \in (0,1).$$

Estimations for two connected intervals

(continuation)

(C) *(BBI & simplified E-RCC)* $\Leftrightarrow$ *(BBI & E-RCC),*
The E-RCC states that, if $Y_1, Y_2 \in \mathbf{R}^{M_1\times M_1}$ *and* $X_1, X_2 \in \mathbb{S}_{M_1}$ *are arbitrary matrices satisfying inequality*

$$\begin{bmatrix} \mathcal{W} & 0 \\ 0 & \mathcal{W} \end{bmatrix} - \alpha \begin{bmatrix} X_1 & Y_1 \\ Y_1^T & 0 \end{bmatrix} - (1-\alpha) \begin{bmatrix} 0 & Y_2 \\ Y_2^T & X_2 \end{bmatrix} \geq 0$$

for $\alpha = 0, 1,$ *then for all* $\alpha \in (0,1)$

$$\begin{aligned} \Omega_B(\alpha, W) &\geq \Omega_3(\alpha, W, X_1, X_2, Y_1, Y_2) \\ &:= \begin{bmatrix} \mathcal{W} + (1-\alpha)X_1 & \alpha Y_1 + (1-\alpha)Y_2 \\ * & \mathcal{W} + \alpha X_2 \end{bmatrix}, \end{aligned}$$

Estimations for two connected intervals

(continuation)

(D) *(BBI & simplified E-RCC)* $\Rightarrow$ *(BBI & the modified E-RCC), but not conversely.*
The modified E-RCC states that, if $Y \in \mathbf{R}^{M_1 \times M_1}$ *is an arbitrary matrix and* $\overline{X}_1 = \mathcal{W} - Y\mathcal{W}^{-1}Y^T$, $\overline{X}_2 = \mathcal{W} - Y^T\mathcal{W}^{-1}Y$, *then, for all* $\alpha \in (0,1)$

$$\Omega_B(\alpha, W) \geq \Omega_4(\alpha, W, Y) := \begin{bmatrix} \mathcal{W} + (1-\alpha)\overline{X}_1 & Y \\ * & \mathcal{W} + \alpha\overline{X}_2 \end{bmatrix}.$$

(E) *(BBI & modified E-RCC)* $\Rightarrow$ *(BBI & RCC).*
RCC : If $Y \in \mathbf{R}^{M_1 \times M_1}$ *is satisfies the inequality of E-RCC with* $X_1 = X_2 = 0$ *and* $Y_1 = Y_2 = Y$, *then*

$$\Omega_B(\alpha, W) \geq \Omega_5(\alpha, W, Y) = \begin{bmatrix} \mathcal{W} & Y \\ Y & \mathcal{W} \end{bmatrix}. \ \forall \alpha \in (0,1)$$

$\Omega_5(\alpha, W, Y) \leq \Omega_4(\alpha, W, Y)$ *if* Y *is chosen as in RCC.*

Synchronization of networks: The problem

(Join work with A. Kazemy & K. Kiss, JFI 355(2018)8934-8956)

Consider the following CDN that consists of N nodes

$$\dot{x}_i(t) = Ax_i(t) + B_f f(x_i(t)) + B_u u_i(t) + c\sum_{j=1}^{N} \ell_{ij} G x_j(t-\tau(t)),$$

$$y_{x_i}(t) = Cx_i(t), \qquad i = 1,\ldots,N,$$

where

$x_i(t) \in \mathbf{R}^{n_x}$	state of the ith node
$u_i(t) \in \mathbf{R}^{n_u}$	control signal of the ith node
$y_i(t) \in \mathbf{R}^{n_y}$	measured output of the ith node
$G \in \mathbf{R}^{n_x \times n_x}$	constant inner-coupling matrix of the nodes
$L = [\ell_{ij}] \in \mathbf{R}^{N\times N}$	outer coupling configuration matrix with properties

$\ell_{ij} = \ell_{ji} > 0$, if there is an interconnection between nodes i and j, and $\ell_{ij} = 0$, otherwise, while the diagonal elements of L are defined by

$$\ell_{ii} = -\sum_{j=1, j\neq i}^{N} \ell_{ij}, \quad i = 1,\ldots,N.$$

Synchronization of networks: The problem

The time-varying delay $\tau(.)$ is supposed to be a differentiable function satisfying conditions

$$0 \le \underline{\tau} \le \tau(t) \le \overline{\tau}, \quad \dot{\tau}(t) \le \mu,$$

with known constant bounds $\underline{\tau} < \overline{\tau}$ and μ. The initial condition: $x_i(t) = \varphi_i(t)$, if $t \in [-\overline{\tau}, 0]$, $\varphi_i \in \mathcal{W}[-\overline{\tau}, 0]$, and $i = 1, \ldots, N$.

Assumption

The continuous function $f : \mathbf{R}^{n_x} \to \mathbf{R}^{n_x}$ satisfies

$$\begin{bmatrix} f(x) - f(y) \\ x - y \end{bmatrix}^T \begin{bmatrix} Q_0 & S_0 \\ S_0^T & R_0 \end{bmatrix} \begin{bmatrix} f(x) - f(y) \\ x - y \end{bmatrix} \ge 0 \quad \forall\, x, y \in \mathbf{R}^{n_x},$$

where $Q_0 \in \mathbf{S}^{n_x}_{<0}$, $S_0 \in \mathbf{R}^{n_x \times n_x}$, $R_0 \in \mathbf{S}^{n_x}_{\ge 0}$ are known matrices.

