Introduction to the semigroup approach for stochastic partial differential equations and their finite element approximation

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Outline

- Preliminaries
- Gaussian measures in Hilbert spaces
- Q-Wiener process
- Wiener-integral
- Semigroup framework for linear SPDEs with additive noise
- Weak solution
- Semigroup framework for semilinear SPDEs with additive noise
- Mild solution
- Example: the stochastic wave equation

Operators on Hilbert spaces: compact operators

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Theorem (Spectral Theorem for compact self-adjoint operators). If $T : U \to U$ is a compact self-adjoint operator on an infinite dimensional Hilbert space U, then there is an orthonormal basis of U consisting of eigenvectors $\{e_k\}$ of T with corresponding real eigenvalues $\{\lambda_k\}$. Furthermore,

$$Tu = \sum_{k=1}^{\infty} \lambda_k \langle u, e_k \rangle e_k, \;\; u \in U.$$

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Corollary (Functional calculus). Let f be a bounded function on $\{\lambda_k\}$. Then, the mapping $\Theta : f \to f(T) \in L(U)$ where

$$f(T)u := \sum_{k=1}^{\infty} f(\lambda_k) \langle u, e_k \rangle e_k, \ u \in U,$$

is an isometric algebra homomorphism. That is, Θ is linear, it maps products to products, $f(\lambda) = \lambda \mapsto T$, $f(\lambda) = 1 \mapsto Id$ and $\|f(T)\| = \sup_{k\in\mathbb{N}} |f(\lambda_k)|$.

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Proof.

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Proof. Linearity is straightforward. If f, g are bounded on $\{\lambda_k\}$, then

$$f(T)g(T)u = f(T)\sum_{l=1}^{\infty} g(\lambda_l)\langle u, e_l\rangle e_l = \sum_{l=1}^{\infty} g(\lambda_l)\langle u, e_l\rangle f(T)e_l$$

$$= \sum_{l=1}^{\infty} g(\lambda_l)\langle u, e_l\rangle \sum_{k=1}^{\infty} f(\lambda_k)\langle e_l, e_k\rangle e_k = \sum_{l=1}^{\infty} g(\lambda_l)f(\lambda_l)\langle u, e_l\rangle e_l = (fg)(T)$$

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Finally, $f(T)e_k = f(\lambda_k)e_k$ and hence $\|f(T)\| \geq \sup_{k\in\mathbb{N}} |f(\lambda_k)|$. On the other hand, by Parseval's identity,

$$\|f(T)u\|^2 = \sum_{k=1}^{\infty} f(\lambda_k)^2\langle u, e_k\rangle^2 \leq \sup_{k\in\mathbb{N}} f(\lambda_k)^2 \sum_{k=1}^{\infty}\langle u, e_k\rangle^2 = \sup_{k\in\mathbb{N}} f(\lambda_k)^2\|u\|^2$$

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- If $\langle Tu, u\rangle \geq 0$ for all $u \in U$, the we write $T \geq 0$ and say that T is **positive semidefinite**. If T is compact, self-adjoint and $T \geq 0$, then the eigenvalues of T are nonnegative. Then, we may define the **square-root** of T by

$$T^{\frac{1}{2}}u = \sum_{k=1}^{\infty} \lambda_k^{\frac{1}{2}}\langle u, e_k\rangle e_k, \;\; u \in U$$

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- $T \in \mathcal{L}_p(U, H)$ if $\sum_j s_j(T)^p < \infty$, $0 < p < \infty$.
- For $p = 1$ and 2 the Schatten p-classes coincide with the nuclear and Hilbert Schmidt operators, respectively.

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- As $v' : U \to \mathbb{R}$ is continuous it is also measurable if we equip U and $\mathbb{R}$ with their respective Borel σ-algebra
- Hence, given a probability measure μ on $(U, \mathcal{B}(U))$, the functional v' can be viewed as a real-valued random variable on the probability space $(U, \mathcal{B}(U), \mu)$.

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Definition. A probability measure μ on $(U, \mathcal{B}(U))$ is **Gaussian** if for all $v \in U$, the random variable $v' : (U, \mathcal{B}(U)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ has a Gaussian law. That is, for all $v \in U$, there are $m_v \in \mathbb{R}$ and $\sigma_v \in \mathbb{R}_+$, such that, if $\sigma_v > 0$,

$$\mu_{v'}(A) = \mu(v'^{-1}(A)) = \mu\left(\{u \in U : v'(u) \in A\}\right) = \frac{1}{\sqrt{2\pi}\sigma_v} \int_A \mathrm{e}^{-\frac{(s-m_v)^2}{2\sigma_v^2}} \, \mathrm{d}s,$$

for all $A \in \mathcal{B}(\mathbb{R})$.

