# Adequate Numerical Methods for Non-linear Parabolic Problems in Mathematical Finance

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# Non-linear Black-Scholes equation

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2\left(S, t, \frac{\partial^2 V}{\partial S^2}\right)S^2\frac{\partial^2 V}{\partial S^2}+(r-q)S\frac{\partial V}{\partial S}-rV=0,\ \ S>0,\ \ 0\le t\le T.$$

$V(S,t)$ - option value;
$t$ - time variable;
$S$ - asset price;
$r>0$ is the interest rate;
$q\ge 0$ is the dividend yield rate;
$T$ - maturity date;
$\sigma>0$ - volatility of the underlying asset price.

M. Koleva, L. Vulkov, Fast Computational Approach to the Delta Greek of Non-linear Black-Scholes Equations, *J. Comput. Appl. Math.* accepted, available on-line.

# i.b.c.

$$
\begin{aligned}
V(S,T) &= g_1(S), \quad S \in (0, S_{\max}), \\
V(0,t) &= g_2(t), \quad t \in (0,T], \\
V(S_{\max},t) &= g_3(t), \quad t \in (0,T],
\end{aligned}
$$

$$
g_1 = \begin{cases} (S-K)^+, & \text{Vanilla call}, \\ (K-S)^+, & \text{Vanilla put}, \\ (S-K_1)^+ - 2(S-K)^+ + (S+K_2)^+ & \text{Butterfly Spread}, \end{cases}
$$

$$
g_2 = \begin{cases} 0, & \text{Vanilla call}, \\ Ke^{-rt}, & \text{Vanilla put}, \\ 0, & \text{Butterfly Spread}, \end{cases} \qquad g_3 = \begin{cases} S_{\max} - Ke^{-rt}, & \text{Vanilla call}, \\ 0, & \text{Vanilla put}, \\ 0, & \text{Butterfly Spread} \end{cases}
$$

where $v^+ = \max\{v, 0\}$ (and $v^- = \max\{-v, 0\}$), $K$, $K_1$ and $K_2$ denote the strike prices of the options.

# Greeks and Delta eqn.

Greek Delta: $\dfrac{\partial V}{\partial S}$;

Greek Gamma: $\dfrac{\partial^2 V}{\partial S^2}$;

Greek Vega: $\dfrac{\partial V}{\partial \sigma}$;

Greek Theta: $-\dfrac{\partial V}{\partial t}$.

Further we introduce $W := V_S$, $t = T - t$, and formally differentiate Black-Sholes eqn and b.c. with respect to $S$ to derive the equation

$$\frac{\partial W}{\partial t} = \frac{1}{2}\frac{\partial}{\partial S}\left[\sigma^2\left(S, t, \frac{\partial W}{\partial S}\right) S^2 \frac{\partial W}{\partial S}\right] + (r - q)S\frac{\partial W}{\partial S} - qW, \qquad (1)$$

for $S \in (0, S_{\max})$, $0 < t \leq T$

# i.b.c

Initial data (here $H(x) = 1_{[0,\infty)}(x)$ stands for Heaviside functions):

$$g_1'(S) = \begin{cases} H(S-K), & \text{Vanilla call,} \\ -H(K-S), & \text{Vanilla put,} \\ H(S-K_1) - 2H(S-K) + H(S-K_2), & \text{Butterfly Spread.} \end{cases} \quad (2)$$

Boundary conditions:

$$g_2^W = \begin{cases} 0, & \text{Vanilla call ,} \\ -1, & \text{Vanilla put,} \\ 0, & \text{Butterfly Spread,} \end{cases} \qquad g_3^W = \begin{cases} 1 & \text{Vanilla call,} \\ 0 & \text{Vanilla put,} \\ 0 & \text{Butterfly Spread.} \end{cases} \quad (3)$$

# Mesh

**In space**: non-uniform grid with $M$ nodes $S_i$ [T. Haentjens, K.J. in 't Hout, 2012] - uniform inside the region $[S_r, S_l] = [m_1 K, m_2 K]$, $m_2 > 1$, $0 < m_1 < 1$; non-uniform outside with stretching parameter $c = K/10$:

$$S_i := \phi(\xi_i) = \begin{cases} S_l + c\sinh(\xi_i), & \xi_{\min} \le \xi_i < 0, \\ S_l + c\xi_i, & 0 \le \xi_i \le \xi_{\text{int}}, \\ S_r + c\sinh(\xi_i - \xi_{\text{int}}), & \xi_{\text{int}} \le \xi_i < \xi_{\max}. \end{cases} \tag{4}$$

