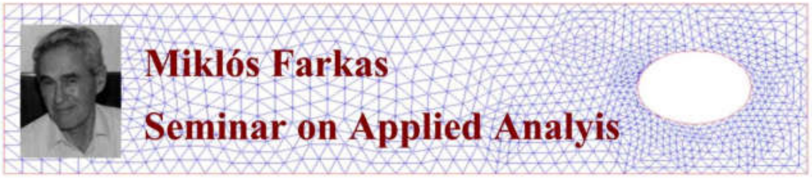
Department of Fluid Mechanics

BME Faculty of Mechanical Engineering 1871

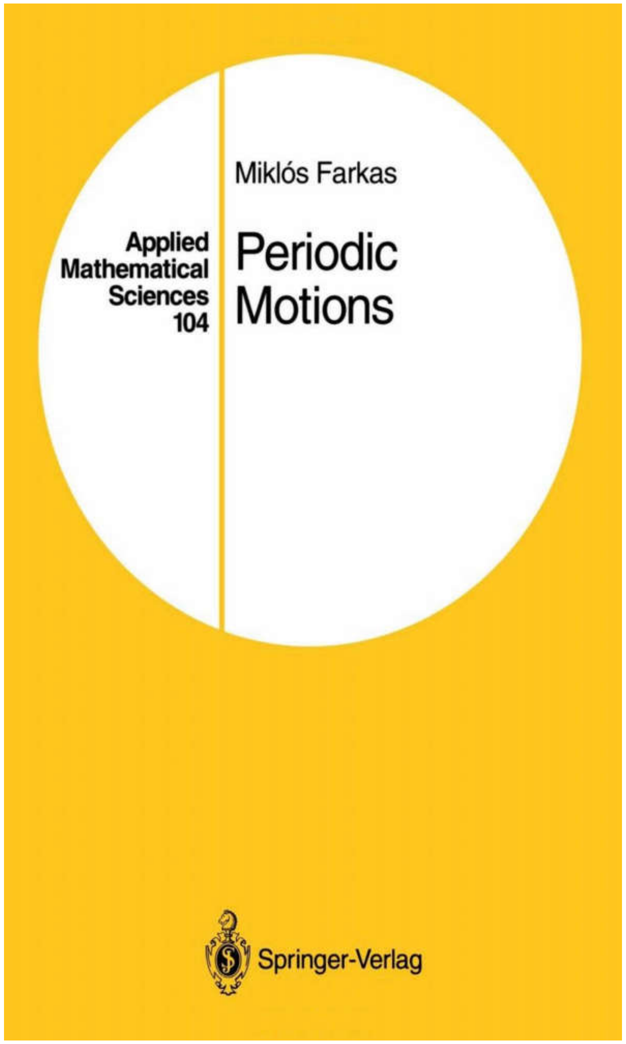
MŰEGYETEM 1782

Tamás Kalmár-Nagy, BME ARA

Having fun on the plane: Poincaré-Lyapunov constants, Jacobians, Quadrics and Jordan Forms



DEDICATION



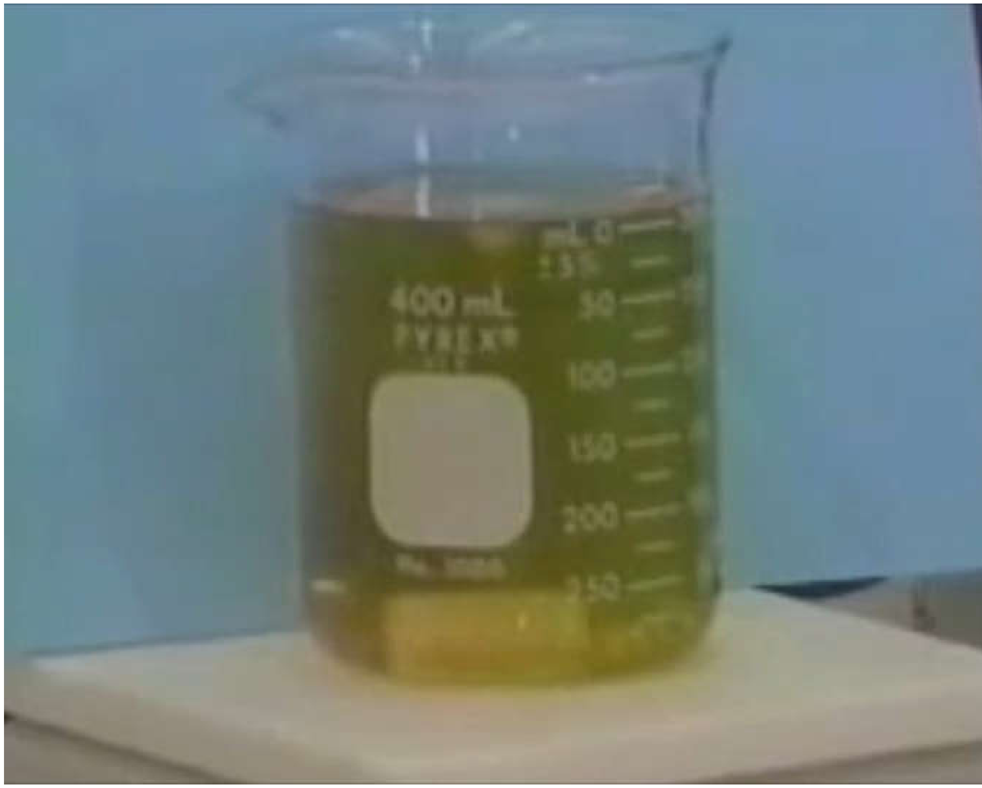
I wish to express my gratitude to those friends and colleagues who helped me in the preparation of this volume. Barna Garay, Viktor Kertész and Gábor Stépán of the Budapest University of Technology and Dana Schlomiuk of l'Université de Montreal read parts of the manuscript, corrected many errors and made invaluable suggestions that improved the text considerably. The participants of the Stability Theory Seminar at the Budapest University of Technology contributed much by their active criticism when listening to talks about some of the topics. My student Tamás Kalmár-Nagy corrected many misprints. Zoltán Farkas drew the figures with great care and much skill. Gábor Salfer solved with ease all the problems caused by our computer dependence in preparing the LaTeXversion of the manuscript. Miss Flóra Géczy has done a fine job bravely learning TeXand typing the text. Karen Kosztolnyik of Springer-Verlag New York organized and arranged everything with utmost efficiency when, finally, the publication of the book gained momentum. I thank them all. Finally, I thank my wife, Kati, for her encouragement, moral support and patience, that is, for playing the role of the Muse superbly; without this, the book probably would have not been written.

Hegyesd, March 1994 Miklós Farkas

OUTLINE

- Intro: Hopf bifurcation, Poincaré-Lyapunov constants
- Hopf Quadric
- 15 Little Jacobians
- Poincaré-Lyapunov constants and linear algebraic properties of Carleman matrices
- Summary