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Question: Given a real separable Hilbert space U are there any Gaussian measures at all on $(U, \mathcal{B}(U))$?

Characterization of Gaussian measures

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Theorem. A finite measure μ on $(U, \mathcal{B}(U))$ is Gaussian if and only if

$$\hat{\mu}(v) := \int_U \mathrm{e}^{\mathrm{i}\langle v,u\rangle_U} \, \mathrm{d}\mu(u) = \mathrm{e}^{\mathrm{i}\langle m,v\rangle_U - \frac{1}{2}\langle Qv,v\rangle_U},$$

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In this case we write $\mu = N(m, Q)$, and m and Q are called the mean and the covariance operator of μ. The measure μ is uniquely determined by m and Q.

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Corollary. Let μ be a Gaussian measure on U with mean m and covariance operator Q. Then, for all $u, v \in U$,

$$\int_U \langle u, v\rangle_U\,\mathrm{d}\mu(u) = \langle m, v\rangle_U,$$

$$\int_U \langle u-m, v\rangle_U \langle u-m, w\rangle_U\,\mathrm{d}\mu(u) = \langle Qv, w\rangle_U,$$

$$\int_U \|u-m\|_U^2\,\mathrm{d}\mu(u) = \mathrm{Tr}(Q).$$

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$$= \int_U \mathrm{e}^{\mathrm{i}t\langle v,u\rangle_U}\,\mathrm{d}\mu(u) = \int_U \mathrm{e}^{\mathrm{i}tv'(u)}\,\mathrm{d}\mu(u) = \mathbf{E}(e^{itv'}).$$

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Finally, by monotone convergence and Parseval's identity,

$$\begin{aligned}\mathrm{Tr}(Q) = \sum_{k=1}^{\infty}\langle Qe_k, e_k\rangle_U &= \sum_{k=1}^{\infty}\int_U \langle u-m, e_k\rangle_U^2\,\mathrm{d}\mu(u)\\ &= \int_U \sum_{k=1}^{\infty}\langle u-m, e_k\rangle_U^2\,\mathrm{d}\mu(u) = \int_U \|u-m\|_U^2\,\mathrm{d}\mu(u).\end{aligned}$$

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Definition. A U-valued random variable X on a probability space $(\Omega, \mathcal{F}, P)$; that is, a measurable mapping $X : (\Omega, \mathcal{F}, P) \to (U, \mathcal{B}(U))$, is **Gaussian** if the law $\mu_X = P \circ X^{-1}$ of X is a Gaussian measure on $(U, \mathcal{B}(U))$. That is, $P \circ X^{-1} \in N(m, Q)$ for some $m \in U$ and $Q \in L(U)$. We call m the mean and Q the covariance operator of X.

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Proof. Change of variables.

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Theorem. Let $m \in U$ and $Q \in L(U)$, $Q \geq 0$, with $\mathrm{Tr}(Q) < \infty$. A U-valued random variable X on $(\Omega, \mathcal{F}, P)$ is Gaussian with $P \circ X^{-1} = N(m, Q)$ if and only if

$$X = m + \sum_{k=1}^{\infty} \sqrt{\lambda_k} \beta_k e_k,$$

where (λ_k, e_k) are the eigenpairs of Q and β_k are independent real random variables with $P \circ \beta_k^{-1} = N(0,1)$ if $\lambda_k > 0$ and $\beta_k = 0$ otherwise. The series converges in $L_2(\Omega, \mathcal{F}, P; U)$.

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Corollary (Existence of Gaussian measures.) For each $m \in U$ and $Q \in L(U)$, $Q \geq 0$, with $\mathrm{Tr}(Q) < \infty$, there exists $\mu = N(m, Q)$.

Remark. This all relies on the nontrivial fact that there exist a probability space $(\Omega, \mathcal{F}, P)$ with infinitely many independent Gaussian random variables (Kolmogorov's Extension Theorem).

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Question: For a given Q are there any Q-Wiener processes?

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Remark. Again, we need the (nontrivial) existence of a probability space $(\Omega, \mathcal{F}, P)$ with countably many independent Brownian motions!

Cylindrical processes

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Often one would like to consider a Wiener process with more general covariance operator Q, such as $Q = I$. If $\mathrm{Tr}(Q) = \infty$, then the sum

$$W(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\beta_k(t)e_k$$

does not even converge in $L_2(\Omega, \mathcal{F}, P; U)$, since

$$\mathbf{E}\Big\| \sum_{j=1}^{\infty} \sqrt{\lambda_j}\beta_j(t)e_j \Big\|_U^2 = \sum_{j=1}^{\infty} \lambda_j \mathbf{E}\big(\beta_j(t)^2\big) = t\sum_{j=1}^{\infty} \lambda_j = t\,\mathrm{Tr}(Q) = \infty.$$

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Remark.