Synchronization of networks: The problem

Let $z(t) \in \mathbf{R}^{n_x}$ be the trajectory of the *unforced isolate node* described by

$$\begin{aligned} \dot{z}(t) &= Az(t) + B_f f(z(t)), \\ y_z(t) &= Cz(t). \end{aligned}$$

The communication structure is as follows:
Measurement time instants: $j\delta$, with $\delta > 0$, $j = 0, 1, \ldots,$
they are transmitted from the isolate node to all nodes, while packet dropouts may happen.
Successfully transmitted measurements: $\{s_k, y_z(s_k)\}$,
$0 = s_0 < s_1 < \cdots < s_k < \ldots$, $\lim_{k\to\infty} s_k = \infty$, $0 < s_{k+1} - s_k \leq \nu$,
ν is given. The measurements of the ith node $\{j\delta, y_{x_i}(j\delta)\}$ are saved and, based on s_k, $\{s_k, y_{x_i}(s_k)\}$ is taken out.

Synchronization of networks: The problem

The control

$$u_k^i = (K_i + \Delta K_{i,k})\,(y_{x_i}(s_k) - y_z(s_k))$$

is computed and applied through a Zero-Order-Hold (ZOH) device from $t_k = s_k + \eta$ till $t_{k+1} = s_{k+1} + \eta$, i.e.

$$u_i(t) = u_k^i, \qquad t \in [t_k, t_{k+1})\,.$$

$\eta > 0$ is a known constant transmission delay.
The time-dependent uncertainty $\Delta K_{i,k}$ represents a possible gain fluctuation satisfying condition

$$\Delta K_{i,k} = D\Delta_{i,k}E_{ai}, \qquad \text{where} \qquad \Delta_{i,k}^T\Delta_{i,k} \leq I, \ \text{for all } i, k,$$

and D, E_{ai} are known constant matrices.
Let $\eta(t) = t - t_k + \eta$, if $t \in [t_k, t_{k+1})$, then $t - \eta(t) = s_k$, if $t \in [t_k, t_{k+1})$, and

$$0 < \eta \leq \eta(t) \leq \eta + \nu =: \eta_M.$$

Synchronization of networks: The problem

Let $r_i(t) = x_i(t) - z(t)$ be the synchronization error of the ith node. Then the synchronization error of the CDN can be written as

$$\dot{r}_i(t) = Ar_i(t) + B_f g(z(t), r_i(t)) + B_u K_i C r_i(t - \eta(t)) + B_u D p_{K_i}(t)$$
$$+ c\ell_{ii} G r_i(t - \tau(t)) + c \sum_{j=1, j \neq i}^{N} \ell_{ij} G r_j(t - \tau(t)),$$
$$t \in [t_k, t_{k+1}),$$
$$r_i(t) = x_i(t) - z(t), \quad t \in (-\overline{\tau}, t_0], \qquad i = 1, \ldots, N,$$

where $g(z(t), r_i(t)) = f(r_i(t) + z(t)) - f(z(t))$.

Definition

Let $t_0^* = t_0 + \overline{\tau}$, and let $r(t) = col\{r_1(t), \ldots, r_N(t)\}$. The CDN is said to be globally exponentially synchronized onto the isolate node, if there exist constants $M > 0$, $\gamma > 0$, such that, for $t \geq t_0^*$ and for any $r_{t_0^*} \in \mathcal{W}[-\overline{\tau}, 0]$, $\|r(t)\| \leq Me^{-\gamma(t - t_0^*)} \left\|r_{t_0^*}\right\|_{\mathcal{W}}$ holds.

Synchronization of networks: The main result

Lyapunov-Krasovskii functional for the error system:

$$V(t, r_t, \dot{r}_t) = \sum_{i=1}^{N} V^i(t, r_t, \dot{r}_t), \qquad t \in [t_k, t_{k+1}),\ k = 0, 1, \dots,$$

where the functionals $\overline{V}^i(t) = V^i(t, r_t, \dot{r}_t)$ are defined as follows.