MOTIVATION: HOPF BIFURCATION ANALYSIS

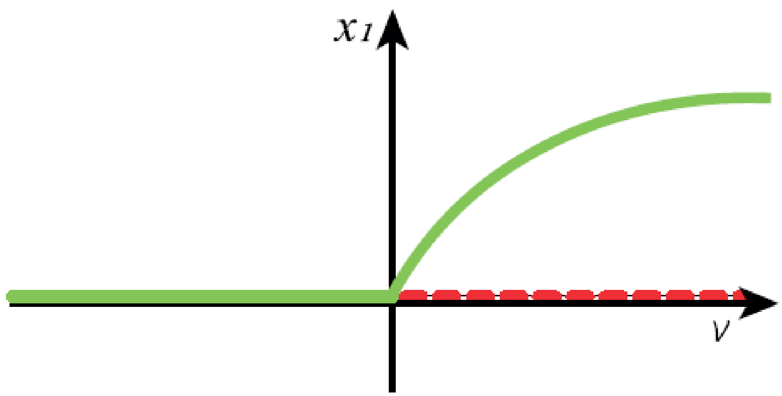


Source: YouTube - Nonlinear Dynamics and Chaos

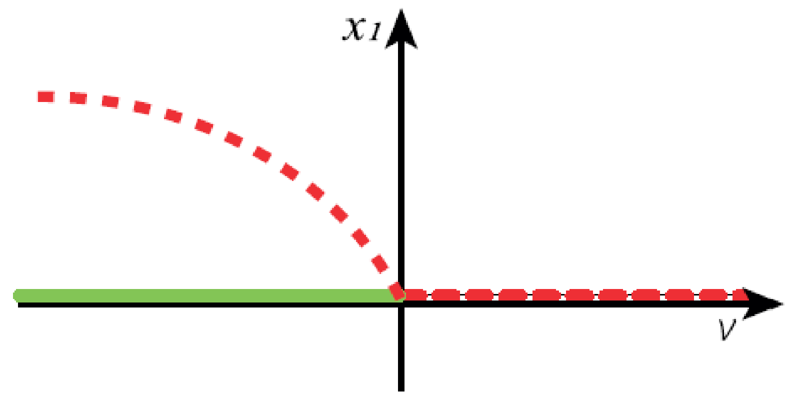
Source : YouTube – Aeroelastic Flutter

POINCARÉ-ANDRONOV-HOPF BIFURCATION

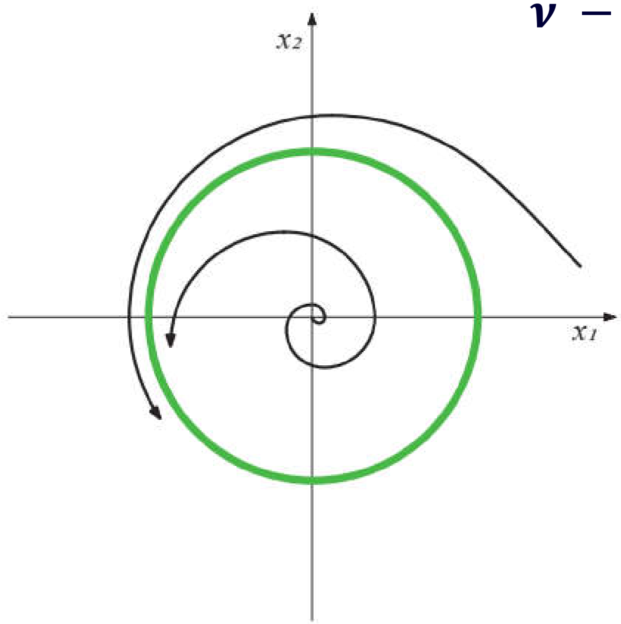
Supercritical case

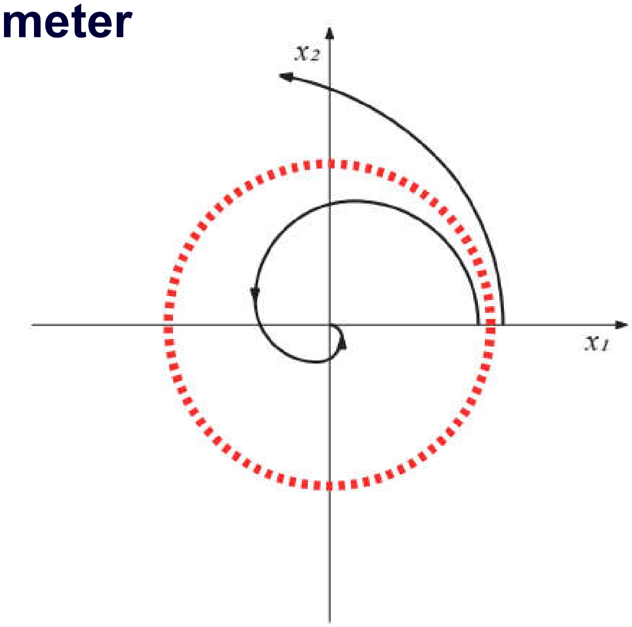


Subcritical case



ν – bifurcation parameter

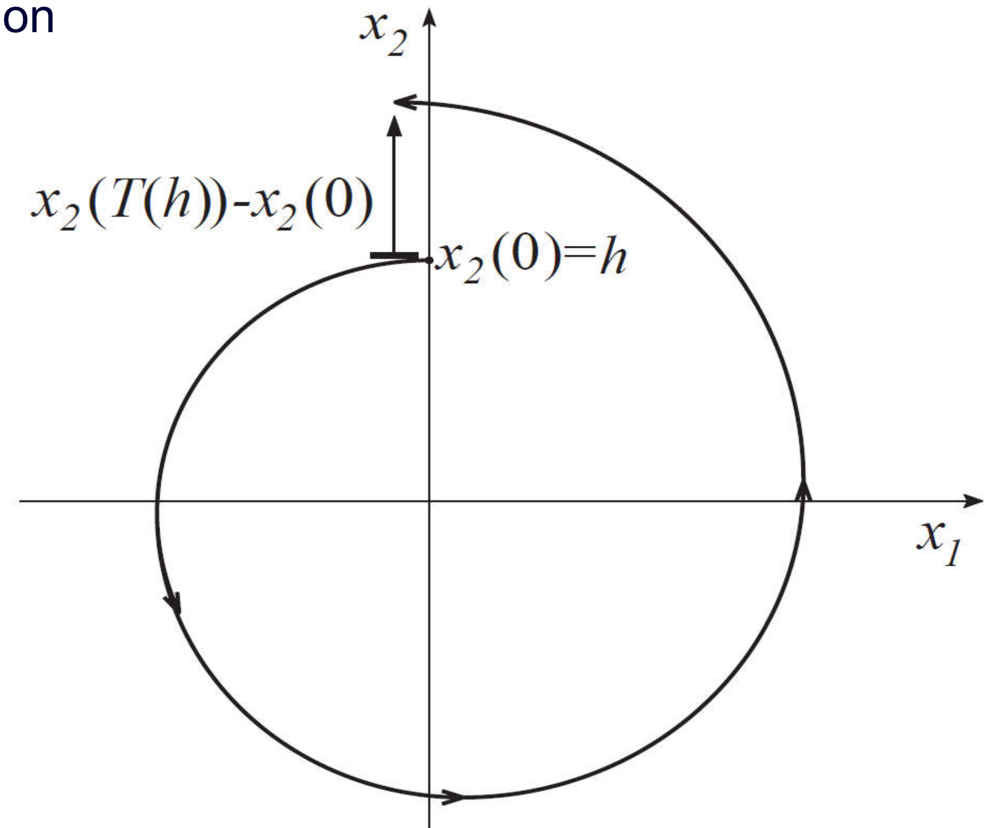




Nonlinear system at Hopf bifurcation

$$\dot{x}_1 = -x_2 + \sum_{i+j\geq 2} a_{ij} x_1^i x_2^j,$$

$$\dot{x}_2 = x_1 + \sum_{i+j\geq 2} b_{ij} x_1^i x_2^j,$$



$$x_1(0) = 0,$$
$$x_2(0) = h.$$

$$x_2(T(h)) - x_2(0) = L_1 h^3 + L_2 h^5 + \cdots + L_m h^{2m+1},$$
$$T(h) = 2\pi + \mathcal{O}(h).$$

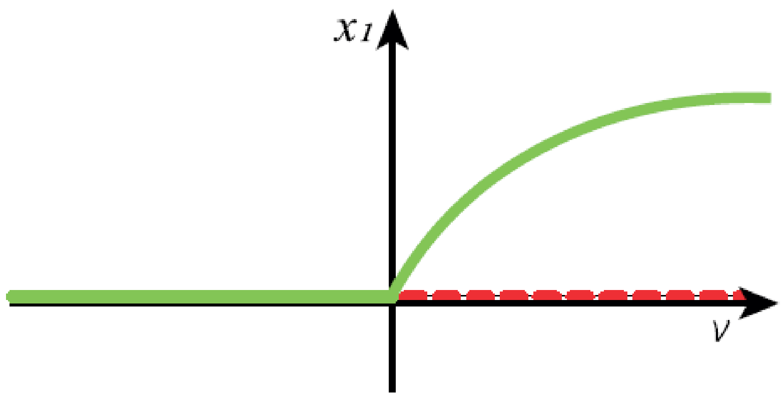
$L_1, \ldots, L_m$ **Poincaré-Lyapunov constants**

CRITICALITY OF HOPF BIFURCATION

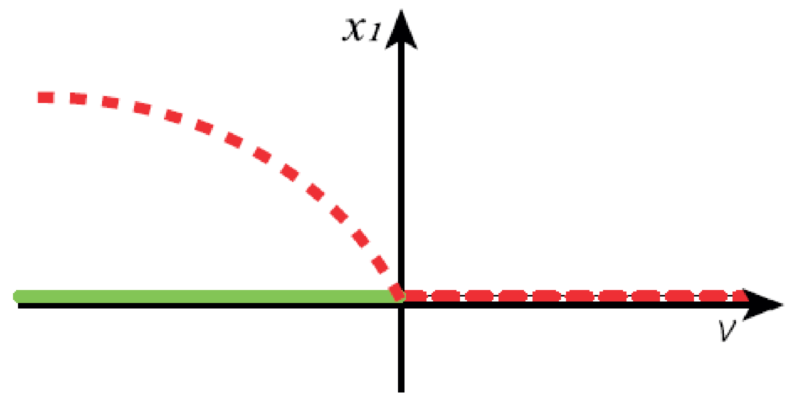
$$\dot{x}_1 = \omega x_2 + a_{20}x_1^2 + a_{11}x_1x_2 + a_{02}x_2^2 + a_{30}x_1^3 + a_{21}x_1^2x_2 + a_{12}x_1x_2^2 + a_{03}x_2^3 + h.o.t$$

$$\dot{x}_2 = -\omega x_1 + b_{20}x_1^2 + b_{11}x_1x_2 + b_{02}x_2^2 + b_{30}x_1^3 + b_{21}x_1^2x_2 + b_{12}x_1x_2^2 + b_{03}x_2^3 + h.o.t$$