- The sum converges in a suitable larger Hilbert space where one obtains a nuclear Wiener process. However, this is not unique.
- The real processes $W_x(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\beta_k(t)\langle e_k, x\rangle_U$ are well-defined real valued Brownian motions with covariance $\mathbf{E}(W_x(t)^2) = t\|Q^{\frac{1}{2}}x\|_U^2$.

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- Let $\{F(t)\}_{t\in[0,T]}$ be a family of linear operators $F(t) : U \to H$ such that $t \mapsto \|F(t)e_k\|_H$ is $L_2[0, T]$ for each $k = 1, 2, \ldots$.

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- Define, first formally,
$$\int_0^t F(s)\,\mathrm{d}W(s) := \sum_{k=1}^{\infty} \sqrt{\lambda_k} \int_0^t F(s)e_k\,\mathrm{d}\beta_k(s), \;\; t \in [0, T]. \tag{2}$$

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- Each term in the expansion in (2) is defined in terms of real-valued Wiener integrals as
$$\int_0^t F(s)e_k\,\mathrm{d}\beta_k(s) := \sum_{j=1}^{\infty} \int_0^t \langle F(s)e_k, \phi_j\rangle\,\mathrm{d}\beta_k(s)\,\phi_j,$$
where $\{\phi_j\}_{j=1}^{\infty}$ is an orthonormal basis for H.

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Parseval's identity and monotone convergence, we have for fixed $t > 0$,

$$\begin{aligned}
\mathbf{E}\Big(\Big\|\int_0^t F(s)e_k\,\mathrm{d}\beta_k(s)\Big\|_H^2\Big) &= \mathbf{E}\Big(\sum_{j=1}^{\infty}\Big|\int_0^t \langle F(s)e_k, \phi_j\rangle\,\mathrm{d}\beta_k(s)\Big|^2\Big) \\
&= \sum_{j=1}^{\infty}\int_0^t |\langle F(s)e_k, \phi_j\rangle|^2\,\mathrm{d}s \\
&= \int_0^t \|F(s)e_k\|_H^2\,\mathrm{d}s < \infty.
\end{aligned}$$

The infinite dimensional Wiener integral and Itô's isometry

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Theorem. Assume that the operator $F(s)Q^{\frac{1}{2}} \in \mathcal{L}_2(U, H)$ for almost all $s \in [0, T]$ and that

$$\int_0^T \|F(s)Q^{\frac{1}{2}}\|_{\mathrm{HS}}^2 \, \mathrm{d}s < \infty.$$

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Then, the following hold.

- The operators $Q_F(t)$ given by

$$Q_F(t)x = \int_0^t F(s)Q^{\frac{1}{2}}(F(s)Q^{\frac{1}{2}})^* x \, \mathrm{d}s, \quad x \in H, \ t \in [0, T],$$

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  are well defined and have finite trace for all $t \in [0,T]$.
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$$\int_0^t F(s)\,\mathrm{d}W(s) \in N(0, Q_F(t)).$$

- The **Itô isometry**

$$\mathbf{E}\Big(\Big\|\int_0^t F(s)\,\mathrm{d}W(s)\Big\|_H^2\Big) = \mathrm{Tr}(Q_F(t)) = \int_0^t \|F(s)Q^{\frac{1}{2}}\|_{\mathrm{HS}}^2 \,\mathrm{d}s \qquad (3)$$

  holds.

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Remark. For fixed s, the operator $F(s)$ does not have to be a bounded operator only the product $F(s)Q^{\frac{1}{2}}$ has to be Hilbert-Schmidt (and hence bounded). If $F(s) \in L(U, H)$ for all s, then

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Proof. That the operator $Q_F(t)$ is well defined follows from the bound

$$\begin{aligned}
&\int_0^t \left\| F(s)Q^{\frac{1}{2}}(F(s)Q^{\frac{1}{2}})^* x \right\|_H \mathrm{d}s \\
&\leq \int_0^t \left\| F(s)Q^{\frac{1}{2}} \right\|_{L(U,H)} \left\| (F(s)Q^{\frac{1}{2}})^* \right\|_{L(H,U)} \|x\|_H \, \mathrm{d}s \\
&= \int_0^t \left\| F(s)Q^{\frac{1}{2}} \right\|_{L(U,H)} \left\| F(s)Q^{\frac{1}{2}} \right\|_{L(U,H)} \|x\|_H \, \mathrm{d}s \\
&= \int_0^t \left\| F(s)Q^{\frac{1}{2}} \right\|_{L(U,H)}^2 \mathrm{d}s \|x\|_H \leq \int_0^t \left\| F(s)Q^{\frac{1}{2}} \right\|_{\mathrm{HS}}^2 \mathrm{d}s \|x\|_H < \infty.
\end{aligned}$$

