Extended finite element methods: a brief introduction

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Outline of the talk

- Preliminaries
  - The classical FEM
  - Why and how to extend it?
- Extended (or generalized) FEMs:
  - XFEM ("extended finite element method")
  - VEM ("virtual element method")
  - CutFEM
  - TraceFEM

The classical finite element method (FEM)

Model problem: linear elliptic BVP in weak form. Find $u \in H$:

$$a(u, v) = \ell v \qquad (\forall v \in H).$$

FEM: for a given finite element subspace $V_h \subset H$, find $u_h \in V_h$:

$$a(u_h, v_h) = \ell v_h \qquad (\forall v_h \in V_h).$$

We seek $u_h = \sum_{j=1}^{n} c_j \varphi_j$ where $\varphi_1, \ldots, \varphi_n$ is a basis in V_h.

Typical properties of V_h and $\varphi_1, \ldots, \varphi_n$:

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The classical FEM

Typical properties:

- The degrees of freedom (e.g. nodal values) come from a conforming (fitted) mesh:
$$\Omega = \bigcup_{s=1}^{M} T_s$$
(or $\Omega \approx \Omega_h = \bigcup_{s=1}^{M} T_s$)
- $T_1, \ldots, T_M$ are triangles/tetrahedra or rectangles/bricks
- $u_h \in C(\overline{\Omega})$ such that all $\varphi_{i\,|T_s}$ are polynomials

The classical FEM

Convergence:

$$|u - u_h|_1 \leq ch^k |u|_{k+1} \qquad \text{i.e. } O(h^k)$$

where $|u|_k := |u|_{H^k} := \|D^k u\|_{L^2(\Omega)}$.

Conditions: $u \in H^{k+1}(\Omega)$, polynomials P^k, regular mesh.

The simplest case ($k = 1$): $|u - u_h|_1 \leq ch|u|_2$.

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- Considered in this talk: extensions motivated by special difficulties to overcome in the physical/enginering problems
- Not considered in this talk: extensions to simplify implementation, such as
  - "partition of unity" (PUFEM) $\rightarrow$ meshfree methods
  - discontinuous Galerkin methods (DG)

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The "extended finite element method" (XFEM)

Motivation: problematic parts for the FEM solution, e.g.

1. discontinuities (e.g. at fractures, cracks)
2. singularities (e.g. at corners)
3. boundary layers (e.g. convection equations)

Traditional ways to handle these:

local refinement of the mesh,

stabilization (modified bilinear form), ...

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Basic idea of the XFEM:

enrichment of the basis,

i.e. including additional (non-polynomial) basis functions, adjusted to the problem.

(XFEM sometimes called: "enriched FEM")

$$\rightarrow \quad u_h = \sum_{i=1}^{n} c_i \varphi_i + \underbrace{\sum_{i=1}^{n_0} d_i \psi_i}_{\substack{\text{a few terms,} \\ \text{supported in the region of interest}}}$$

$\rightarrow$ no need to refine the mesh locally

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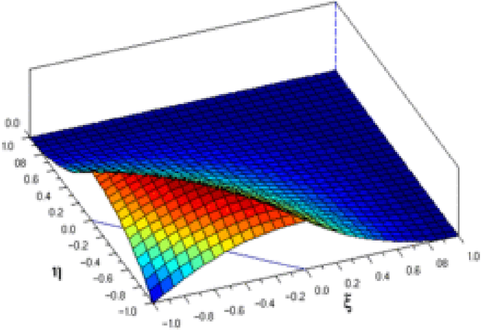
The "extended finite element method" (XFEM)

Some examples of such new shape functions:

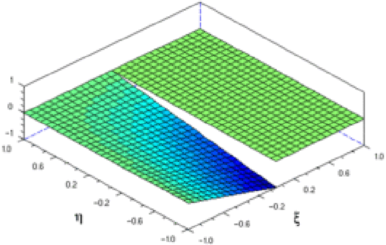
The "extended finite element method" (XFEM)

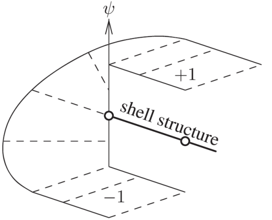


A "kink shape function"

[Inst. Comput. Mech., TU Munich]

The "extended finite element method" (XFEM)

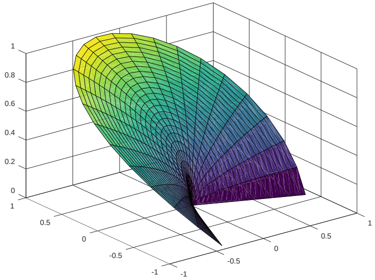




"Jump shape functions"