Supercritical case: $L_1 < 0$



Subcritical case: $L_1 > 0$

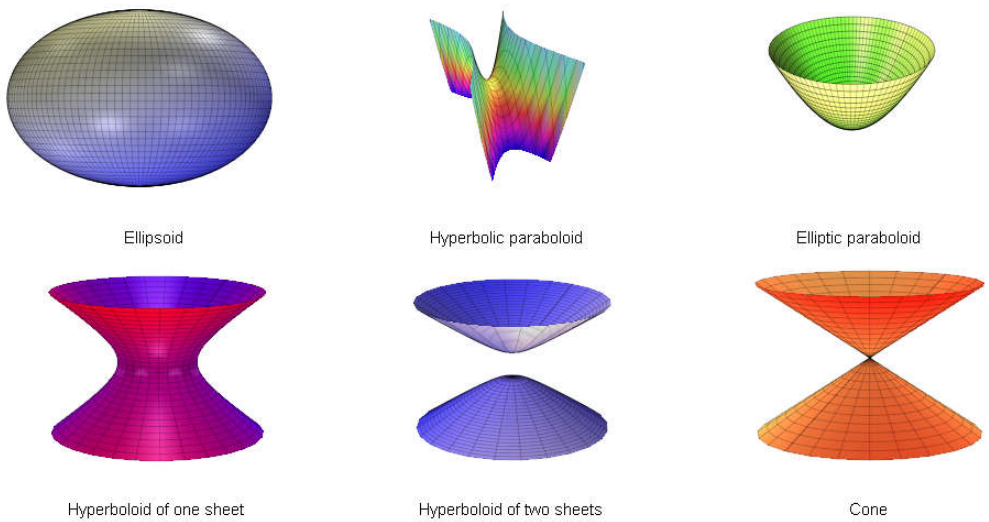


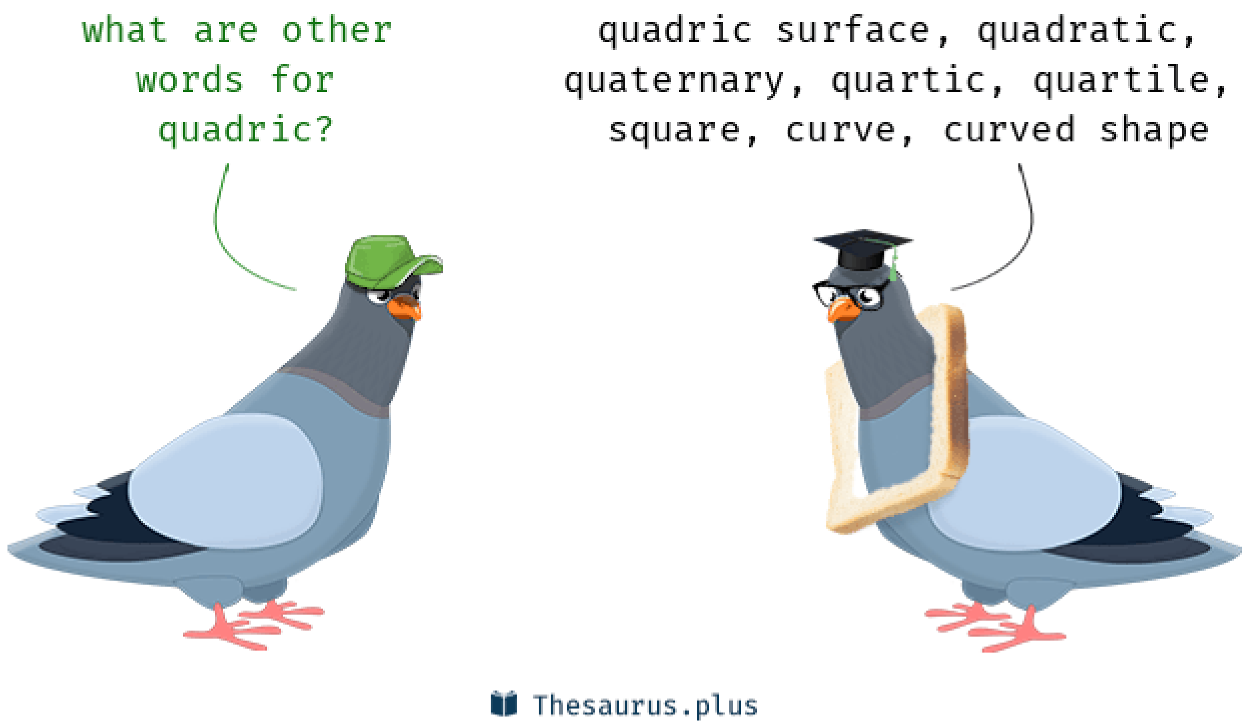
ν – bifurcation parameter

First Poincaré-Lyapunov constant

$$\boxed{L_1 = \frac{1}{8\omega}\left[(a_{20}+a_{02})(b_{20}-b_{02}-a_{11}) + (b_{20}+b_{02})(a_{20}-a_{02}+b_{11})\right] + \frac{1}{8}(3a_{30}+a_{12}+b_{21}+3b_{03})}$$

HOPF QUADRIC
WITH ALEXEI UTESHEV – ST. PETERSBURG STATE UNI.





First Poincaré-Lyapunov constant

$$\boxed{\begin{aligned} L_1 = {} & \frac{1}{8\omega}\left[(a_{20}+a_{02})(b_{20}-b_{02}-a_{11}) + (b_{20}+b_{02})(a_{20}-a_{02}+b_{11})\right] + \\ & + \frac{1}{8}(3a_{30}+a_{12}+b_{21}+3b_{03}) \end{aligned}}$$

15 parameters (ω, 7 a's and 7 b's) in the planar differential

equation, but L_1 is a function of only 11 parameters.

L_1 is a quadratic form

$$L_1 = \frac{1}{8\omega}\left[(a_{20}+a_{02})(b_{20}-b_{02}-a_{11}) + (b_{20}+b_{02})(a_{20}-a_{02}+b_{11})\right] + \\ + \frac{1}{8}(3a_{30}+a_{12}+b_{21}+3b_{03})$$

$$\mathbf{u} = (a_{11}, a_{12}, a_{20}, a_{02}, a_{30}, b_{11}, b_{21}, b_{20}, b_{02}, b_{03})^{\mathsf{T}}$$

$$L_1(\mathbf{u}) = \mathbf{u}^{\mathsf{T}}\mathbf{M}\mathbf{u} + 2\mathbf{n}^{\mathsf{T}}\mathbf{u}$$

$$\mathbf{M} = \frac{1}{8\omega}\begin{pmatrix} 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{n} = \frac{1}{16}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 3 \end{pmatrix}$$

Eigenvalues of **M**: $\frac{1}{8\omega}\sqrt{\frac{3}{2}}\{-1,-1,1,1,0,0,0,0,0,0\}$

$$\mathbf{A} = \mathbf{S}^{\mathsf{T}}\mathbf{M}\mathbf{S} = \frac{1}{8\omega}\sqrt{\frac{3}{2}}\,\mathrm{diag}(-1,-1,1,1,0,0,0,0,0,0)$$

$$\mathbf{b}^{\mathsf{T}} = \mathbf{n}^{\mathsf{T}}\mathbf{S}^{\mathsf{T}} = \frac{1}{16}(0,0,0,0,3,0,1,0,3,1)$$

$$\boxed{H(\mathbf{x}) := \mathbf{x}^{\mathsf{T}}\mathbf{A}\mathbf{x} + 2\mathbf{b}^{\mathsf{T}}\mathbf{x} = 0}$$

HOPF QUADRIC

Question: can we find the distance between a given parameter point $x_0 \in R^8$ and the Hopf quadric?
This distance would be a
measure of the "criticality" of the Hopf bifurcation.

$$\boxed{H(\mathbf{x}) := \mathbf{x}^{\mathsf{T}}\mathbf{A}\mathbf{x} + 2\mathbf{b}^{\mathsf{T}}\mathbf{x} = 0}$$

HOPF QUADRIC

The stationary points of the squared distance function from $x_0 \in R^n$ to the quadric are the real zeros of a univariate algebraic equation

$$f(\mu) = A_0\mu^4 + A_1\mu^3 + A_2\mu^2 + A_3\mu + A_4 \in \mathbb{C}[\mu],\ A_0 \neq 0$$

Discriminant: $\mathcal{D}_\mu(f(\mu)) := 4\mathfrak{I}_2^3 - 27\mathfrak{I}_3^2$

$$\mathfrak{I}_2 := 4A_0A_4 - A_1A_3 + \frac{1}{3}A_2^2,$$

$$\mathfrak{I}_3 := -A_0A_3^2 - A_1^2A_4 + \frac{8}{3}A_0A_2A_4 + \frac{1}{3}A_1A_2A_3 - \frac{2}{27}A_2^3.$$

The condition $\mathcal{D}_\mu(f(\mu)) = 0$ is necessary and sufficient for the existence of a multiple zero of f.

$$\mu_* = \frac{2A_1\mathfrak{J}_2^2+(3A_1A_2-18A_0A_3)\mathfrak{J}_3}{(24A_0A_2-9A_1^2)\mathfrak{J}_3-8A_0\mathfrak{J}_2^2}$$

Theorem 1 *The square of the distance between the quadric (15) and a point* $\mathbf{x}_0$ *outside the quadric* ($H(\mathbf{x}_0) \neq 0$) *is the minimal positive zero* z_* *of the* ***distance equation***

$$\mathcal{F}(z) := \mathcal{D}_\mu(\Phi(\mu, z)) = 0,$$

provided that this zero is not a multiple one. Here

$$\Phi(\mu, z) := \det\left(\left(\begin{array}{cc} \mathbf{A} & \mathbf{b} \\ \mathbf{b}^{\mathrm{T}} & 0 \end{array}\right) + \mu\left(\begin{array}{cc} -\mathbf{I} & \mathbf{x}_0 \\ \mathbf{x}_0^{\mathrm{T}} & z - \mathbf{x}_0^{\mathrm{T}}\mathbf{x}_0 \end{array}\right)\right)$$

and $\mathbf{I}$ *stands for the nth order identity matrix. The coordinates of the point in the quadric nearest to* $\mathbf{x}_0$ *are:*

$$\mathbf{x}_* = (\mu_*\mathbf{I} - \mathbf{A})^{-1}(\mathbf{b} + \mu_*\mathbf{x}_0)\,.$$

Here μ_* *stands for the multiple zero of the polynomial* $\Phi(\mu, z_*)$.