$$\overline{V}^i(t) = \sum_{j=1}^{3} \overline{V}^i_{1j}(t) + \sum_{j=1}^{4} \left(\overline{V}^i_{2j}(t) + \overline{V}^i_{3j}(t)\right) + \overline{V}^i_4(t) + \iota(\alpha)\overline{V}^i_5(t),$$

$$\overline{V}^i_{11}(t) = r_i(t)^T P_{i1} r_i(t), \quad \overline{V}^i_{12}(t) = \rho_{i1}(t)^T P_{i2} \rho_{i1}(t),$$

$$\overline{V}^i_{13}(t) = \rho_{i2}(t)^T P_{i3} \rho_{i2}(t),$$

$$\rho_{i1}(t) = \mathrm{col}\left\{ r_i(t), \frac{1}{\underline{\tau}} \int_{t-\underline{\tau}}^{t} r_i(s)ds, \frac{1}{\overline{\tau} - \underline{\tau}} \int_{t-\overline{\tau}}^{t-\underline{\tau}} r_i(s)ds \right\},$$

$$\rho_{i2}(t) = \mathrm{col}\left\{ r_i(t), \frac{1}{\eta} \int_{t-\eta}^{t} r_i(s)ds, \frac{1}{\eta_M - \eta} \int_{t-\eta_M}^{t-\eta} r_i(s)ds \right\},$$

$$\overline{V}_{21}^i(t) = \int_{t-\underline{\tau}}^{t} e^{2\alpha(s-t)} r_i(s)^T Q_{i1} r_i(s) ds,$$

$$\overline{V}_{22}^i(t) = \int_{t-\overline{\tau}}^{t-\underline{\tau}} e^{2\alpha(s-t)} r_i(s)^T Q_{i2} r_i(s) ds,$$

$$\overline{V}_{23}^i(t) = \int_{t-\eta}^{t} e^{2\alpha(s-t)} r_i(s)^T Q_{i3} r_i(s) ds,$$

$$\overline{V}_{24}^i(t) = \int_{t-\eta_M}^{t-\eta} e^{2\alpha(s-t)} r_i(s)^T Q_{i4} r_i(s) ds,$$

$$\overline{V}_{31}^i(t) = \underline{\tau} \int_{-\underline{\tau}}^{0} \int_{t+\theta}^{t} e^{2\alpha(s-t)} \dot{r}_i(s)^T R_{i1} \dot{r}_i(s) ds d\theta,$$

$$\overline{V}_{32}^i(t) = (\overline{\tau} - \underline{\tau}) \int_{-\overline{\tau}}^{\underline{\tau}} \int_{t+\theta}^{t} e^{2\alpha(s-t)} \dot{r}_i(s)^T R_{i2} \dot{r}_i(s) ds d\theta,$$

$$\overline{V}_{33}^i(t) = \eta \int_{-\eta}^{0} \int_{t+\theta}^{t} e^{2\alpha(s-t)} \dot{r}_i(s)^T R_{i3} \dot{r}_i(s) ds d\theta,$$

$$\overline{V}_{34}^i(t) = (\eta_M - \eta)\int_{-\eta_M}^{-\eta}\int_{t+\theta}^{t} e^{2\alpha(s-t)}\dot{r}_i(s)^T R_{i4}\dot{r}_i(s)dsd\theta,$$

$$\overline{V}_4^i(t) = \int_{t-\tau(t)}^{t} e^{2\alpha(s-t)} r_i(s)^T S_{i1} r_i(s)ds$$

$$+ \int_{t-\overline{\tau}}^{t-\tau(t)} e^{2\alpha(s-t)} r_i(s)^T S_{i2} r_i(s)ds$$

$$+ \frac{2\epsilon_1}{N-1}\sum_{j=1,j\neq 1}^{N}\int_{t-\tau(t)}^{t} e^{2\alpha(s-t)} r_j(s)^T S_{j1} r_j(s)ds,$$

$$\overline{V}_5^i(t) = (\eta_M - \eta)^2\int_{t_k-\eta}^{t}\dot{r}_i(s)^T S_{i0}\dot{r}_i(s)ds$$

$$-\frac{\pi^2}{4}\int_{t_k-\eta}^{t-\eta}(r_i(s_k) - r_i(s))^T S_{i0}(r_i(s_k) - r_i(s))ds,$$

$$P_{i1}, Q_{ij}, R_{ij}, S_{i0}, S_{i1}, S_{i2} \in \mathbf{S}_{>0}^{n_x}, \quad P_{i2}, P_{i3} \in \mathbf{S}^{3n_x},$$

$$i = 1, \ldots, N,\ j = 1, \ldots, 4$$

Synchronization of networks: The main result

Proposition

Let $\kappa_1 = e^{-2\alpha\underline{\tau}}$, $\kappa_2 = e^{-2\alpha\overline{\tau}}$, $\kappa_3 = e^{-2\alpha\eta}$, *and* $\kappa_4 = e^{-2\alpha\eta_M}$. *If the adjustable matrices satisfy the prescribed conditions and the LMIs*

$$P_{i2} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \underline{\tau}\kappa_1 Q_{i1} & 0 \\ 0 & 0 & (\overline{\tau} - \underline{\tau})\kappa_2 Q_{i2} \end{bmatrix} > 0,$$

$$P_{i3} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \eta\kappa_3 Q_{i3} & 0 \\ 0 & 0 & (\eta_M - \eta)\kappa_4 Q_{i4} \end{bmatrix} > 0,$$

then for $\overline{V}(t) = V(t, r_t, \dot{r}_t)$ *there exist constants* $c_1 > 0$ *and* $c_2 > 0$ *such that*

$$c_1 \|r(t)\|^2 \le \overline{V}(t) \le c_2 \|r_t\|^2_{\mathcal{W}},$$

and $\lim_{t \to t_k^-} \overline{V}(t) \ge \lim_{t \to t_k^+} \overline{V}(t)$.