The uniform partition of $[\xi_{\min}, \xi_{\max}]$ is defined trough $\xi_{\min} = \xi_0 < \cdots < \xi_M = \xi_{\max}$:

$$\xi_{\min} = \sinh^{-1}\left(\frac{-S_l}{c}\right), \ \xi_{\text{int}} = \frac{S_r - S_l}{c}, \ \xi_{\max} = \xi_{\text{int}} + \sinh^{-1}\left(\frac{S_{\max} - S_r}{c}\right).$$

**In time**: non-uniform, $\{t_n\}_{n=0}^{N}$, $t_N = T$ and

$\triangle t_n = t_{n+1} - t_n, \ \ n = 0, 1, \ldots, N-1.$

Upwind scheme + 'maximal use of central differencing' [J. Wang, P. Forsyth, SIAM J. Numer. Anal., 2008] for the convection term, $\theta \in [0,1]$

$$
\begin{aligned}
\frac{W_i^{n+1}-W_i^n}{\triangle t_n} & -\frac{\theta}{2\hbar_i}\left[S^2_{i+1/2}\widehat{\sigma}^{2,n+1}_{i+1/2}(W_S)_i^{n+1} - S^2_{i-1/2}\widehat{\sigma}^{2,n+1}_{i-1/2}(W_{\bar{S}})_i^{n+1}\right] \\
& -\theta(r-q)^+ S_i[\chi_i^+(W_S)_i^{n+1} + (1-\chi_i^+)(W_{\mathring{S}})_i^{n+1}] \\
& +\theta(r-q)^- S_i[\chi_i^-(W_{\bar{S}})_i^{n+1} + (1-\chi_i^-)(W_{\mathring{S}})_i^{n+1}] + q\theta W_i^{n+1} \\
= \frac{(1-\theta)}{2\hbar_i} & \left[S^2_{i+1/2}\widehat{\sigma}^{2,n}_{i+1/2}(W_S)_i^{n} - S^2_{i-1/2}\widehat{\sigma}^{2,n}_{i-1/2}(W_{\bar{S}})_i^{n}\right] \\
& +(1-\theta)(r-q)^+ S_i[\chi_i^+(W_S)_i^{n} + (1-\chi_i^+)(W_{\mathring{S}})_i^{n}] \\
& -(1-\theta)(r-q)^- S_i[\chi_i^-(W_{\bar{S}})_i^{n} + (1-\chi_i^-)(W_{\mathring{S}})_i^{n+1}] + q(1-\theta)W
\end{aligned}
$$

$$
\widehat{\sigma}^{2,n+1}_{i+1/2} := \sigma^2\left(S_{i+1/2}, t_{n+1}, (W_S)_i^{n+1}\right),\ \widehat{\sigma}^{2,n+1}_{i-1/2} := \sigma^2\left(S_{i-1/2}, t_{n+1}, (W_{\bar{S}})_i^{n+1}\right),
$$

$$
\chi_i^+ = \begin{cases} 0, & h_i < \dfrac{S^2_{i-1/2}\widehat{\sigma}^{2,n+1}_{i-1/2}}{(r-q)^+ S_i}, \\ 1, & \text{otherwise}, \end{cases} \quad \chi_i^- = \begin{cases} 0, & h_{i-1} < \dfrac{S_{i+1/2}\widehat{\sigma}^{2,n+1}_{i+1/2}}{(r-q)^-}, \\ 1, & \text{otherwise}. \end{cases}
$$

# Basic property of the volatility

(**P**) *Function* $\sigma^2(S,t,U)\,U$ *is non-decreasing with respect to* $U$, $U = \frac{\partial W}{\partial S}$.