HOPF QUADRIC

$$\Phi(\mu, z) = A_0\,(z)\,\mu^4 + A_1\mu^3 + A_2\,(z)\,\mu^2 + A_3\mu + A_4,$$

where

$$A_0\,(z) = 8192\,\omega^2 z,\;\; A_1 = 8192\omega^2 L_1(\mathbf{x}_0),$$

$$A_2\,(z) = 32[20\omega^2 - 6z + 3(b_{20}+b_{02})^2 + 3(a_{20}+a_{02})^2 + 2(a_{20}-a_{02}+b_{11})^2 + 2(b_{20}-b_{02}+a_{11})^2],$$

$$A_3 = -24(3a_{30} + a_{12} + b_{21} + 3b_{03}),\;\; A_4 = -15\,.$$

Theorem 2 *For a specific $\mathbf{x}_0$ the square of the distance from it to the Hopf quadric equals the minimal positive zero of the equation*

$$\mathcal{F}(z) := \sum_{j=0}^{5} C_j z^{5-j} = 0\,.$$

The coefficient C_j is a polynomial of the degree $2j$ in ω and components of the vector $\mathbf{x}_0$:

$$C_0 = 12960,$$

$$C_1 = 216\big[800\omega^2 - 3(3a_{30} + a_{12} + b_{21} + 3b_{03})^2$$

$$-200((a_{20}+a_{02})^2 + (b_{20}+b_{02})^2) + 80((b_{11}+2a_{20})(2a_{02}-b_{11}) + (2b_{20}-a_{11})(a_{11}+2b_{02}))\big]$$

$$\vdots$$

HOPF QUADRIC - EXAMPLE

Example *For* $\mathbf{x}_0 = (5, -2, 4, -6, -3, -7, 2, 4, -2, 7)^T$, *one has*

$$\mathcal{F}(z) = 12960\, z^5 + 3456(50\omega^2 - 137)z^4 + 1152(750\omega^4 - 4340\omega^2 + 1980\omega + 5442)z^3$$

$$+3072(625\omega^6 - 6050\omega^4 - 2475\omega^3 - 5608\omega^2 - 16245\omega - 11884)z^2$$

$$+512(3125\omega^8 - 53500\omega^6 - 103500\omega^5 + 725610\omega^4 + 212040\omega^3 + 1655540\omega^2 + 528084\omega + 163097)z$$

$$-2048(625\omega^6 - 6675\omega^4 - 16200\omega^3 + 111081\omega^2 + 33372\omega + 13189)(3\omega + 1)^2\,.$$

For $\omega = 2$ *the distance equation possesses a single real zero, namely*

$$z_* \approx 3.803941\,.$$

Therefore, the distance from $u^{(0)}$ *to the Hopf quadric equals* $\sqrt{z_*} \approx 1.950369$.
For $\omega = 1/2$, *the distance equation has three real zeros, namely*

$$z_* \approx 2.016646, 14.061803, 14.252662\,.$$

In this case, the distance to the Hopf quadric equals $\sqrt{z_*} \approx 1.420086$.

Can we determine stability from eigenvalues around a nonhyperbolic equilibrium?

Ghaffari and Lasemi considered the autonomous system

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}\right),$$

where $\mathbf{f}$ is continuously differentiable, and the origin $\mathbf{x} = \mathbf{0}$ is an isolated nonhyperbolic equilibrium point. Let $\mathbf{x}_0$ be a point inside of a punctured neighborhood N of the origin and $J = \left.\frac{d\mathbf{f}(\mathbf{x})}{d\mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}_0}$. Their theorem states that the origin is asymptotically stable if for all $\mathbf{x}_0$ in N the matrix A is stable, otherwise it is unstable.

Let us consider

$$\dot{x}_1 = \omega x_2 + a_{20}x_1^2 + a_{11}x_1x_2 + a_{02}x_2^2 + a_{30}x_1^3 + a_{21}x_1^2x_2 + a_{12}x_1x_2^2 + a_{03}x_2^3,$$

$$\dot{x}_2 = -\omega x_1 + b_{20}x_1^2 + b_{11}x_1x_2 + b_{02}x_2^2 + b_{30}x_1^3 + b_{21}x_1^2x_2 + b_{12}x_1x_2^2 + b_{03}x_2^3.$$

$$J = \begin{pmatrix} 2a_{20}x_1 + a_{11}x_2 + 3a_{30}x_1^2 + 2a_{21}x_1x_2 + a_{12}x_2^2 & \omega + a_{11}x_1 + 2a_{02}x_2 + a_{21}x_1^2 + 2a_{12}x_1x_2 + 3a_{03}x_2^2 \\ -\omega + 2b_{20}x_1 + b_{11}x_2 + 3b_{30}x_1^2 + 2b_{21}x_1x_2 + b_{12}x_2^2 & b_{11}x_1 + 2b_{02}x_2 + b_{21}x_1^2 + 2b_{12}x_1x_2 + 3b_{03}x_2^2 \end{pmatrix}$$

$$\omega^2 = \left.\mathrm{Det}\, J\right|_{\mathbf{x}=0}$$

$$\mathrm{Tr}J = (2a_{20} + b_{11})x_1 + (a_{11} + 2b_{02})x_2 + (3a_{30} + b_{21})x_1^2 + 2(a_{21} + b_{12})x_1x_2 + (a_{12} + 3b_{03})x_2^2$$

$$\mathrm{Tr}J = \sum_{i+j=1,2} t_{ij}x_1^i x_2^j$$

$$t_{10} = 2a_{20} + b_{11}, \quad t_{01} = a_{11} + 2b_{02},$$

$$t_{20} = 3a_{30} + b_{21}, \quad t_{11} = 2\left(a_{21} + b_{12}\right), \quad t_{02} = a_{12} + 3b_{03}$$

Let us define

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

$$C = \begin{pmatrix} 3a_{30} + b_{21} & a_{21} + b_{12} \\ a_{21} + b_{12} & a_{12} + 3b_{03} \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} a_{20} + \frac{b_{11}}{2} \\ \frac{a_{11}}{2} + b_{02} \end{pmatrix}$$

$$\mathrm{Tr}\, J = \mathbf{x}^T C \mathbf{x} + \mathbf{2b}^T \mathbf{x}$$

$$\mathrm{Tr}\, C = t_{20} + t_{02} = 3a_{30} + b_{21} + a_{12} + 3b_{03}$$

$$L_1 = \frac{1}{8\omega}[(a_{20}+a_{02})(b_{20}-b_{02}-a_{11}) + (b_{20}+b_{02})(a_{20}-a_{02}+b_{11})] + \\ + \frac{1}{8}(3a_{30}+a_{12}+b_{21}+3b_{03})$$

Can we express the Poincaré-Lyapunov constant in terms of TrJ and DetJ evaluated at different points?

