Logarithmic norms and Quadratic forms.

On the relation between logarithmic norms and Calculus of Variation

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The logarithmic norm was introduced in 1958 independently by Germund Dahlquist and Sergey Lozinskij. It is a real-valued functional on operators, quantifying the notions of *definiteness* for matrices; *monotonicity* for nonlinear maps; and *ellipticity* for differential operators. It is defined either in terms of an inner product in Hilbert space, or in terms of the operator norm on a Banach space. Originally, it was only introduced for matrices, but today there is a modern theory also covering nonlinear operators, infinite dimensional spaces, bounded and unbounded linear operators, such as differential operators. Thus the logarithmic norm has a wide range of applications in matrix theory, stability theory and numerical analysis, e.g. offering various quantitative bounds on (functions of) operators, operator spectra, resolvents, Rayleigh quotients and the numerical range, stability and error bounds in initial as well as boundary value problems and their discretizations. Some special fields in mathematics, such as semigroup theory, rely on notions that are strongly related to the logarithmic norm.

In this talk, we shall only work within a Hilbert space setting and note that the lower and upper logarithmic norms of an operator $\mathcal{A}$ are the best bounds such that

$$m_2[\mathcal{A}] \cdot \|u\|_2^2 \;\le\; (u, \mathcal{A}u) \;\le\; M_2[\mathcal{A}] \cdot \|u\|_2^2$$

holds for all u and a given inner product $(\cdot, \cdot)$. This extends to all types of linear operators, but for nonlinear operators, where there is no longer any equivalence between boundedness and continuity, one has a choice to either emphasize either one. Mostly, continuity is emphasized, considering the applications in stability theory, and for nonlinear operators it is common to define the logarithmic Lipschitz constants as the best bounds satisfying

$$m_2[\mathcal{F}] \cdot \|u-v\|_2^2 \;\le\; (u-v, \mathcal{F}(u) - \mathcal{F}(v)) \;\le\; M_2[\mathcal{F}] \cdot \|u-v\|_2^2$$

To give two important applications of how the logarithmic norm plays a significant role in stability theory, let us first consider a nonlinear initial value problem

$$\begin{aligned} u_t &= \mathcal{F}(u) + p(t) \\ v_t &= \mathcal{F}(v). \end{aligned}$$

Using the upper Dini derivative, we have the differential inequality

$$D_t^+ \|u-v\| \le M[\mathcal{F}] \cdot \|u-v\| + \|p(t)\|.$$

By integration, one finds that

$$\|u(t) - v(t)\|_2 \le \mathrm{e}^{tM_2[\mathcal{F}]} \|u(0) - v(0)\|_2 + \int_0^t \mathrm{e}^{(t-\tau)M_2[\mathcal{F}]} \|p(\tau)\|_2 \,\mathrm{d}\tau.$$

In case $p \equiv= 0$, one finds a bound on how initial values are propagated; in case $u(0) = v(0)$, one finds a bound on how far a perturbation can force two solutions apart. In both cases, one finds that the solution depends continuously on the data.

For operator equations, independent of time, the logarithmic norm is equally important. Assuming that $m[\mathcal{F}] > 0$, the operator $\mathcal{F}$ is invertible. Considering two neighboring problems

$$\mathcal{F}(x + \delta x) = y + \delta y$$
$$\mathcal{F}(x) = y$$

one finds that $\delta x = \mathcal{F}^{-1}(y+\delta y) - \mathcal{F}(y)$. Therefore, by the Uniform Monotonicity Theorem (Browder & Minty 1963) we have

$$\|\delta x\| \leq L[\mathcal{F}^{-1}] \cdot \|\delta y\| \leq \frac{\|\delta y\|}{m[\mathcal{F}]},$$

where $L[\cdot]$ is the (least upper bound) Lipschitz constant. Thus $\mathcal{F}$ has a Lipschitz inverse. One of the prime applications of this result is in the classical convergence theory of discretizations, due to Lax and Stetter, and in (some variants of) the Lax–Milgram lemma in finite element theory.

In (static) differential equations, the condition $m[\mathcal{F}] > 0$ typically represents ellipticity. In this talk, we shall develop some aspects of elliptic differential operators. The reason why logarithmic norms play an important role is that they are easily used to derive bounds in perturbation theory. Thus, some of the most important general properties of the logarithmic norm are

- $-l[\mathcal{F}] \leq M[\mathcal{F}] \leq L[\mathcal{F}]$
- $M[\alpha \mathcal{F}] = \alpha \cdot M[\mathcal{F}]\,, \qquad \alpha \geq 0$
- $m[\mathcal{F}] = -M[-\mathcal{F}]$
- $M[\mathcal{F} + z] = M[\mathcal{F}] + \operatorname{Re} z$
- $M[\mathcal{F}] + m[\mathcal{G}] \leq M[\mathcal{F} + \mathcal{G}] \leq M[\mathcal{F}] + M[\mathcal{G}]$,

where $l[\cdot]$ is the (greatest lower bound) Lipschitz constant.