The infinite dimensional Wiener integral and Itô's isometry

We can calculate the trace of $Q_F(t)$ using monotone convergence as

$$
\begin{aligned}
\mathrm{Tr}(Q_F(t)) &= \sum_{k=1}^{\infty} \langle Q_F(t)\phi_k, \phi_k\rangle_H = \sum_{k=1}^{\infty} \langle \int_0^t F(s)Q^{\frac12}(F(s)Q^{\frac12})^*\phi_k \,\mathrm{d}s, \phi_k\rangle_H \\
&= \sum_{k=1}^{\infty} \int_0^t \langle F(s)Q^{\frac12}(F(s)Q^{\frac12})^*\phi_k, \phi_k\rangle_H \,\mathrm{d}s \\
&= \sum_{k=1}^{\infty} \int_0^t \langle (F(s)Q^{\frac12})^*\phi_k, (F(s)Q^{\frac12})^*\phi_k\rangle_U \,\mathrm{d}s \\
&= \sum_{k=1}^{\infty} \int_0^t \|(F(s)Q^{\frac12})^*\phi_k\|_U^2 \,\mathrm{d}s = \int_0^t \sum_{k=1}^{\infty} \|(F(s)Q^{\frac12})^*\phi_k\|_U^2 \,\mathrm{d}s \\
&= \int_0^t \|(F(s)Q^{\frac12})^*\|_{\mathrm{HS}}^2 \,\mathrm{d}s = \int_0^t \|F(s)Q^{\frac12}\|_{\mathrm{HS}}^2 \,\mathrm{d}s,
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where $\{\phi_j\}_{j=1}^{\infty}$ is an orthonormal basis of H. This is the last equality in (3).

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&= \sum_{k=1}^{\infty} \int_0^t \langle F(s)Q^{\frac{1}{2}}(F(s)Q^{\frac{1}{2}})^*\phi_k, \phi_k\rangle_H \,\mathrm{d}s \\
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&= \sum_{k=1}^{\infty} \int_0^t \|(F(s)Q^{\frac{1}{2}})^*\phi_k\|_U^2 \,\mathrm{d}s = \int_0^t \sum_{k=1}^{\infty} \|(F(s)Q^{\frac{1}{2}})^*\phi_k\|_U^2 \,\mathrm{d}s \\
&= \int_0^t \|(F(s)Q^{\frac{1}{2}})^*\|_{\mathrm{HS}}^2 \,\mathrm{d}s = \int_0^t \|F(s)Q^{\frac{1}{2}}\|_{\mathrm{HS}}^2 \,\mathrm{d}s,
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where $\{\phi_j\}_{j=1}^{\infty}$ is an orthonormal basis of H. This is the last equality in (3).

Next we show that the series in (2) converges in $L_2(\Omega, \mathcal{F}, P; H)$ (omitting the subscript H from norms and scalar products in the calculation):

The infinite dimensional Wiener integral and Itô's isometry

$$
\begin{aligned}
&\mathbf{E}\Big(\Big\|\sum_{k=m}^{n}\lambda_k^{\frac12}\int_0^t F(s)e_k\,\mathrm{d}\beta_k(s)\Big\|^2\Big)\\
&=\mathbf{E}\Big(\sum_{j=1}^{\infty}\Big|\sum_{k=m}^{n}\lambda_k^{\frac12}\int_0^t\langle F(s)e_k,\phi_j\rangle\,\mathrm{d}\beta_k(s)\Big|^2\Big)\\
&=\sum_{j=1}^{\infty}\mathbf{E}\Big(\Big|\sum_{k=m}^{n}\lambda_k^{\frac12}\int_0^t\langle F(s)e_k,\phi_j\rangle\,\mathrm{d}\beta_k(s)\Big|^2\Big)\\
&=\sum_{j=1}^{\infty}\sum_{k,l=m}^{n}\lambda_k^{\frac12}\lambda_l^{\frac12}\mathbf{E}\Big(\int_0^t\langle F(s)e_k,\phi_j\rangle\,\mathrm{d}\beta_k(s)\int_0^t\langle F(s)e_l,\phi_j\rangle\,\mathrm{d}\beta_l(s)\Big)\\
&\{\text{Independence of } \beta_k,\beta_l \text{ and real Itô isometry}\}\\
&=\sum_{j=1}^{\infty}\sum_{k=m}^{n}\lambda_k\int_0^t|\langle F(s)e_k,\phi_j\rangle|^2\,\mathrm{d}s=\sum_{j=1}^{\infty}\sum_{k=m}^{n}\int_0^t|\langle F(s)Q^{\frac12}e_k,\phi_j\rangle|^2\,\mathrm{d}s\\
&=\int_0^t\sum_{k=m}^{n}\sum_{j=1}^{\infty}|\langle F(s)Q^{\frac12}e_k,\phi_j\rangle|^2\,\mathrm{d}s=\int_0^t\sum_{k=m}^{n}\|F(s)Q^{\frac12}e_k\|^2\mathrm{d}s
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As