15 LITTLE JACOBIANS

$$\mathrm{Tr}J = \begin{pmatrix} x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \end{pmatrix} \begin{pmatrix} 2a_{20}+b_{11} \\ a_{11}+2b_{02} \\ 3a_{30}+b_{21} \\ 2\left(a_{21}+b_{12}\right) \\ a_{12}+3b_{03} \end{pmatrix}$$

$$\frac{1}{2}\left(t_{01}-\frac{d_{10}}{\omega}\right) = b_{02}+b_{20},$$

$$\frac{1}{2}\left(t_{10}+\frac{d_{01}}{\omega}\right) = a_{02}+a_{20}$$

$$\omega^2 = \mathrm{Det}\, J\big|_{\mathbf{x}=0}$$

$$\mathrm{PL} = \frac{1}{8}\left(3a_{30}+b_{21}+a_{12}+3b_{03}\right)+\frac{1}{8\omega}\left[(b_{02}+b_{20})(2a_{20}+b_{11})-(a_{02}+a_{20})(a_{11}+2b_{02})\right] =$$

$$\frac{1}{8}\left(t_{20}+t_{02}\right)+\frac{1}{8\omega}\left[\frac{1}{2}\left(t_{01}-\frac{d_{10}}{\omega}\right)t_{10}-\frac{1}{2}\left(t_{10}+\frac{d_{01}}{\omega}\right)t_{01}\right] =$$

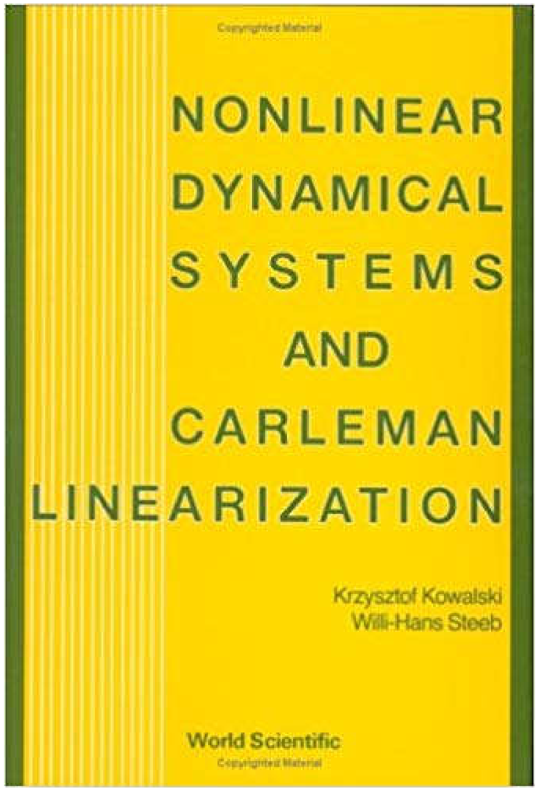
$$\frac{1}{8}\left(t_{20}+t_{02}-\frac{d_{10}t_{10}+d_{01}t_{01}}{2d_{00}}\right)$$

$$\mathrm{Tr} J = \lambda_1 + \lambda_2$$

$$\mathrm{Det}\, J = \lambda_1 \lambda_2$$

Yes, we can determine stability from eigenvalues at 15 points around a nonhyperbolic equilibrium!

- 1932, Carleman method based on Poincaré's idea
- 1986, Tsiligiannis & Lyberatos, theorems about Hopf bifurcation using Carleman matrices
- 1991, Steeb and Kowalski book



Goal: *Use of linear algebraic methods in bifurcation analysis*

Research question:

What is the connection between the Poincaré-Lyapunov constants and the linear algebraic properties of the Carleman matrices?

Let us consider

$$\dot{x}_1 = -x_2 + \sum_{i+j\geq 2} a_{ij} x_1^i x_2^j,$$

$$\dot{x}_2 = x_1 + \sum_{i+j\geq 2} b_{ij} x_1^i x_2^j,$$

Introducing the monomials of x_1 and x_2

$$\mathbf{x}^{[j]} = (x_1^j,\ x_1^{j-1} x_2^1,\ x_1^{j-2} x_2^2,\ \ldots,\ x_1^2 x_2^{j-2},\ x_1^1 x_2^{j-1}, x_2^j)^{\mathrm{T}}.$$

e.g.

$$\mathbf{x}^{[1]} = (x_1,\ x_2)^{\mathrm{T}},$$

$$\mathbf{x}^{[2]} = \left(x_1^2,\ x_1 x_2,\ x_2^2\right)^{\mathrm{T}},$$

$$\mathbf{x}^{[3]} = \left(x_1^3,\ x_1^2 x_2,\ x_1 x_2^2,\ x_2^3\right)^{\mathrm{T}},$$

$$\ldots$$

DERIVATION OF THE VECTORS OF MONOMIALS

For example, differentiating $\mathbf{x}^{[3]}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}^{[3]} = \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1^3 \\ x_1^2 x_2 \\ x_1 x_2^2 \\ x_2^3 \end{pmatrix} = \begin{pmatrix} 3x_1^2 \dot{x}_1 \\ 2x_1 x_2 \dot{x}_1 + x_1^2 \dot{x}_2 \\ x_2^2 \dot{x}_1 + 2x_1 x_2 \dot{x}_2 \\ 3x_2^2 \dot{x}_2 \end{pmatrix}.$$

In general,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}^{[j]} = \left(\mathrm{N}_1(j)\,\dot{x}_1\right)\frac{\mathbf{x}^{[j]}}{x_1} + \left(\mathrm{N}_2(j)\,\dot{x}_2\right)\frac{\mathbf{x}^{[j]}}{x_2},$$

where

$$\mathrm{N}_1(j) = \left(\begin{array}{cccc:c} j & 0 & \dots & 0 & 0 \\ 0 & j-1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ \hdashline 0 & 0 & \dots & 0 & 0 \end{array}\right), \quad \mathrm{N}_2(j) = j\mathbf{I} - \mathrm{N}_1(j).$$

The Carleman embedding

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}^{[j]} = \sum_{k=j}^{n} \mathbf{A}_{j,k}\mathbf{x}^{[k]} \ , j = 1, \ldots, n.}$$

The matrices $\mathbf{A}_{j,k}$

$$\mathbf{A}_{j,k} = \mathrm{N}_1(j)\begin{pmatrix} a_{l,0} & a_{l-1,1} & \cdots & a_{1,l-1} & a_{0,l} & 0 & 0 & 0 \\ 0 & a_{l,0} & a_{l-1,1} & \cdots & a_{1,l-1} & a_{0,l} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & a_{l,0} & a_{l-1,1} & \cdots & a_{1,l-1} & a_{0,l} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} +$$

$$+\mathrm{N}_2(j)\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b_{l,0} & b_{l-1,1} & \cdots & b_{1,l-1} & b_{0,l} & 0 & 0 & 0 \\ 0 & b_{l,0} & b_{l-1,1} & \cdots & b_{1,l-1} & b_{0,l} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & b_{l,0} & b_{l-1,1} & \cdots & b_{1,l-1} & b_{0,l} \end{pmatrix},$$

where $l = k - j + 1.$

The Carleman embedded system

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \mathbf{x}^{[1]} \\ \mathbf{x}^{[2]} \\ \vdots \\ \mathbf{x}^{[n]} \end{pmatrix}}_{\dot{\mathbf{y}}} = \underbrace{\begin{pmatrix} \mathbf{A}_{1,1} & \mathbf{A}_{1,2} & \dots & \mathbf{A}_{1,n} \\ \mathbf{0} & \mathbf{A}_{2,2} & \dots & \mathbf{A}_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{n,n} \end{pmatrix}}_{\mathbf{C}_n} \underbrace{\begin{pmatrix} \mathbf{x}^{[1]} \\ \mathbf{x}^{[2]} \\ \vdots \\ \mathbf{x}^{[n]} \end{pmatrix}}_{\mathbf{y}} + \mathcal{O}\left(\mathbf{x}^{[n+1]}\right).$$

and the vector of the initial conditions

$$\mathbf{y}_0 = \left(0,\, h,\, 0,\, 0,\, h^2,\, 0,\, 0,\, 0,\, h^3,\, \dots,\, h^n\right)^{\mathrm{T}}.$$

The linearized system

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \mathbf{x}^{[1]} \\ \mathbf{x}^{[2]} \\ \vdots \\ \mathbf{x}^{[n]} \end{pmatrix}}_{\dot{\mathbf{y}}} = \underbrace{\begin{pmatrix} \mathbf{A}_{1,1} & \mathbf{A}_{1,2} & \dots & \mathbf{A}_{1,n} \\ \mathbf{0} & \mathbf{A}_{2,2} & \dots & \mathbf{A}_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{n,n} \end{pmatrix}}_{\mathbf{C}_n} \underbrace{\begin{pmatrix} \mathbf{x}^{[1]} \\ \mathbf{x}^{[2]} \\ \vdots \\ \mathbf{x}^{[n]} \end{pmatrix}}_{\mathbf{y}}.$$

Strategy to solve the linearized system:

1. Calculating the eigenvalues of the Carleman matrix
2. Calculating the eigenvectors of the Carleman matrix
3. Constructing the solution as linear combinations

1. EIGENVALUES OF THE CARLEMAN MATRIX

The Carleman matrix:

$$\mathbf{C}_n = \begin{pmatrix} \mathbf{A}_{1,1} & \mathbf{A}_{1,2} & \cdots & \mathbf{A}_{1,n} \\ \mathbf{0} & \mathbf{A}_{2,2} & \cdots & \mathbf{A}_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{n,n} \end{pmatrix}.$$

Lemma (non-trivial):

$$\mathrm{sp}\left(\mathbf{A}_{n,n}\right) = \begin{cases} \{0, \pm 2i, \ldots, \pm ni\} & 2 \mid n \\ \{\pm i, \pm 3i, \ldots, \pm ni\} & 2 \nmid n \end{cases}.$$

Lemma (trivial):

$$\mathrm{sp}\left(\mathbf{C}_n\right) = \bigcup_{i=1}^{n} \mathrm{sp}\left(\mathbf{A}_{i,i}\right).$$

It is true since $\mathbf{C}_n$ is an upper-triangular matrix.