Synchronization of networks: The main result

Sufficient condition for exponential synchronization can be derived by the estimation of $\frac{d}{dt}\overline{V}(t)$. This estimation can be given in terms of LMIs, but it is very technical, therefore we refer those who are interested for details to the cited paper.

Advantages of the proposed method are the reduced conservatism and the reduced number of decision variables (NoDV).

The number of decision variables is

$3Nn_x(5Nn_x+3)/2+Nn_x^2$

in Lee et al. AMC 219(2012)1354-1366,

$31.5(Nn_x)^2+6.5Nn_x+3Nn_x^2+2$

in Liu et al. IEEE Trans. NNLS 29(2018)118-128,

$N((25.5+0.5\iota(\alpha))n_x^2+(8.5+0.5\iota(\alpha))n_x+2)$

in the present work presumed that $n_u = n_y = n_x$.

Finite frequency H_∞ control design: The problem

(Join work with A. Kazemy T.& Takács, ISA Trans.
doi:10.1016/j.isatra.2019.06.005)

Contributions :

1. Finite-frequency $\mathcal{H}_\infty$ control is designed for linear systems via dynamic output feedback.
2. Practical hard constraints are considered in the design problem.

Consider a linear dynamic system as

$$
\begin{aligned}
\dot{x}(t) &= A_x x(t) + B_x u(t) + E_x f(t), \qquad x(0) = x_0,\\
z(t) &= C_z x(t) + B_z u(t) + E_z f(t),\\
y(t) &= C_y x(t),\\
v(t) &= C_v x(t),
\end{aligned}
$$

where

$x(t) \in \mathbf{R}^{n_x}$	state,	$f(t) \in \mathbf{L}_2[0, T)\ \forall T > 0$	external disturbance,
$u(t) \in \mathbf{R}^{n_u}$	control signal,	$z(t) \in \mathbf{R}^{n_z}$	penalty output,
$y(t) \in \mathbf{R}^{n_y}$	measured output,	$v(t) \in \mathbf{R}^{n_v}$	output to be constrained,

Finite frequency H_∞ control design: The problem

Hard constraints:

$$\begin{aligned} |v_i(t)| &\leq 1,\ i=1,\dots,n_v, \\ |u(t)_j| &\leq u_{j\max},\ \ j=1,\dots,n_u,\ \ \text{where}\ \ u_{j\max}\ \text{is a given constant.} \end{aligned}$$

Admissible external disturbance: f satisfying

$$\int_0^\infty f(t)^2 dt \leq f_{\max}^2, \qquad \text{where}\ \ f_{\max}\ \ \text{is a given constant.}$$

The dynamic output feedback controller:

$$\begin{aligned} \dot{\hat{x}}(t) &= A_c \hat{x}(t) + B_c y(t), \\ u(t) &= C_c \hat{x}(t) + D_c y(t), \end{aligned}$$

$\mathcal{K} = [A_c, B_c, C_c, D_c]$ is referred to as controller gain matrix.

Finite frequency H_∞ control design: The problem

Define $\xi(t) = \left[x^T(t), \widehat{x}^T(t)\right]^T \in \mathbf{R}^{2n_x}$.
The closed-loop systems is

$$
\begin{aligned}
\dot{\xi}(t) &= \mathcal{A}\xi(t) + \mathcal{B}f(t), \qquad \xi(0) = \xi_0 = \left[x_0^T, 0^T\right]^T, \\
\zeta(t) &= \mathcal{C}\xi(t) + \mathcal{D}f(t),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{A} &= \begin{bmatrix} A_x + B_x D_c C_y & B_x C_c \\ B_c C_y & A_c \end{bmatrix}, \qquad \mathcal{B} = \begin{bmatrix} E_x \\ 0 \end{bmatrix}, \\
\mathcal{C} &= \begin{bmatrix} C_z + B_z D_c C_y & B_z C_c \end{bmatrix}, \qquad \mathcal{D} = E_z.
\end{aligned}
$$

Let $e_j \in \mathbf{R}^{1\times n_u}$ be the ith unit vector. The hard constraints are

$$
\begin{aligned}
\xi(t)^T \mathcal{C}_{vi}^T \mathcal{C}_{vi} \xi(t) &\leq 1, \qquad i = 1, \ldots, n_v, \\
\xi(t)^T \kappa^T e_j^T e_j \kappa \xi(t) &\leq u_{j\max}^2, \qquad j = 1, \ldots, n_u,
\end{aligned}
$$

where $\kappa = [D_c C_y \;\; C_c]$, $\mathcal{C}_v = [C_v \;\; 0]$ and $\mathcal{C}_{vi}$ is the i^{th} row of $\mathcal{C}_v$.