[Inst. Comput. Mech., TU Munich] [A. Legay, IJNME (2015)]

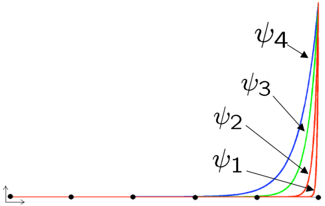
The "extended finite element method" (XFEM)



A "corner function": $r^\beta \sin(\beta\theta)$ for some $0 < \beta < 1$ [Cai, SINUM (2001)]

Around cracks: $\sqrt{r}\sin(\theta/2)$, $\sqrt{r}\sin(\theta/2)\sin\theta$ etc. [Loehnert et al (2014)]

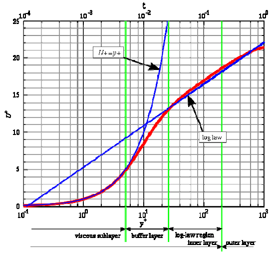
The "extended finite element method" (XFEM)



Enrichment functions in a boundary layer [T-P. Fries et al., WCCM (2008)]

in an 1D model, $\psi_i(x) \approx \frac{e^{n_i x}-1}{e^{n_i}-1}$

The "extended finite element method" (XFEM)



A "wall function" (turbulent flow near a wall/boundary) [Tominaga (2000)]

Exponential formula [Krank et al., Comp Fluids (2018)]

The "extended finite element method" (XFEM)

Typical new basis functions: from the standard ones,
$\psi_i := \phi_i \cdot u_0$

Advantages of the XFEM:

1. no need to refine in the problematic subdomain
2. represents well the solution (but behaviour must be known)

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Convergence:

1. Original idea (already in [Strang-Fix]):
   if $u = u^{reg} + u^{sing}$, we may only approximate u^{reg} .
   (If even $u_h = u_h^{reg} + u^{sing}$, then $u - u_h = u^{reg} - u_h^{reg}$!)
2. A typical theorem: for linear elements for the Poisson or elasticity problem, using Heaviside and corner functions,

$$|u - u_h|_1 \leq ch\,|u^{reg}|_2 \,.$$

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Some important papers:

Chessa, Jack; Smolinski, Patrick; Belytschko, Ted, The extended finite element method (XFEM) for solidification problems, Internat. J. Numer. Methods Engrg. 53 (2002), no. 8, 1959-1977.

Chahine, Elie; Laborde, Patrick; Renard, Yves A quasi-optimal convergence result for fracture mechanics with XFEM. C. R. Math. Acad. Sci. Paris 342 (2006), no. 7, 527-532.

Fries, Thomas-Peter; Belytschko, Ted, The extended/generalized finite element method: an overview of the method and its applications, Internat. J. Numer. Methods Engrg. 84 (2010), no. 3, 253-304.

Nicaise, Serge; Renard, Yves; Chahine, Elie, Optimal convergence analysis for the extended finite element method. Internat. J. Numer. Methods Engrg. 86 (2011), no. 4-5, 528-548.

The virtual element method (VEM)

Basic idea: use

polygonal elements

instead of only triangular/rectangular ones in 2D
(and similarly, use polyhedral elements in 3D)

1. Various versions and names:
   Voronoi cell FEM, Polygonal FEM,...
   Related FDM: the Mimetic FDM
2. A general framework: VEM [B. da Veiga et al. (2013)]

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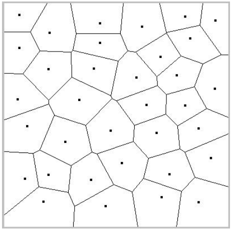
The virtual element method (VEM)

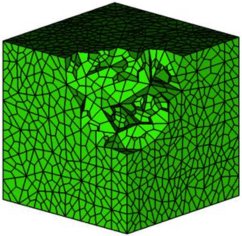
Motivation for allowing polygonal/polyhedral elements:

1. useful flexibility for generating meshes,
   e.g. Voronoi cells for heterogeneous materials
2. no problems with hanging nodes

The virtual element method (VEM)



A polygonal mesh
with Voronoi cells



A polyhedral mesh
[UC Davis]

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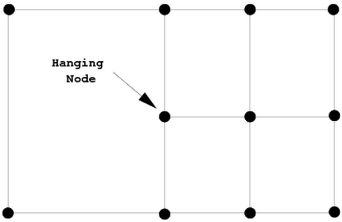