The original application to matrices was straightforward, but the extension to non-numeric objects such as nonlinear maps and differential operators has required some modifications of the theory. The simplest application to differential operators is to consider the 1D Poisson equation, $-u'' = f$ with Dirichlet conditions $u(0) = u(1) = 0$. since this is an operator equation, we are interested in whether $\mathcal{L} = -\mathrm{d}^2/\mathrm{d}x^2$ is "invertible" on $H_0^1[0, 1]$. Although this is well known and classical theory, we shall compute the logarithmic norm of the operator. Thus, integrating by parts, using the Poincaré-Sobolev inequality, we find

$$(u, \mathcal{L}u) = -\int_0^1 uu''\,\mathrm{d}x = -(u, u'') = (u', u') \;\geq\; \pi^2 \|u\|^2_{L^2[0,1]}$$

This implies that $\mathcal{L} = -\mathrm{d}^2/\mathrm{d}x^2$ is elliptic on $H_0^1[0, 1]$, with

$$m_2[\mathcal{L}] = m_2\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right] = \pi^2 > 0.$$

This can also be formulated as a variational problem, as

$$\min_u \int_0^1 |u'|^2 \, \mathrm{d}x$$

subject to $\|u\|_{L^2} = 1$ and the boundary conditions $u(0) = u(1) = 0$. With a Lagrangian $L(u, u') = (u')^2 - \lambda u^2$, this leads to the standard Sturm–Liouville problem $-u'' = \lambda u$, whose eigenfunctions are $u_k(x) = \sin k\pi x$ with eigenvalues $\lambda_k = k^2\pi^2$. Unlike in variational calculus, where we normally seek the minimizing function, we here seek the minimum itself, and $m[-\mathrm{d}^2/\mathrm{d}x^2]$ equals the smallest eigenvalue, λ_1, which is π^2.

There are many variants, and changing to Neumman conditions $u'(0) = u(0) = 0$ leads to another Strum–Liouville problem, with $m[-\mathrm{d}^2/\mathrm{d}x^2] = \pi^2/4$. Likewiese, with periodic boundary conditions, $u(0) = u(1)$ and $u'(0) = u'(1)$ we get yet another Sturm–Liouville problem with $m[-\mathrm{d}^2/\mathrm{d}x^2] = 4\pi^2$. Thus the logarithmic norm of a differential operator depends on the space of functions over which it is computed.

We then extended this theory to non-selfadjoint convection-diffusion operators, demonstrating how to construct symmetrizers. Thus, for $\mathcal{L}_{\mathrm{cd}}u = -u'' + au'$ with $u(0) = u(1) = 0$, the integrating factor $w = \exp(-ax)$ leads to the selfadjoint operator $w\mathcal{L}_{\mathrm{cd}}u = (wu')'$. By further symmetrizing this operator into the selfadjoint operator $\mathcal{A} = w^{1/2}\mathcal{L}_{\mathrm{cd}}w^{-1/2}$, the logarithmic norm of $\mathcal{A}$ as well as of $\mathcal{L}_{cd}$ are easily computed, and a "tailor-made" inner product is introduced to obtain sharp bounds, such that $m[\mathcal{A}] = \lambda_1[\mathcal{L}_{\mathrm{cd}}] = \pi^2 + a^2/4$. The same result is obntained if the standard inner product (u, v) is replaced by the weighted inner product, (u, wv).

The theory is finally illustrated for a singular problem, the axisymmetric Bessel operator

$$\mathcal{L}_0 u = -u'' - \frac{u'}{r},$$

with Neumann conditions $u'(0) = u(1) = 0$. This problem is not elliptic with respect to the standard L^2 inner product, but after symmetrization with $w(r) = r$ it can be shown that the operator is elliptic with respect to a weighted inner product. For the non-axisymmetric operator

$$\mathcal{L}_m u = -u'' - \frac{u'}{r} - \frac{m^2}{r^2}u,$$

with Dirichelt conditions $u(0) = u(1) = 0$, the problem is elliptic, and the logarithmic norm in L^2 is estimated by simple use of submultipliciativity and integration by parts.

Due to time restrictions, applications to fourth order operators, such as the biharmonic equation, were left out. These lead to more difficult eigenvelaue problems, and variational formulations outside the classical Euler–Lagrange theory.