Finite frequency H_∞ control design: The problem

Consider the finite-frequency $\mathcal{H}_\infty$ performance index

$$\sup_{\varpi_1<\omega<\varpi_2} \|\mathcal{G}(j\omega)\|_\infty < \gamma, \quad (j = \sqrt{-1})$$

where

$\varpi_1,\ \varpi_2$:	lower and upper bound of the concerned frequency,
γ :	positive scalar,
$\mathcal{G}(j\omega)$:	transfer function matrix of the closed-loop system,

Problem statement: Design an appropriate dynamic output feedback controller gain matrix $\mathcal{K} = [A_c, B_c, C_c, D_c]$ such that,

- the closed-loop system is asymptotically stable, if $f(t) \equiv 0$,
- the finite-frequency $\mathcal{H}_\infty$ performance index is guaranteed with a γ as small as possible,
- the hard constraints are met.

Finite frequency H_∞ control design: GKYP lemma

Lemma (GKYP Lemma, Iwasaki & Hara, (2005), Pipeleers & Vandenberghe (2011))

Let $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$, $C \in \mathbf{R}^{p\times n}$, $D \in \mathbf{R}^{p\times m}$, a $\gamma \in \mathbf{R}_+$, $\mathcal{I}_\omega = \{\omega \in \mathbf{R} : \ \overline{\omega}_1 \leq \omega \leq \overline{\omega}_2\}$, be given Suppose that A has no eigenvalues on the imaginary axis, and $D^T D - \gamma^2 I < 0$. Then for $\mathcal{G}(j\omega) = C(j\omega I - A)^{-1} B + D$ the following statements are equivalent:

(i)

$$\begin{bmatrix} \mathcal{G}(j\omega) \\ I \end{bmatrix}^* \Pi \begin{bmatrix} \mathcal{G}(j\omega) \\ I \end{bmatrix} < 0, \quad \text{for all} \quad \omega \in \mathcal{I}_\omega.$$

(ii) There exist real symmetric matrices P and Q with $Q > 0$ such that

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T \Xi \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^T \Pi \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} < 0,$$

(continuation)

where

$$\Xi = \begin{bmatrix} -Q & P + j\omega_c Q \\ P - j\omega_c Q & -\overline{\omega}_1 \overline{\omega}_2 Q, \end{bmatrix},$$

$$\Pi = \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix}$$

$$\omega_c = \frac{1}{2}\left(\overline{\omega}_1 + \overline{\omega}_2\right).$$

Finite frequency H_∞ control design: Main result

Proposition (1)

Let $\mathcal{I}_\omega = \{\omega \in \mathbf{R} : \ \overline{\omega}_1 \leq \omega \leq \overline{\omega}_2\}$ *and let* $\mathcal{G}(j\omega)$ *be the transfer function of the closed-loop system. Suppose that* $\mathcal{A}$ *has no eigenvalues on the imaginary axis, and* $\mathcal{D}^T\mathcal{D} - \gamma^2 I < 0$. *Then*

$$\sup_{\omega \in \mathcal{I}_\omega} \|\mathcal{G}(j\omega)\|_\infty < \gamma,$$

if and only if there exist matrices $\mathcal{P} \in \mathbf{S}_{2n_x}$, $\mathcal{Q} \in \mathbf{S}^+_{2n_x}$, $\mathcal{W}_r, \mathcal{W}_{im} \in \mathbf{R}^{2n_x \times 2n_x}$ *such that the following matrix inequality holds:*

$$\begin{bmatrix} \widehat{\Omega}_r & \Gamma^T & \widehat{\Omega}_{im} & 0 \\ \Gamma & -I & 0 & 0 \\ -\widehat{\Omega}_{im} & 0 & \widehat{\Omega}_r & \Gamma^T \\ 0 & 0 & \Gamma & -I \end{bmatrix} < 0,$$

(continuation)

where $\Gamma = [0\ \mathcal{C}\ \mathcal{D}]$,

$$
\begin{aligned}
\widehat{\Omega}_r &= \begin{bmatrix} -\mathcal{Q} & \mathcal{P} - \mathcal{W}_r & 0 \\ \mathcal{P} - \mathcal{W}_r^T & \Omega_{1r} & \mathcal{W}_r^T \mathcal{B} \\ 0 & \mathcal{B}^T \mathcal{W}_r & -\gamma^2 I \end{bmatrix}, \\
\Omega_{1r} &= \mathcal{A}^T \mathcal{W}_r + \mathcal{W}_r^T \mathcal{A} - \overline{\omega}_1 \overline{\omega}_2 \mathcal{Q}, \\
\widehat{\Omega}_{im} &= \begin{bmatrix} 0 & \omega_c \mathcal{Q} - \mathcal{W}_{im} & 0 \\ -\omega_c \mathcal{Q} + \mathcal{W}_{im}^T & \Omega_{1im} & -\mathcal{W}_{im}^T \mathcal{B} \\ 0 & \mathcal{B}^T \mathcal{W}_{im} & 0 \end{bmatrix}, \\
\Omega_{1im} &= \mathcal{A}^T \mathcal{W}_{im} - \mathcal{W}_{im}^T \mathcal{A}.
\end{aligned}
$$

Finite frequency H_∞ control design: Main result

Divide the selectable variables into two groups:

(1) $\Psi_0 = [\mathcal{P}, \mathcal{Q}, \mathcal{W}_r, \mathcal{W}_{im}, \overline{\gamma}]$, where $\overline{\gamma} = \gamma^2$, and

(2) $\mathcal{K} = [A_c, B_c, C_c, D_c]$.