2. EIGENVECTORS OF THE CARLEMAN MATRIX

Eigenproblem

$$\mathbf{C}_n \mathbf{u}_n = \lambda \mathbf{u}_n .$$

Block matrix structure

$$\underbrace{\left(\begin{array}{c|c} & \mathbf{A}_{1,n} \\ \mathbf{C}_{n-1} & \vdots \\ & \mathbf{A}_{n-1,n} \\ \hline \mathbf{0} & \mathbf{A}_{n,n} \end{array}\right)}_{\mathbf{C}_n} \underbrace{\begin{pmatrix} \mathbf{u}_{n,1} \\ \vdots \\ \mathbf{u}_{n,n-1} \\ \mathbf{u}_{n,n} \end{pmatrix}}_{\mathbf{u}_n} = \lambda \underbrace{\begin{pmatrix} \mathbf{u}_{n,1} \\ \vdots \\ \mathbf{u}_{n,n-1} \\ \mathbf{u}_{n,n} \end{pmatrix}}_{\mathbf{u}_n} .$$

Starting point

$$\mathrm{sp}(\mathbf{C}_1) = \{\pm i\}, \mathbf{C}_1 \mathbf{u}_1 = i \mathbf{u}_1$$

that yields

$$\mathbf{u}_1 = \begin{pmatrix} i \\ 1 \end{pmatrix} .$$

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2. EIGENVECTORS OF THE CARLEMAN MATRIX

The equations to solve are

$$\boxed{\begin{gathered} \mathbf{A}_{n,n}\mathbf{u}_{n,n} = \lambda \mathbf{u}_{n,n}, \qquad (1) \\ (\mathbf{C}_{n-1} - \lambda \mathbf{I}) \begin{pmatrix} \mathbf{u}_{n,1} \\ \vdots \\ \mathbf{u}_{n,n-1} \end{pmatrix} = - \begin{pmatrix} \mathbf{A}_{1,n} \\ \vdots \\ \mathbf{A}_{n-1,n} \end{pmatrix} \mathbf{u}_{n,n}. \qquad (2) \end{gathered}}$$

1. Case of eigenvalues with already calculated eigenvectors $\mathbf{u}_{n-1}$

   $\lambda \notin \mathrm{sp}(\mathbf{A}_{n,n})$, $\lambda \in \mathrm{sp}(\mathbf{C}_{n-1})$, i.e. $\lambda \in \{(n-1)i,\ (n-3)i,\ \ldots\}$.

The eigenvector reads

$$\mathbf{u}_n = \begin{pmatrix} \mathbf{u}_{n-1} \\ \mathbf{0}_{n+1} \end{pmatrix}.$$

2. EIGENVECTORS OF THE CARLEMAN MATRIX

$$\boxed{\begin{aligned} &\mathbf{A}_{n,n}\mathbf{u}_{n,n} = \lambda \mathbf{u}_{n,n}, \\ &(\mathbf{C}_{n-1} - \lambda \mathbf{I}) \begin{pmatrix} \mathbf{u}_{n,1} \\ \vdots \\ \mathbf{u}_{n,n-1} \end{pmatrix} = - \begin{pmatrix} \mathbf{A}_{1,n} \\ \vdots \\ \mathbf{A}_{n-1,n} \end{pmatrix} \mathbf{u}_{n,n}. \end{aligned}}$$

(1)

(2)

2. Case of new, single eigenvalues, $\lambda \in \mathrm{sp}(\mathbf{A}_{n,n})$, $\lambda \notin \mathrm{sp}(\mathbf{C}_{n-1})$, i.e. $\lambda = \pm ni$.

$$\begin{pmatrix} \mathbf{u}_{n,1} \\ \vdots \\ \mathbf{u}_{n,n-1} \end{pmatrix} = (\lambda \mathbf{I} - \mathbf{C}_{n-1})^{-1} \begin{pmatrix} \mathbf{A}_{1,n} \\ \vdots \\ \mathbf{A}_{n-1,n} \end{pmatrix} \mathbf{u}_{n,n}.$$

Thus,

$$\mathbf{u}_n = \begin{pmatrix} (\lambda \mathbf{I} - \mathbf{C}_{n-1})^{-1} \begin{pmatrix} \mathbf{A}_{1,n} \\ \vdots \\ \mathbf{A}_{n-1,n} \end{pmatrix} \\ \mathbf{I} \end{pmatrix} \mathbf{u}_{n,n}.$$

2. EIGENVECTORS OF THE CARLEMAN MATRIX

3. Case of repeated eigenvalues,

$$\lambda \in \mathrm{sp}(\mathbf{A}_{n,n}),\ \lambda \in \mathrm{sp}(\mathbf{C}_{n-1}),\ \text{i.e. } \lambda \in \{(n-2)i,\ (n-4)i,\ \ldots\}.$$

a) Similarly to the first case

$$\mathbf{u}_n = \begin{pmatrix} \mathbf{u}_{n-1} \\ \mathbf{0}_{n+1} \end{pmatrix}.$$

b)

$$\boxed{\begin{gathered} \mathbf{A}_{n,n}\mathbf{u}_{n,n} = \lambda \mathbf{u}_{n,n}, \\ (\mathbf{C}_{n-1} - \lambda \mathbf{I}) \begin{pmatrix} \mathbf{u}_{n,1} \\ \vdots \\ \mathbf{u}_{n,n-1} \end{pmatrix} = - \begin{pmatrix} \mathbf{A}_{1,n} \\ \vdots \\ \mathbf{A}_{n-1,n} \end{pmatrix} \mathbf{u}_{n,n}. \end{gathered}}$$

$\mathbf{u}_{n,n}$

Inhomogeneous eigenvalue problem

TRICKY CASE: INHOMOGENEOUS EIGENVALUE PROBLEM

In case of zero eigenvalue

$$(\mathbf{C}_{n-1} \cancel{- \lambda \mathbf{I}}) \begin{pmatrix} \mathbf{u}_{n,1} \\ \vdots \\ \mathbf{u}_{n,n-1} \end{pmatrix} = - \begin{pmatrix} \mathbf{A}_{1,n} \\ \vdots \\ \mathbf{A}_{n-1,n} \end{pmatrix} \mathbf{u}_{n,n} \, ,$$

utilizing the blockmatrix structure of $\mathbf{C}_{n-1}$

$$\begin{aligned} \mathbf{A}_{n-1,n-1}\mathbf{u}_{n,n-1} &= -\mathbf{A}_{n-1,n}\mathbf{u}_{n,n}, \\ \mathbf{A}_{n-2,n-2}\mathbf{u}_{n,n-2} &= -\mathbf{A}_{n-2,n-1}\mathbf{u}_{n-1} - \mathbf{A}_{n-2,n}\mathbf{u}_{n,n}, \\ &\vdots \\ \mathbf{A}_{2,2}\mathbf{u}_{n,2} &= -\mathbf{A}_{2,3}\mathbf{u}_{n,3} - \mathbf{A}_{2,4}\mathbf{u}_{n,4} - \cdots - \mathbf{A}_{2,n}\mathbf{u}_{n,n}, \\ \mathbf{A}_{1,1}\mathbf{u}_{n,1} &= -\mathbf{A}_{1,2}\mathbf{u}_{n,2} - \mathbf{A}_{1,3}\mathbf{u}_{n,3} - \cdots - \mathbf{A}_{1,n}\mathbf{u}_{n,n}. \end{aligned}$$

Since $\lambda = 0 \notin \mathrm{sp}\left(\mathbf{A}_{n-1,n-1}\right)$

$$\mathbf{u}_{n,n-1} = -\mathbf{A}_{n-1,n-1}^{-1}\mathbf{A}_{n-1,n}\mathbf{u}_{n,n}.$$

APPLICATION OF THE FREDHOLM ALTERNATIVE

We want to solve the following equation

$$\mathbf{A}_{n-2,n-2}\mathbf{u}_{n,n-2} = \underbrace{-\mathbf{A}_{n-2,n-1}\mathbf{u}_{n,n-1} - \mathbf{A}_{n-2,n}\mathbf{u}_{n,n}}_{\mathbf{b}_2}.$$

Let $\mathbf{z}_2 \in \mathcal{N}\left(\mathbf{A}^{\mathrm{T}}_{n-2,n-2}\right)$ then

$$\mathbf{z}_2^{\mathrm{T}}\mathbf{b}_2 = c_1 L_1$$

If $L_1 \neq 0$. there exists no solution. Otherwise,

$$\mathbf{u}_{n,n-3} = -\mathbf{A}^{-1}_{n-3,n-3}\left(\mathbf{A}_{n-3,n-2}\mathbf{u}_{n,n-2} + \mathbf{A}_{n-3,n-1}\mathbf{u}_{n,n-1} + \mathbf{A}_{n-3,n}\mathbf{u}_{n,n}\right),$$

and

$$\begin{aligned}&\mathbf{A}_{n-4,n-4}\mathbf{u}_{n,n-4} = \\ &= \underbrace{-\mathbf{A}_{n-4,n-3}\mathbf{u}_{n,n-3} - \mathbf{A}_{n-4,n-2}\mathbf{u}_{n-2} - \mathbf{A}_{n-4,n-1}\mathbf{u}_{n,n-1} - \mathbf{A}_{n-4,n}\mathbf{u}_{n,n}}_{\mathbf{b}_4},\end{aligned}$$

we examine the scalar product ($\mathbf{z}_4 \in \mathcal{N}\left(\mathbf{A}^{\mathrm{T}}_{n-4,n-4}\right)$)

$$\mathbf{z}_4^{\mathrm{T}}\mathbf{b}_4 = c_2 L_2.$$

...