No problem with a hanging node: quadrangle $\rightarrow$ pentagon

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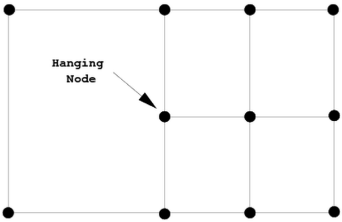

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The virtual element method (VEM)

Construction: implicitly. E.g., for the Poisson equation:

In an element T:

1. for $k = 1$: $v_{|e}$ is linear ($\forall$ edge e), $\Delta v = 0$ in T
2. for $k \geq 2$: $v_{|e} \in \mathbb{P}^k$ ($\forall$ edge e), $\Delta v \in \mathbb{P}^{k-2}$ in T

(in 3D: also on the faces)

"Virtual" element = we don't use v in T explicitly

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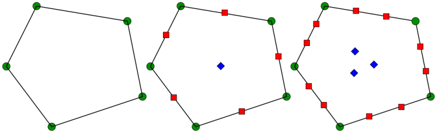

$k=1$ $k=2$ $k=3$

[da Veiga, 2014]

The virtual element method (VEM)

Construction: how to form the stiffness matrix?

1. For polynomials $p \in \mathbb{P}^{k-2}$:

$$\int_T \nabla p \cdot \nabla v = - \underbrace{\int_T \Delta p \; v}_{\text{known from moments}} + \underbrace{\int_{\partial T} \partial_\nu p \; v}_{\text{known from edge DOFs}}$$

2. For any virtual functions u, v: additional terms involving $R_h u := u - \Pi_h u$, where $\Pi_h u \in \mathbb{P}^k$ is a projection.

   E.g. for $k = 1$: letting $\varphi_i = p_i + r_i$ (where $p_i := \Pi_h \varphi_i$),

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The CutFEM

Motivation:

The standard FEM adjusts the mesh to the domain, i.e. uses boundary-fitted meshes.

This may be complicated in many situations:

1. complex geometry
2. evolving geometry: moving domains,
   maybe even with topological changes
3. several BVPs
   (e.g. looking for an optimally located object)

The CutFEM

Main idea of CutFEM:

1. boundary-unfitted mesh: create a mesh for a larger (usually fixed and simpler) domain Ω^* containing Ω
2. "cut" shape functions: define shape functions first on Ω^* and then restrict them to Ω

On figures:

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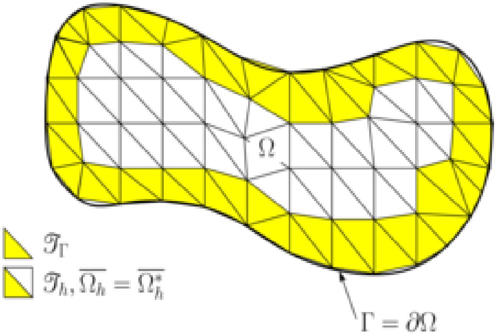
On figures:

The CutFEM

A standard FEM mesh (boundary-fitted): [Ins. Comp. Mech., TU Munich]



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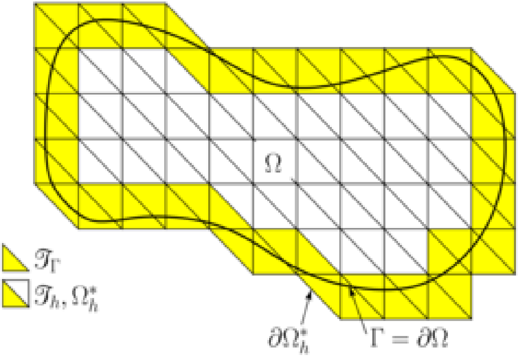
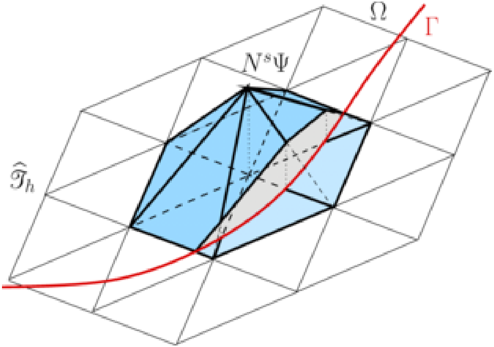
A CutFEM mesh (boundary-unfitted): [Ins. Comp. Mech., TU Munich]



The CutFEM

A CutFEM shape function:



[Inst. Comput. Mech., TU Munich]

The CutFEM

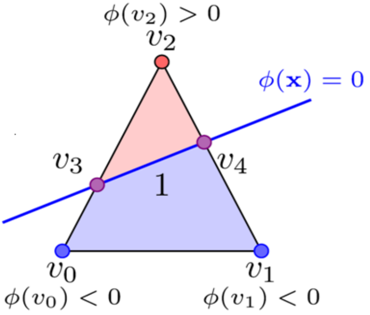
Construction. How to work with "cut" functions?