Formally, the inequality of Proposition 1 can be written as

$$\mathcal{L}_0 \left(\Psi_0, \mathcal{K} \right) < 0,$$

which is LMI with respect to Ψ_0 by fixing the matrices in $\mathcal{K}$, and it is LMI with respect to $\mathcal{K}$ by fixing the matrices in Ψ_0.

Finite frequency H_∞ control design: Main result

For given $\mathcal{R} \in \mathbf{S}^+_{2n_x}$ and $\alpha > 0$, introduce the ellipsoid

$$\mathcal{E}_\alpha(\mathcal{R}) = \left\{ \xi \in \mathbf{R}^{2n_x} : \xi^T \mathcal{R} \xi \leq \alpha \right\}.$$

Proposition (2)

Let $\alpha_0 > 0$, $\nu > 0$ be given, and consider the closed-loop system with admissible disturbances. Suppose that there exists a matrix $\mathcal{R} \in \mathbf{S}^+_{2n_x}$ such that

$$\begin{bmatrix} \mathcal{A}^T\mathcal{R} + \mathcal{R}\mathcal{A} & \mathcal{B}^T\mathcal{R} \\ \mathcal{R}\mathcal{B} & -\nu I \end{bmatrix} < 0,$$

$$\begin{bmatrix} \mathcal{R} & \overline{\alpha}\kappa^T e_j^T / \sqrt{u_{jmax}} \\ \overline{\alpha} e_j \kappa / \sqrt{u_{jmax}} & \overline{\alpha} u_{max} I \end{bmatrix} \geq 0, \qquad j = 1, \ldots, n_u,$$

$$\begin{bmatrix} \mathcal{R} & \overline{\alpha}\mathcal{C}_{v_i}^T \\ \overline{\alpha}\mathcal{C}_{v_i} & \overline{\alpha} I \end{bmatrix} \geq 0, \qquad i = 1, \ldots, n_v,$$

(continuation)

where $\overline{\alpha} = \alpha_0 + \nu f_{max}^2$.
Then

- *the closed-loop system is asymptotically stable for* $f(t) \equiv 0$;
- *if* f *is an admissible disturbance and* $\xi_0 \in \mathcal{E}_{\alpha_0}(\mathcal{R})$, *then*
  - $\xi(t) \in \mathcal{E}_{\overline{\alpha}}(\mathcal{R})$ *for all* $t \geq 0$,
  - *and the hard constraints are satisfied.*

Divide the selectable variables into two groups: $\Psi_1 = [\mathcal{R},\ \nu]$ and $\mathcal{K} = [A_c,\ B_c,\ C_c,\ D_c]$. Formally, one can write inequalities of Proposition 2 as

$$\mathcal{L}_1(\Psi_1, \mathcal{K}) < 0, \quad \mathcal{L}_2(\Psi_1, \mathcal{K}) \geq 0, \quad \mathcal{L}_3(\Psi_1, \mathcal{K}) \geq 0,$$

which are LMIs with respect to Ψ_1 by fixing $\mathcal{K}$, and they are also LMIs with respect to $\mathcal{K}$ by fixing Ψ_1.

Finite frequency H_∞ control design: Main result

- How to solve the obtained system of BMIs?
  Idea: find an initial guess for (A_c, B_c, C_c, D_c), then reduce γ by iteratively solving the obtained bilinear inequalities alternately fixing one or the other group of the decision variables.
- How to obtain a suitable initial guess?
  Fix γ_0, find the solution of the $\mathcal{H}_\infty$-problem on the *entire* frequency domain $\omega \in \mathcal{R}$. If it has a feasible solution, then it is a feasible solution of the $\mathcal{H}_\infty$-problem on the restricted frequency domain.

The construction can be done by an approach frequently applied since a seminal paper of Gahinet & Apkarjan.
(The details are given in A. Kazemy et al. ISA Trans.
doi:10.1016/j.isatra.2019.06.005)

Finite frequency H_∞ control design: Main result

Algorithm

Step 0. Chose $\alpha_0 > 0$, $\overline{\gamma}_0 > 0$. Find the solution of the H_∞ problem. If it has a feasible solution, then let $k = 1$, $\overline{\gamma}^{(0)} = \overline{\gamma}_0$, $\mathcal{K}^{(0)} = \{A_c, B_c, C_c, D_c\}$. Choose a $\gamma_{\min} > 0$, and $N_{max} \in \mathbf{N}^+$.

Step k. (i) If $\mathcal{K}^{(k-1)}$ is known, solve problem P1 for Ψ_0, Ψ_1 :

$$P1: \quad \min \ \overline{\gamma}, \quad \text{with respect to}$$

$$\mathcal{L}_0\left(\Psi_0, \mathcal{K}^{(k-1)}\right) < 0, \ \mathcal{L}_1\left(\Psi_1, \mathcal{K}^{(k-1)}\right) < 0,$$

$$\mathcal{L}_2\left(\Psi_1, \mathcal{K}^{(k-1)}\right) \geq 0, \ \mathcal{L}_3\left(\Psi_1, \mathcal{K}^{(k-1)}\right) \geq 0,$$

Let $\Psi_0^{(k)}$, $\Psi_1^{(k)}$ be defined as the solution.