$$\int_0^t \sum_{k=1}^{\infty} \left\| F(s) Q^{\frac{1}{2}} e_k \right\|^2 \mathrm{d}s = \int_0^t \left\| F(s) Q^{\frac{1}{2}} \right\|_{\mathrm{HS}} \mathrm{d}s < \infty,$$

the series in (2) converges in $L_2(\Omega, \mathcal{F}, P; H)$ to a random variable, which is zero-mean Gaussian, because it is the limit of zero-mean Gaussian random variables (similarly to the real case).

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$$\mathbf{E}\Big(\Big\| \int_0^t F(s)\, \mathrm{d}W(s) \Big\|_H^2 \Big) = \int_0^t \|F(s) Q^{\frac{1}{2}}\|_{\mathrm{HS}}^2 \, \mathrm{d}s.$$

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Finally, a similar calculation shows that

$$\mathbf{E}\Big(\Big\langle \int_0^t F(s)\, \mathrm{d}W(s), x \Big\rangle_H \Big\langle \int_0^t F(s)\, \mathrm{d}W(s), y \Big\rangle_H \Big) = \langle Q_F(t) x, y \rangle_H, \quad x, y \in H,$$

so that the covariance operator of $\int_0^t F(s)\, \mathrm{d}W(s)$ is indeed $Q_F(t)$. $\square$

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Remark. Even when the Fourier expansion (1) of $W(t)$ does not converge in $L_2(\Omega, \mathcal{F}, P; U)$ ($\mathrm{Tr}(Q) = \infty$), the expansion (2) of the stochastic integral still converges in $L_2(\Omega, \mathcal{F}, P; H)$ provided that $\int_0^t \| F(s) Q^{\frac{1}{2}} \|_{\mathrm{HS}}^2 \,\mathrm{d}s < \infty$. Hence the integral may be defined even when $W(t)$ itself does not exist in U.

Semigroup approach to SPDEs: the linear case

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Linear SPDEs with additive noise:

$$\begin{cases} \mathrm{d}X(t) + AX(t)\,\mathrm{d}t = B\,\mathrm{d}W(t), \ t > 0 \\ X(0) = X_0 \end{cases}$$

- H, U Hilbert spaces
- $W(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\beta_k(t)e_k$, Q-Wiener process on U, $Qe_k = \lambda_k e_k$
- Filtration $\mathcal{F}_s := \bigcap_{r>s} \tilde{\mathcal{F}}_r^0$, where

  $$\mathcal{N} := \{C \in \mathcal{F} : P(C) = 0\},\ \tilde{\mathcal{F}}_s := \sigma(\beta_k(r) : r \leq s, k \in \mathbf{N}),\ \tilde{\mathcal{F}}_s^0 := \sigma(\mathcal{N} \cup \tilde{\mathcal{F}}_s)$$

- $B \in L(U, H)$
- $\{X(t)\}_{t\geq 0}$, H-valued stochastic process
- X_0 is $\mathcal{F}_0$-measurable

- $-A : \mathcal{D}(A) \subset H \to H$ is linear operator, generating a strongly continuous semigroup (C_0-semigroup) of bounded linear operators $\{S(t)\}_{t\geq 0} \subset L(H)$; that is,
  - $S(0) = I$;
  - $S(t+s) = S(t)S(s)$ for all $s, t \geq 0$;
  - $\{S(t)\}_{t\geq 0}$ is strongly continuous on $[0, \infty)$, that is, $t \mapsto S(t)x$ is continuous on $[0, \infty)$ for all $x \in H$;
  - $\lim_{h\to 0^+} \frac{S(h)x - x}{h} = -Ax$ for all $x \in \mathcal{D}(A)$;

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  - $\lim_{h\to 0^+} \frac{S(h)x - x}{h} = -Ax$ for all $x \in \mathcal{D}(A)$;

In this case $u(t) = S(t)x$ is the unique (mild) solution of the deterministic equation

$$u(t) + A\int_0^t u(s)ds = x, \ x \in H, \ t \geq 0,$$

and if $x \in \mathcal{D}(A)$, then u is the unique (strong) solution of

$$\dot{u}(t) + Au(t) = 0, \ t > 0; \ u(0) = x.$$

Sometimes the semigroup $S(t)$ generated by A is also written as $S(t) = e^{-tA}$ in analogy with matrix exponentials. In several cases this can be made rigorous using a functional calculus.