Some typical issues:

1. level set method to describe the geometry
2. weak Dirichlet b.c. (Nitsche's approach)
3. stabilization: ghost penalty

The CutFEM

1. The level set method to describe the geometry



$$
\begin{aligned}
x \in \Omega \quad &\Leftrightarrow \quad \phi(x) < 0 \\
x \in \partial\Omega \quad &\Leftrightarrow \quad \phi(x) = 0 \\
x \notin \overline{\Omega} \quad &\Leftrightarrow \quad \phi(x) > 0
\end{aligned}
$$

[Burman et al., IJNME (2014)]

$\rightarrow$ computer geometry (CAD)

The CutFEM

2. Weak Dirichlet boundary conditions (Nitsche's approach)

Problem: how to enforce Dirichlet b.c. with "cut" functions?

Example: consider a Poisson equation

$$\begin{cases} -\Delta u = f \\ u_{|\partial\Omega} = g. \end{cases}$$

The CutFEM

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The CutFEM

Consistent weak form for $v \in H^1(\Omega^*)$:

$$\int_\Omega \nabla u \cdot \nabla v - \int_{\partial\Omega} \partial_\nu u \, v \qquad\qquad = \int_\Omega fv$$

Using $u_{|\partial\Omega} = g$:

The CutFEM

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Consistent weak form for $v \in H^1(\Omega^*)$:

$$\int_\Omega \nabla u \cdot \nabla v - \int_{\partial\Omega} \partial_\nu u \, v - \int_{\partial\Omega} u \, \partial_\nu v + \int_{\partial\Omega} \frac{\gamma}{h} u v = \int_\Omega f v - \int_{\partial\Omega} g \, \partial_\nu v + \int_{\partial\Omega} \frac{\gamma}{h} g v$$

Using $u_{|\partial\Omega} = g$:

(where $\gamma, h > 0$ are parameters)

The CutFEM

Consistent weak form for $v \in H^1(\Omega^*)$:

$$\int_\Omega \underbrace{\nabla u \cdot \nabla v - \int_{\partial\Omega} \partial_\nu u \, v - \int_{\partial\Omega} {\color{green} u} \, \partial_\nu v + \int_{\partial\Omega} \frac{\gamma}{h} {\color{red} u} v}_{a(u,v)} = \underbrace{\int_\Omega f v - \int_{\partial\Omega} {\color{green} g} \, \partial_\nu v + \int_{\partial\Omega} \frac{\gamma}{h} {\color{red} g} \, v}_{\ell v}$$

FEM problem: $\quad a(u_h, v_h) = \ell v_h \quad (\forall v_h \in V_h).$

Role of the two new terms for $a(u_h, v_h)$:

symmetry

stability (coercivity)

The CutFEM

Consistent weak form for $v \in H^1(\Omega^*)$:

$$\underbrace{\int_\Omega \nabla u \cdot \nabla v - \int_{\partial\Omega} \partial_\nu u\, v - \int_{\partial\Omega} u\, \partial_\nu v + \int_{\partial\Omega} \frac{\gamma}{h} u v}_{a(u,v)} = \underbrace{\int_\Omega fv - \int_{\partial\Omega} g\, \partial_\nu v + \int_{\partial\Omega} \frac{\gamma}{h} g\, v}_{\ell v}$$

FEM problem: $\quad a(u_h, v_h) = \ell v_h \quad (\forall v_h \in V_h).$

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stability (coercivity)

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Proof of coercivity: [P. Hansbo, GAMM-Mitt. (2005)]:

$$a(u_h, u_h) = \int_\Omega |\nabla u_h|^2 - 2\int_{\partial\Omega} \partial_\nu u_h\, u_h + \int_{\partial\Omega} \frac{\gamma}{h} u_h^2$$

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Remarks:

- Similar to a penalty method in DG.
- The inverse inequality on a cell K for linear FEM:

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$$\Updownarrow$$

$$h_e^2 \leq C_I|K|, \quad \text{i.e.} \quad \frac{h_e^2}{|K|} \leq C_I .$$

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Remarks:

- The constant C_I in the inverse inequality:
  $\exists$ computable bounds, but C_I depends on the shape regularity of the "cut" elements $K \cap \Omega$.