(ii) If $\Psi_0^{(k)}$, $\Psi_1^{(k)}$ is known, solve problem P2 for $\mathcal{K}$ and $\varepsilon > 0$:

$$P2: \quad \min\left(-\varepsilon\right), \quad \text{with respect to}$$

$$\mathcal{L}_0\left(\Psi_0^{(k)}, \mathcal{K}\right) < -\varepsilon, \; \mathcal{L}_1\left(\Psi_1^{(k)}, \mathcal{K}\right) < -\varepsilon,$$

$$\mathcal{L}_2\left(\Psi_1^{(k)}, \mathcal{K}\right) \geq 0, \; \mathcal{L}_3\left(\Psi_1^{(k)}, \mathcal{K}\right) \geq 0.$$

Let $\mathcal{K}^{(k)}$ be defined as the solution.

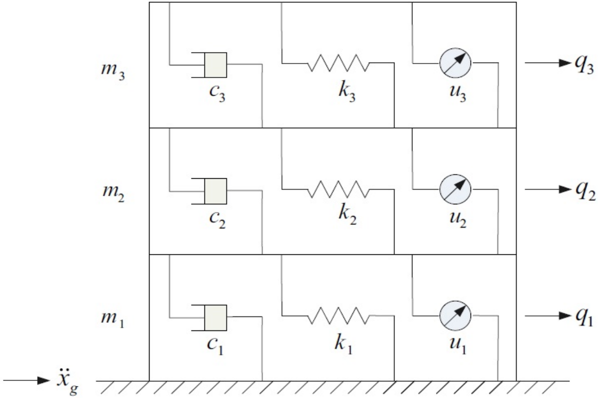
If $\overline{\gamma}^{(k-1)} > \overline{\gamma}^{(k)} > \gamma_{\min}$ and $k < N_{max}$, then set $k = k+1$, and repeat step k, otherwise stop.

Theorem

If the LMIs in Step 0 have a feasible solution, then problems P1 and P2 are feasible, Step k defines a strictly decreasing sequence $\overline{\gamma}^{(k)}$, and the algorithm terminates in finitely many steps yielding a suboptimal solution of the formulated problem.

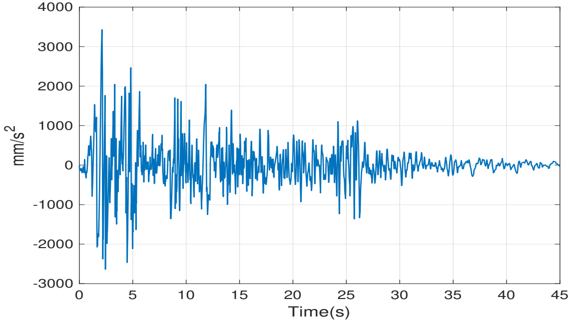
Finite frequency H_∞ control design: Example 1

Example 1. Consider a three-storey building model drawn in the next Figure.



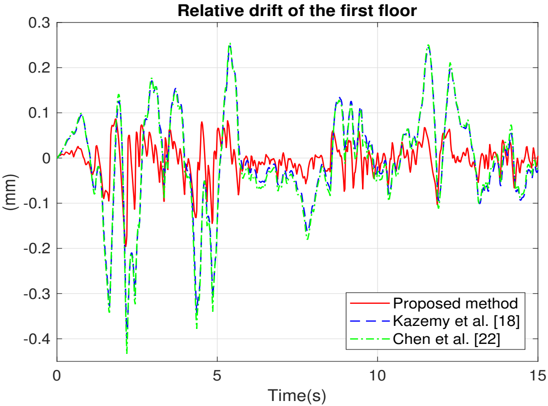
Finite frequency H_∞ control design: Example 1

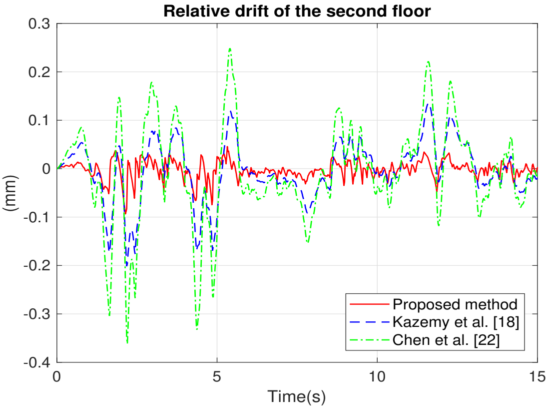
In this model, all three storeys are supposed to be identical, $n_x = 6, n_u = 3$, The coefficient matrices of the model can be computed from the physical parameters given in the literature. Parameter $z_{\max}$ is the maximum allowable relative drift between the floors with value 0.02 m. The 1940 El-Centro earthquake real data is utilized, for which $\varpi_1 = 0.3$ and $\varpi_2 = 8.8$.