Weak solution

Weak solution

Definition. An H-valued process $\{X(t)\}_{t\in[0,T]}$ is a **weak solution** of the linear SPDE if $X(t)$ is $\mathcal{F}_t$-measurable ($t \in [0,T]$), $\{X(t)\}_{t\in[0,T]}$ has Bochner integrable trajectories P-almost surely and

$$\langle X(t), \eta\rangle + \int_0^t \langle X(s), A^*\eta\rangle \, \mathrm{d}s = \langle \xi, \eta\rangle + W_{B^*\eta}(t)$$

$$P\text{-a.s.}, \ \forall \eta \in \mathcal{D}(A), \ t \in [0,T].$$

Recall that

$$W_{B^*\eta}(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\beta_k(t)\langle e_k, B^*\eta\rangle_U = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\beta_k(t)\langle Be_k, \eta\rangle_H.$$

Note, that $W_{B^*\eta}(t) = \int_0^t l_\eta B \, \mathrm{d}W(s)$, where

$$l_\eta : H \to \mathbf{R}, \quad l_\eta(h) := \langle h, \eta\rangle, \ h \in H.$$

Weak solution

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The obvious candidate for the solution is given by the variation of constants formula

$$X(t) = S(t)\xi + \int_0^t S(t-s)B \, \mathrm{d}W(s).$$

Weak solution: existence and uniqueness

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Theorem. If

$$\int_0^T \|S(r)BQ^{1/2}\|_{\mathrm{HS}}^2 \, \mathrm{d}r < \infty,$$

then

$$X(t) = S(t)\xi + \int_0^t S(t-s)B \, \mathrm{d}W(s)$$

is a weak solution of the linear SPDE and it is unique up to modification. That is, if $Y(t)$ is another weak solution then $X(t) = Y(t)$, P-a.s.

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Remark. The concept of weak solution is necessary for two reasons.

- The relation $X(t) \in \mathcal{D}(A)$ is seldom true
- For the integral $\int_0^t B\,\mathrm{d}W(t)$ to exist one needs $\|BQ^{1/2}\|_{\mathrm{HS}}^2 < \infty$

Semigroup approach to SPDEs: the semilinear case

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Here we consider equations written formally as

$$\begin{aligned} &\mathrm{d}X(t) + AX(t)\,\mathrm{d}t = f(X(t))\,\mathrm{d}t + B\,\mathrm{d}W(t), \quad 0 < t < T, \\ &X(0) = \xi. \end{aligned} \tag{4}$$

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- H, U real, separable Hilbert spaces
- $W(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\beta_k(t)e_k$, Q-Wiener process on U, $Qe_k = \lambda_k e_k$
- Filtration $\mathcal{F}_s := \bigcap_{r>s} \tilde{\mathcal{F}}_r^0$, where

  $\mathcal{N} := \{C \in \mathcal{F} : P(C) = 0\}$, $\tilde{\mathcal{F}}_s := \sigma(\beta_k(r) : r \leq s, k \in \mathbf{N})$, $\tilde{\mathcal{F}}_s^0 := \sigma(\mathcal{N} \cup \tilde{\mathcal{F}}_s)$
- $-A$ generates a C_0-semigroup $\{S(t)\}_{t\geq 0}$
- $B \in L(U, H)$
- $f : H \to H$
- $\{X(t)\}_{t\geq 0}$, H-valued stochastic process
- X_0 is $\mathcal{F}_0$-measurable

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The main difference when dealing with this kind of equations compared to the one before is that, in general, there is no explicit representation of the solution of (4). Another solution concept is more convenient in this case.