  Hard to ensure in advance $\rightarrow$ "small cell problem".
- Also suitable for interface problems
  when the jump $[[u]]_{|\Gamma} := u^+_{|\Gamma} - u^-_{|\Gamma} = g$ on some $\Gamma \subset \Omega$

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3. The "small cell problem":
the "cut" elements $K \cap \Omega$ may be not regular

Consequences:

(i) C_I becomes large, non-uniform

Further stabilization: ghost penalty (GP), i.e. we add

$$j(u_h, v_h) := \gamma \sum_{e \in \mathcal{E}_T} \int_e h \, [[\partial_\nu u_h]] \, [[\partial_\nu v_h]]$$

where $\mathcal{E}_T$:= edges adjacent to $\partial\Omega$ ("ghost edges").

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(ii) Ill-conditioned linear systems

$\rightarrow$ cell agglomeration [Kummer et al., IJNME, 2018]

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Convergence for linear elements: as for the standard FEM,

$$|u - u_h|_1 \leq ch\,|u|_2 \qquad \text{if } u \in H^2(\Omega)$$

[Burman–Hansbo (2012)].

For higher order elements: $\sim O(h^k)$ holds,
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The CutFEM

Implementation:

1. Multigrid works: with additional smoothing around Γ
   $\Rightarrow$ convergence independent of Γ [Gross et al. (2017)])
2. Advanced software: open source library libCutFEM (described in [Burman et al., IJNME (2014)])

The CutFEM

Important papers:

Hansbo, P., Nitsche's method for interface problems in computational mechanics, GAMM-Mitt. 28 (2005), no. 2, 183-206.

Burman, E.; Hansbo, P., Fictitious domain finite element methods using cut elements: II. A stabilized Nitsche method, Appl. Numer. Math. 62 (2012), no. 4, 328-341.

Burman, E.; Claus, S.; Hansbo, P.; Larson, M. G.; Massing, A., CutFEM: discretizing geometry and partial differential equations, Int. J. Numer. Methods Engrg. 104 (2015), no. 7, 472-501.

Massing, A.; Schott, B.; Wall, W. A., A stabilized Nitsche cut finite element method for the Oseen problem, Comput. Methods Appl. Mech. Engrg. 328 (2018), 262300.

The TraceFEM

TraceFEM = special CutFEM: for PDEs posed on a surface Γ

Main ideas similar to those of CutFEM:

1. surface-unfitted mesh: create a mesh for a "bulk" domain Ω^* containing Γ
2. "cut=trace" shape functions: define shape functions first on Ω^* and then restrict them to Γ

Typical motivation: when Γ is moving $\rightarrow$ the mesh is the same

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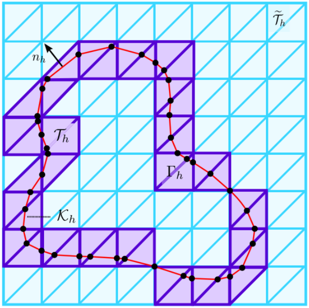

[Burman et al., CMAME (2016)]

The TraceFEM

Typical PDEs:

1. Elliptic model PDE on a closed surface Γ:

$$-\Delta_\Gamma u = f \qquad \text{on } \Gamma$$

   where Δ_Γ is the Laplace-Beltrami operator;

2. coupled "bulk-surface" equations:

   a PDE on Ω $\quad + \quad$ a PDE on Γ

   e.g. cell + membrane $\;(\Gamma := \partial\Omega)$;

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Construction. Level sets: $x \in \Gamma \;\Leftrightarrow\; \phi(x) = 0$

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where $c > 0$ is independent of Γ and how it cuts the mesh [Olshanskii–Reusken (2017)].

Higher order: also depending on the approximation of the surface and the stabilization term

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Burman, E.; Hansbo, P.; Larson, M. G.; Massing, A.; Zahedi, S., Full gradient stabilized cut finite element methods for surface partial differential equations, Comput. Methods Appl. Mech. Engrg. 310 (2016), 278-296

Olshanskii, M. A.; Reusken, A., Trace finite element methods for PDEs on surfaces, Lect. Notes Comput. Sci. Eng., 121, Springer, 2017.

Massing, A., A cut discontinuous Galerkin method for coupled bulk-surface problems, Lect. Notes Comput. Sci. Eng., 121, Springer, 2017.

Grande, J.; Lehrenfeld, Ch.; Reusken, A., Analysis of a high-order trace finite element method for PDEs on level set surfaces, SIAM J. Numer. Anal. 56 (2018), no. 1, 228-255.

The TraceFEM

Thank you for your attention!