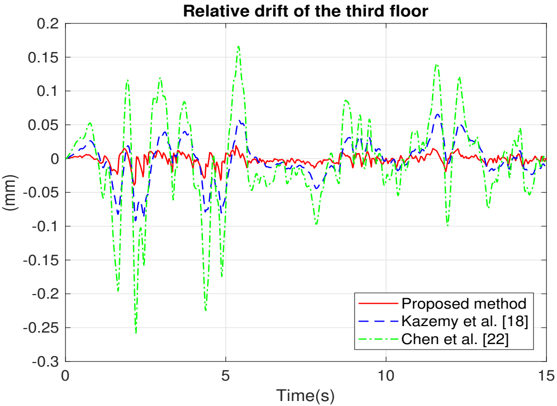


Finite frequency H_∞ control design: Example 1

Simulation results:

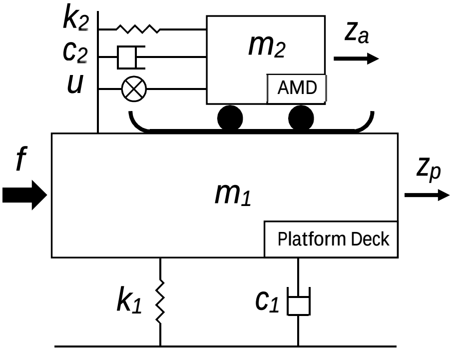



Relative drift of the second floor
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
(mm)
0
5
10
15
Time(s)
Proposed method
Kazemy et al. [18]
Chen et al. [22]


Relative drift of the third floor
0.2
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
-0.2
-0.25
-0.3
(mm)
0
5
10
15
Time(s)
Proposed method
Kazemy et al. [18]
Chen et al. [22]

Finite frequency H_∞ control design: Example 2

Example 2. In this example, an offshore platform with active mass damper (AMD) is considered. A simplified model of this platform is drawn in the next Figure.

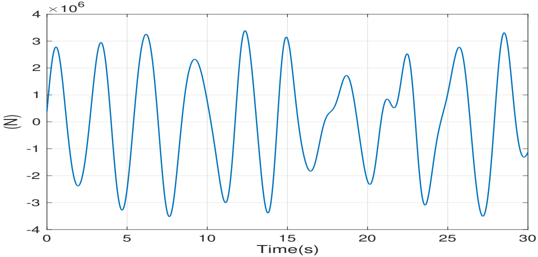


Finite frequency H_∞ control design: Example 2

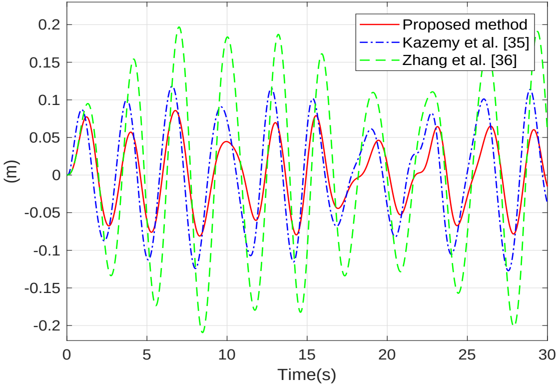
Hard constraints:

- $z_{dmax} = 25$ m is the maximum deflection between the AMD and the platform deck,
- $z_{pmax} = 0.2$ m is the maximum deviation of the platform,
- $u_{\max} = 7.6 \times 10^6$.

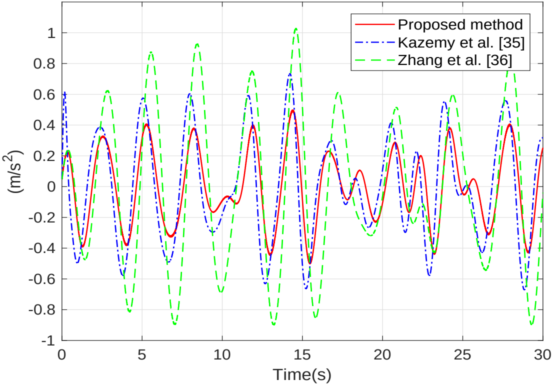
The frequency limits $\varpi_1 = 0.25$, and $\varpi_2 = 5$ are considered, a corresponding wave force has been generated and shown in the next Figure:



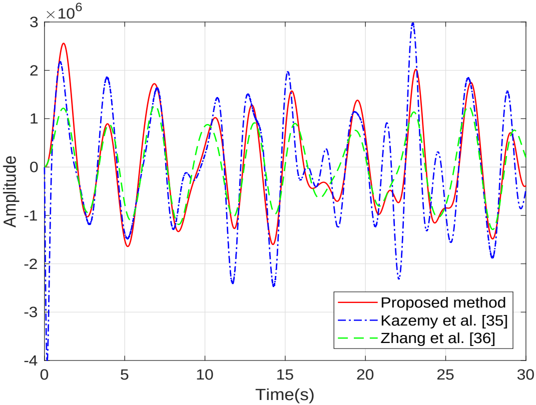
Simulation results with the computed controller:



ábra: Displacement of the platform deck



ábra: Acceleration of the platform deck



ábra: Control signal generated by different controllers