Mild solution

Mild solution

Definition. An H-valued process $\{X(t)\}_{t\in[0,T]}$ is a **mild solution** of (4) if $X(t)$ is $\mathcal{F}_t$-measurable ($t \in [0,T]$),

$$X \in C\big([0,T]; L_2(\Omega, \mathcal{F}, P; H)\big)$$

and, for all $t \in [0,T]$,

$$X(t) = S(t)\xi + \int_0^t S(t-s)f(X(s))\,\mathrm{d}s + \int_0^t S(t-s)B\,\mathrm{d}W(s) \quad P\text{-a.s.}$$

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Theorem. If $\xi \in L_2(\Omega, \mathcal{F}_0, P; H)$,

$$\int_0^T \|S(s)BQ^{1/2}\|_{\mathrm{HS}}^2\,\mathrm{d}s < \infty$$

and $f : H \to H$ satisfies the global Lipschitz condition

$$\|f(x) - f(y)\|_H \le K\|x - y\|_H, \quad \forall x, y \in H,$$

for some $K > 0$, then there is a unique mild solution of (4).

Mild solution

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Proof (**Sketch.**) The proof is a fixed point argument.

First, it is not difficult to show that

$$Z_{[a,b]} := \Big\{ X \in C\big([a,b]; L_2(\Omega, \mathcal{F}, P; H)\big) : X(t) \text{ is } \mathcal{F}_t\text{-measurable } (t \in [a,b]) \Big\}$$

with norm $\|Y\|_{Z_{[a,b]}} = \sup_{t\in[a,b]}(\mathbf{E}\|Y(t)\|_H^2)^{1/2}$ is a Banach space.

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Then, define

$$F(Y)(t) := S(t)\xi + \int_0^t S(t-s)f(Y(s))\mathrm{d}s + \int_0^t S(t-s)B\,\mathrm{d}W(s)$$

and show that $F : Z_{[0,\tau]} \to Z_{[0,\tau]}$ for some $\tau > 0$ and that it is a contraction; that is,

$$\|F(Y_1) - F(Y_2)\|_{Z_{[0,\tau]}} \leq L\|Y_1 - Y_2\|_{Z_{[0,\tau]}}, \;\; L < 1.$$

This yields a unique fixed point of F and hence a unique mild solution on $[0,\tau]$. Finally, repeat the argument on $[\tau, 2\tau]$, $[2\tau, 3\tau]$ and so on, to get a unique solution on $[0, T]$.

Example: linear stochastic wave equation

Example: linear stochastic wave equation

We consider the stochastic wave equation

$$\begin{aligned}
&\mathrm{d}\dot{u} - \Delta u \,\mathrm{d}t = \mathrm{d}W && \text{in } \mathcal{D} \times \mathbf{R}_+,\\
&u = 0 && \text{on } \partial\mathcal{D} \times \mathbf{R}_+,\\
&u(\cdot, 0) = u_0,\ \dot{u}(\cdot, 0) = u_1 && \text{in } \mathcal{D}.
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Let $\dot{H}^{-1} = (H_0^1(\mathcal{D}))^*$. We let $\Lambda = -\Delta$ with $\mathcal{D}(\Lambda) = H_0^1$ and we regard Λ as an operator $H_0^1 \subset \dot{H}^{-1} \to \dot{H}^{-1}$ by

$$(\Lambda u)(v) = \langle \nabla u, \nabla v \rangle_{L_2(\mathcal{D})}.$$

Let $U = L_2(\mathcal{D})$ and W be a Q-Wiener process on U. We put

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} := \begin{bmatrix} u \\ \dot{u} \end{bmatrix}, \quad \xi =: \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}, \quad H = L_2(\mathcal{D}) \times \dot{H}^{-1}.$$

Example: linear stochastic wave equation

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Now we can write

$$\begin{aligned}
\mathrm{d}X = \begin{bmatrix} \mathrm{d}u \\ \mathrm{d}\dot{u} \end{bmatrix} &= \begin{bmatrix} \dot{u}\,\mathrm{d}t \\ \Delta u\,\mathrm{d}t + \mathrm{d}W \end{bmatrix} \\
&= \begin{bmatrix} X_2 \\ -\Lambda X_1 \end{bmatrix} \mathrm{d}t + \begin{bmatrix} 0 \\ I \end{bmatrix} \mathrm{d}W \\
&= \begin{bmatrix} 0 & I \\ -\Lambda & 0 \end{bmatrix} X\,\mathrm{d}t + \begin{bmatrix} 0 \\ I \end{bmatrix} \mathrm{d}W \\
&= -AX\,\mathrm{d}t + B\,\mathrm{d}W,
\end{aligned}$$

where

$$A = \begin{bmatrix} 0 & -I \\ \Lambda & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ I \end{bmatrix}.$$

So we have

$$\begin{aligned}
&\mathrm{d}X + AX\,\mathrm{d}t = B\,\mathrm{d}W, \quad t > 0, \\
&X(0) = \xi,
\end{aligned} \tag{5}$$

where

$$\mathcal{D}(A) = \left\{ x \in H : Ax = \begin{bmatrix} -x_2 \\ \Lambda x_1 \end{bmatrix} \in H = L_2(\mathcal{D}) \times \dot{H}^{-1} \right\} = H_0^1(\mathcal{D}) \times L_2(\mathcal{D}).$$