GENERALIZED EIGENVECTORS

The Carleman matrix $\mathbf{C}_n$ is defective.

The number of distinct eigenvalues is $p = 2n + 1$.

Generalized eigenvector of rank k

$$(\mathbf{C}_n - \lambda\mathbf{I})^k \mathbf{w}_n = \mathbf{0},$$

but

$$(\mathbf{C}_n - \lambda\mathbf{I})^{k-1} \mathbf{w}_n \neq \mathbf{0}.$$

The rank 1 generalized eigenvector is the normal eigenvector $\mathbf{u}_n$. Generalized eigenvectors can be calculated from

$$(\mathbf{C}_n - \lambda\mathbf{I})\mathbf{w}_n = \mathbf{w}_{n-1}.$$

SOLUTION OF THE LINEAR SYSTEM

$\boxed{\alpha_{\lambda_j} \text{ - algebraic multiplicity of the eigenvalue } \lambda_j}$

Introducing the notation of the eigenvectors

$$\mathbf{v}(\mathbf{C}_n,\, \lambda_j,\, k)\,,$$

where

- $\mathbf{C}_n$ is the Carleman matrix of order n,
- λ_j is the jth eigenvalue,
- k denotes the rank k generalized eigenvector.

The linear combination of all the linearly independent solutions of the eigenvalues gives

$$\mathbf{y}(t) = \sum_{j=1}^{p} \sum_{k=1}^{\alpha_{\lambda_j}} \sum_{l=1}^{k} c_{j,k}(h) e^{\lambda_j t} \frac{t^{k-l}}{(k-l)!} \mathbf{v}(\mathbf{C}_n,\, \lambda_j,\, k)\,.$$

SOLUTION OF THE LINEAR SYSTEM

$$x_2(T(h)) - x_2(0) = L_1 h^3 + L_2 h^5 + \cdots + L_m h^{2m+1},$$
$$T(h) = 2\pi + \mathcal{O}(h).$$ Kuznetsov & Leonov (2007, 2008)

Difference between the solution and the intial conditions

$$x_1(2\pi) - x_1(0) = \mathbf{e}_1^{\mathrm{T}}\left(\mathbf{y}(2\pi) - \mathbf{y}(0)\right) = \sum_{j=1}^{p}\sum_{k=1}^{\alpha_{\lambda_j}}\sum_{l=1}^{k-1} c_{j,k}(h)\frac{(2\pi)^{k-l}}{(k-l)!}\mathbf{e}_1^{\mathrm{T}}\mathbf{v}\left(\mathbf{C}_n, \lambda_j, l\right) = M_1 h^3 + M_2 h^5 + \cdots + M_m h^n,$$

$$x_2(2\pi) - x_2(0) = \mathbf{e}_2^{\mathrm{T}}\left(\mathbf{y}(2\pi) - \mathbf{y}(0)\right) =$$

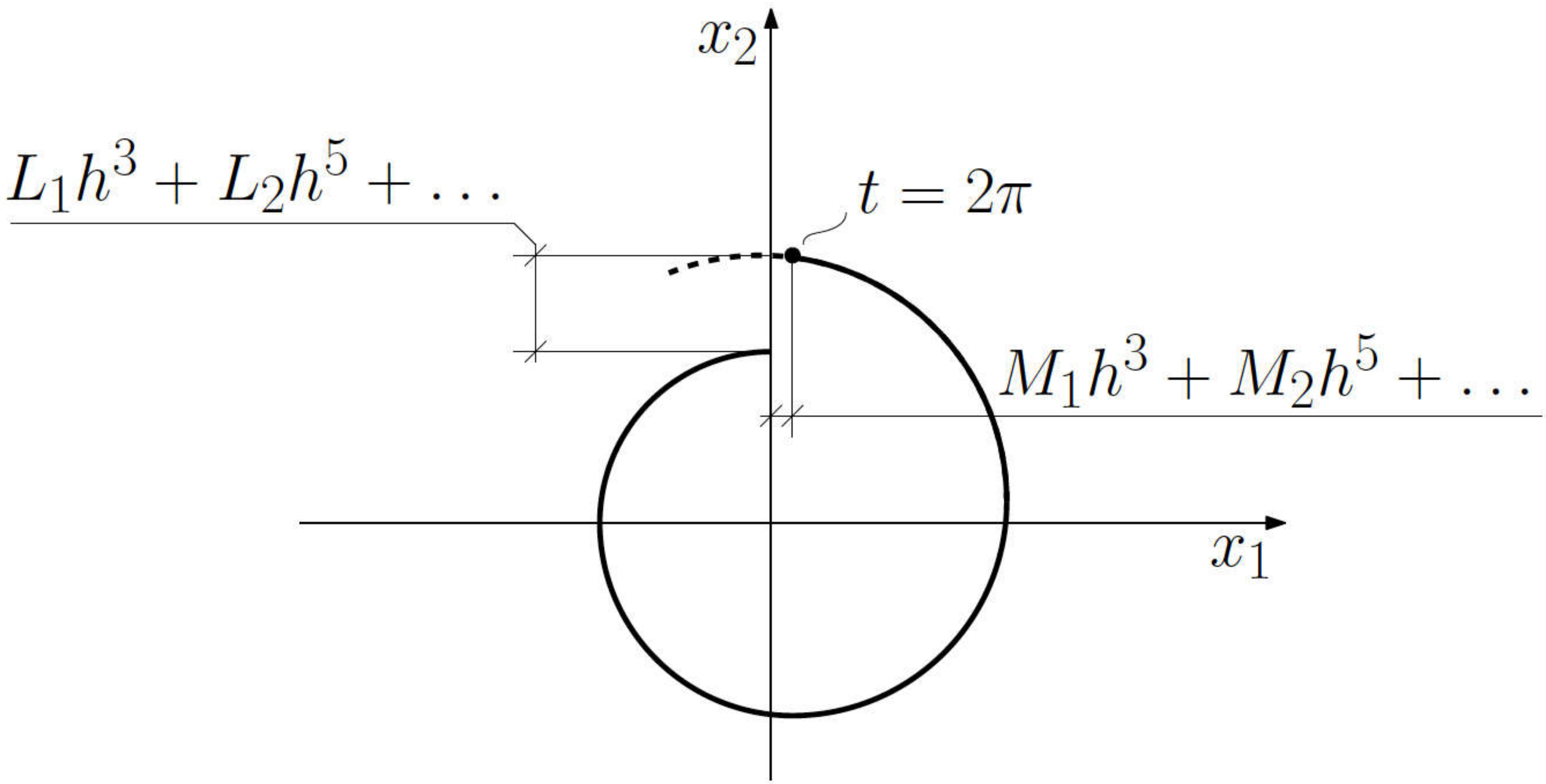
$$= \sum_{j=1}^{p}\sum_{k=1}^{\alpha_{\lambda_j}}\sum_{l=1}^{k-1} c_{j,k}(h)\frac{(2\pi)^{k-l}}{(k-l)!}\mathbf{e}_2^{\mathrm{T}}\mathbf{v}(\mathbf{C}_n,\, \lambda_j,\, k) = L_1 h^3 + L_2 h^5 + \cdots + L_m h^n + \mathcal{O}(h^{n+1}),$$

Where

$\mathbf{e}_2^{\mathrm{T}} = (0,\, 1,\, 0,\, \ldots,\, 0)$ is the standard basis vector,

$L_1,\, \ldots,\, L_m$ are the Poincaré-Lyapunov constants.

PHASE PORTRAIT



EXAMPLE

The third order Carleman matrix is written as

$$\mathbf{C}_3 = \begin{pmatrix} 0 & -1 & a_{2,0} & a_{1,1} & a_{0,2} & a_{3,0} & a_{2,1} & a_{1,2} & a_{0,3} \\ 1 & 0 & b_{2,0} & b_{1,1} & b_{0,2} & b_{3,0} & b_{2,1} & b_{1,2} & b_{0,3} \\ 0 & 0 & 0 & -2 & 0 & 2a_{2,0} & 2a_{1,1} & 2a_{0,2} & 0 \\ 0 & 0 & 1 & 0 & -1 & b_{2,0} & a_{2,0}+b_{1,1} & a_{1,1}+b_{0,2} & a_{0,2} \\ 0 & 0 & 0 & 2 & 0 & 0 & 2b_{2,0} & 2b_{1,1} & 2b_{0,2} \\ 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \end{pmatrix},$$

its spectrum is

$$\mathrm{sp}\,(\mathbf{C}_3) = \{0,\ \pm i[2],\ \pm 2i,\ \pm 3i\},$$

and the corresponding eigenvectors are the following.

$$\mathbf{v}(\mathbf{C}_3,\, 0) = (-b_{0,2} - b_{2,0},\ a_{2,0} + a_{0,2},\ 1,\ 0,\ 1,\ 0,\ 0,\ 0,\ 0)^{\mathrm{T}},$$

$$\mathbf{v}(\mathbf{C}_3,\, i) = (i,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0)^{\mathrm{T}},$$

$$\mathbf{v}(\mathbf{C}_3,\, 2i) = \frac{1}{3}\begin{pmatrix} -b_{2,0} + 2a_{1,1} + b_{0,2} \\ a_{2,0} + 2b_{1,1} - a_{0,2} \\ -3 \\ 0 \\ 3 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + i\frac{1}{3}\begin{pmatrix} 2a_{2,0} + b_{1,1} - 2a_{0,2} \\ 2b_{2,0} - a_{1,1} - 2b_{0,2} \\ 0 \\ 3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\mathbf{v}(\mathbf{C}_3,\, 3i) = \ldots$$

$$
\mathbf{v}(\mathbf{C}_3,\, i,\, 2) = \frac{2}{d}\left(\left(\begin{matrix} \frac{1}{2} \\ 10a_{0,2}b_{0,2} + 14a_{2,0}b_{0,2} + a_{1,1}b_{1,1} - 4a_{0,2}b_{2,0} + 4a_{2,0}b_{2,0} - 3b_{1,1}b_{0,2} + 9b_{0,3} - 3b_{1,1}b_{2,0} + 3b_{2,1} \\ 4a_{1,1} + 14b_{0,2} + 10b_{2,0} \\ -5a_{0,2} - 7a_{2,0} + b_{1,1} \\ 2a_{1,1} + 4b_{0,2} - 4b_{2,0} \\ -9 \\ 0 \\ -3 \\ 0 \end{matrix}\right) + \right.
$$

$$
\left. +i\left(\begin{matrix} 0 \\ -2a_{1,1}b_{0,2} + 5a_{0,2}b_{1,1} + 7a_{2,0}b_{1,1} - 4a_{1,1}b_{2,0} - 4b_{0,2}^2 - 10b_{2,0}b_{0,2} - b_{1,1}^2 - 10b_{2,0}^2 + 3b_{1,2} + 9b_{3,0} \\ -4a_{0,2} + 4a_{2,0}32b_{1,1} \\ a_{1,1} - 7b_{0,2} - 5b_{2,0} \\ 10a_{0,2} + 14a_{2,0} + 4b_{1,1} \\ 0 \\ 3 \\ 0 \\ 9 \end{matrix}\right)\right),
$$

where

$$
d = \frac{12}{\pi}(M_1 + iL_1),
$$

$$
\begin{aligned}
M_1 &= \frac{\pi}{12}(-a_{0,2}b_{1,1} - 5a_{1,1}b_{0,2} - 5a_{2,0}b_{1,1} - a_{1,1}b_{2,0} + 10a_{0,2}^2 + 10a_{2,0}a_{0,2} + a_{1,1}^2 + 4a_{2,0}^2 + 9a_{0,3} + 3a_{2,1} + 4b_{0,2}^2 + \\
&\quad + b_{1,1}^2 + 10b_{2,0}^2 - 3b_{1,2} + 10b_{0,2}b_{2,0} - 9b_{3,0}) \\
L_1 &= \frac{\pi}{4}\left(2a_{0,2}b_{0,2} - 2a_{2,0}b_{2,0} + a_{0,2}a_{1,1} + a_{2,0}a_{1,1} + a_{1,2} + 3a_{3,0} + 3b_{0,3} - b_{0,2}b_{1,1} - b_{1,1}b_{2,0} + b_{2,1}\right).
\end{aligned}
$$

EXAMPLE: CONSTRUCTING A SOLUTION

The subsolutions if $d \neq 0$

$$
\begin{aligned}
&\mathbf{s}_{1,1}(t) = \mathbf{v}\left(\mathbf{C}_3,\, 0\right),\\
&\mathbf{s}_{2,1}(t) = e^{it}\mathbf{v}\left(\mathbf{C}_3,\, i\right),\ \mathbf{s}_{2,2}(t) = e^{it}\left(t\mathbf{v}\left(\mathbf{C}_3,\, i\right) + \mathbf{v}\left(\mathbf{C}_3,\, i,\, 2\right)\right),\\
&\mathbf{s}_{3,1}(t) = e^{-it}\overline{\mathbf{v}}\left(\mathbf{C}_3,\, i\right),\ \mathbf{s}_{3,2}(t) = e^{-it}\left(t\overline{\mathbf{v}}\left(\mathbf{C}_3,\, i\right) + \overline{\mathbf{v}}\left(\mathbf{C}_3,\, i,\, 2\right)\right),\\
&\mathbf{s}_{4,1}(t) = e^{2it}\mathbf{v}\left(\mathbf{C}_3,\, 2i\right),\\
&\mathbf{s}_{5,1}(t) = e^{-2it}\overline{\mathbf{v}}\left(\mathbf{C}_3,\, 2i\right),\\
&\mathbf{s}_{6,1}(t) = e^{3it}\mathbf{v}\left(\mathbf{C}_3,\, 3i\right),\\
&\mathbf{s}_{7,1}(t) = e^{-3it}\overline{\mathbf{v}}\left(\mathbf{C}_3,\, 3i\right).
\end{aligned}
$$

The solution $\mathbf{y}(t)$ of system

$$\mathbf{y}(t) = \sum_{j=1}^{7}\sum_{k=1}^{\alpha_{\lambda_j}} c_{j,k}(h)s_{j,k}(t),$$

where

$$\begin{pmatrix} c_{1,1}\\ c_{2,1}\\ c_{2,2}\\ c_{3,1}\\ c_{3,2}\\ c_{4,1}\\ c_{5,1}\\ c_{6,1}\\ c_{7,1} \end{pmatrix} = \mathbf{W}^{-1}\mathbf{y}_0$$

SOLUTION OF THE EXAMPLE

$$
\mathbf{y}(t) = \begin{pmatrix}
\frac{\pi}{12}h^3\left(-a_{0,2}b_{1,1} - 5a_{1,1}b_{0,2} - 5a_{2,0}b_{1,1} - a_{1,1}b_{2,0} + 10a_{0,2}^2 + 10a_{2,0}a_{0,2} + a_{1,1}^2 + 4a_{2,0}^2 + 9a_{0,3} + 3a_{2,1} + 4b_{0,2}^2 + b_{1,1}^2 + 10b_{2,0}^2 - 3b_{1,2} + 10b_{0,2}b_{2,0} - 9b_{3,0}\right) \\
h + \frac{\pi}{4}h^3\left(2a_{0,2}b_{0,2} - 2a_{2,0}b_{2,0} + a_{0,2}a_{1,1} + a_{2,0}a_{1,1} + a_{1,2} + 3a_{3,0} + 3b_{0,3} - b_{0,2}b_{1,1} - b_{1,1}b_{2,0} + b_{2,1}\right) \\
0 \\
0 \\
h^2 \\
0 \\
0 \\
0 \\
h^3
\end{pmatrix}
$$

$$
L_1 = \frac{\pi}{4}(-2a_{20}b_{20} + a_{11}a_{20} + a_{11}a_{02} - b_{11}b_{20} - b_{11}b_{02} + \\
+ 2b_{02}a_{02} + 3a_{30} + a_{12} + b_{21} + 3b_{03})
$$

LINEAR ALGEBRAIC PROP. OF CARLEMAN MATRICES

Lemma: Suppose $\mathbf{C}_{2m+1}$ is the $(2m+1)$th order Carleman matrix and $\lambda_j \in \mathbb{C} \setminus \{0\}$ are the pure imaginary eigenvalues of that. Then generalized eigenvectors of the eigenvalues may exists in the form

$$\mathbf{v}(\mathbf{C}_{2m+1},\, \lambda_j,\, k) = \frac{1}{M_{k-1} + iL_{k-1}}(.),\; k \leq m$$

Lemma: Suppose $\mathbf{C}_{2m+1}$ is the $(2m+1)$th order Carleman matrix then generalized eigenvectors of the eigenvalue $\lambda = 0$

$$\mathbf{v}(\mathbf{C}_{2m+1},\, 0,\, k) = \frac{1}{L_{k-1}}(.),\; k \leq m.$$

LINEAR ALGEBRAIC PROP. OF CARLEMAN MATRICES

The row echelon form of a Carleman matrix

$$\mathbf{C}_{2k} = \left(\begin{array}{ccccc} \circ & \square & \square & \dots & \square \\ 0 & \circ & \square & \dots & \square \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & c\frac{L_i}{\pi} \\ \hdashline \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \end{array} \right),$$

where $L_i \ (i \leq k-1)$ is the i^{th} non-vanishing Poincaré-Lyapunov constant.

Lemma:

$$\boxed{\gamma_0(\mathbf{C}_{2m}) = \gamma_0(\mathbf{C}_{2m+1}) = \underbrace{\#\{i|\, L_i = 0, i = 1, \dots, m-1\}}_{\text{Number of zero Poincaré-Lyapunov constants}} + 1.}$$

The research question is answered: we found the connection between the Poincaré-Lyapunov constants and the linear algebraic properties of the Carleman matrices.

Thank you for your attention!