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Hence, in this case, $U \neq H$ and $B \neq I$. In order to see what $S(t) = e^{-tA}$ is, we note that $y(t) = S(t)x$ is the solution of

$$\dot{y} + Ay = 0; \quad y(0) = x,$$

that is,

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We solve it using an eigenfunction expansion:

$$\begin{aligned} y_1(t) &= \sum_{j=1}^{\infty} \cos(\sqrt{\mu_j}t)\langle x_1, \phi_j\rangle \phi_j + \frac{1}{\sqrt{\mu_j}} \sin(\sqrt{\mu_j}t)\langle x_2, \phi_j\rangle \phi_j \\ &= \cos(t\Lambda^{1/2})x_1 + \Lambda^{-1/2} \sin(t\Lambda^{1/2})x_2, \end{aligned}$$

and

$$y_2 = \dot{y}_1(t) = -\Lambda^{1/2} \sin(t\Lambda^{1/2})x_1 + \cos(t\Lambda^{1/2})x_2.$$

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Now we can write the semigroup as

$$S(t) = \mathrm{e}^{-tA} = \begin{bmatrix} \cos(t\Lambda^{1/2}) & \Lambda^{-1/2}\sin(t\Lambda^{1/2}) \\ -\Lambda^{1/2}\sin(t\Lambda^{1/2}) & \cos(t\Lambda^{1/2}) \end{bmatrix}.$$

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With $\xi = 0$ the evolution problem (5) has the unique weak solution

$$\begin{aligned} X(t) &= \int_0^t S(t-s)B\,\mathrm{d}W(s) \\ &= \begin{bmatrix} \int_0^t \Lambda^{-1/2}\sin((t-s)\Lambda^{1/2})\,\mathrm{d}W(s) \\ \int_0^t \cos((t-s)\Lambda^{1/2})\,\mathrm{d}W(s) \end{bmatrix}. \end{aligned}$$

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For the existence and uniqueness of mild solutions one needs

$$\int_0^T \|S(t)BQ^{1/2}\|_{\mathrm{HS}}^2\,\mathrm{d}t < \infty.$$

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We have,

$$\int_0^T \|S(t)BQ^{\frac{1}{2}}\|_{\mathrm{HS}}^2 \,\mathrm{d}t = \int_0^T \sum_k \|S(t)BQ^{\frac{1}{2}} f_k\|_H^2 \,\mathrm{d}t$$

$$= \int_0^T \sum_k \left(\|\Lambda^{-\frac{1}{2}} \sin(t\Lambda^{\frac{1}{2}})Q^{\frac{1}{2}} f_k\|_{L_2(\mathcal{D})}^2 + \|\cos(t\Lambda^{\frac{1}{2}})Q^{\frac{1}{2}} f_k\|_{\dot{H}^{-1}}^2 \right) \mathrm{d}t$$

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This must be finite.

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This must be finite.

- If $\mathrm{Tr}(Q) < \infty$:

$$\|\Lambda^{-\frac{1}{2}} \sin(t\Lambda^{\frac{1}{2}}) Q^{\frac{1}{2}}\|_{\mathrm{HS}}^2 \le \|\Lambda^{-\frac{1}{2}}\|_{L(L_2(\mathcal{D}))}^2 \|\sin(t\Lambda^{\frac{1}{2}})\|_{L(L_2(\mathcal{D}))}^2 \,\mathrm{Tr}(Q) < \infty,$$

  and similarly for cosine, so the condition holds in any spatial dimension.

Example: linear stochastic wave equation

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- For $Q = I$ we have

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  Similarly for the cosine operator. We have that

$$\|A^{-1/2}\|_{\mathrm{HS}}^2 = \sum_k \mu_k^{-1} \sim \sum_k k^{-2/d}.$$

  This is finite if and only if $d = 1$. Thus white noise is too irregular in higher spatial dimensions.

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This is finite if and only if $d = 1$. Thus white noise is too irregular in higher spatial dimensions.

Note: this is where one needs the choice $H = L_2(\mathcal{D}) \times \dot{H}^{-1}$. Otherwise, if one takes $H = H_0^1(\mathcal{D}) \times L_2(\mathcal{D})$, then in case $Q = I$ one would need, for example,

$$\int_0^T \|\cos(t\Lambda^{\frac{1}{2}})\|_{\mathrm{HS}}^2 \, \mathrm{d}t < \infty$$

which does not hold in any spatial dimension.