- **Barles-Soner model**. The volatility is given by

$$\sigma^2(S,t,U) = \sigma_0^2\left(1 + \psi\left[e^{rt}a^2S^2U\right]\right).$$

$\sigma_0^2$ - the historical volatility, $a = \kappa\sqrt{\widetilde{\gamma}\widetilde{N}}$ with $\kappa$ - transaction cost parameter, $\widetilde{\gamma}$ - risk aversion factor, $\widetilde{N}$- the number of options to be sold.

$$\psi'(z) = \frac{\psi(z)+1}{2\sqrt{z\psi(z)} - z} \quad \text{for} \quad z \neq 0 \quad \text{and} \quad \psi(0) = 0. \tag{5}$$

# Basic property of the volatility

(**P**) *Function* $\sigma^2(S,t,U)\,U$ *is non-decreasing with respect to* $U$, $U=\frac{\partial W}{\partial S}$.

An implicit exact solution of is obtained in [R. Company et. al., 2008]

$$\sqrt{z}=\frac{-\text{arcsinh}\sqrt{\psi}}{\sqrt{\psi+1}}+\sqrt{\psi}\ \text{ for }\ z>0,\quad \psi(z)>0,$$

$$\sqrt{-z}=\frac{\text{arcsin}\sqrt{-\psi}}{\sqrt{\psi+1}}-\sqrt{-\psi}\ \text{ for }\ z<0,\quad -1<\psi(z)<0,$$

$$-1<\psi(z)<\infty,\quad z\in\mathbb{R},\quad \text{and}\quad \psi'(z)>0\ \text{ for }\ z\neq 0.$$

In [D. Lesmana, S Wang, 2013]: $\left(1+\psi\left[e^{rt}a^2S^2U\right]\right)U$ is increasing with respect to $U$, therefore the property (P) is satisfied for all options.

# Basic property of the volatility

(**P**) *Function* $\sigma^2(S,t,U)\,U$ *is non-decreasing with respect to* $U$, $U = \frac{\partial W}{\partial S}$.

- **Leland model**:

$$\sigma^2(U) = \sigma_0^2\left(1 + \mathrm{Le} \times \mathrm{sign}(U)\right), \quad \mathrm{Le} = \sqrt{\frac{2}{\pi}}\left(\frac{\kappa}{\sigma_0\sqrt{\delta t}}\right),$$

$0 < \mathrm{Le} < 1$ is the Leland number, $\delta t$ - transaction frequency and $\kappa$ - transaction cost measure.
It is evidently that $\sigma^2(U)\,U$ is non-decreasing function with respect to $U$, i.e. the property (P) is fulfilled.

# Basic property of the volatility

(**P**) *Function* $\sigma^2(S,t,U)\,U$ *is non-decreasing with respect to* $U$, $U=\frac{\partial W}{\partial S}$.

• **Uncertain volatility model**: best/worst case for an investor with a long position in the option:

$$\sigma^2(U)=\begin{cases}\sigma^2_{\text{max}}, & U\le 0,\\ \sigma^2_{\text{min}}, & U>0,\end{cases}\qquad \sigma^2(U)=\begin{cases}\sigma^2_{\text{max}}, & U>0,\\ \sigma^2_{\text{min}}, & U\le 0,\end{cases}$$

$$\sigma_{\text{min}}\le\sigma(U)\le\sigma_{\text{min}}.$$

Property (P) is fulfilled.

# Stability

## Lemma

*If the following restriction is fulfilled*

$$\triangle t_n \leq \frac{1}{1-\theta}\Bigg(\frac{1}{2\hbar_i h_{i-1}}S^2_{i-1/2}\widehat{\sigma}^{2,n}_{i-1/2} + \frac{1}{2\hbar_i h_i}S^2_{i+1/2}\widehat{\sigma}^{2,n}_{i+1/2}$$
$$+(r-q)^- S_i\left[\frac{\chi_i^-}{h_{i-1}} + \frac{(1-\chi_i^-)h_i}{2\hbar_i}\left(\frac{h_i}{h_{i-1}} - \frac{h_{i-1}}{h_i}\right)\right]$$
$$+(r-q)^+ S_i\left[\frac{\chi_i^+}{h_i} + \frac{(1-\chi_i^+)h_{i-1}}{2\hbar_i}\left(\frac{h_{i-1}}{h_i} - \frac{h_i}{h_{i-1}}\right)\right] + q\Bigg)^{-1},$$

*then the solution of the discretization satisfies the estimate*

$$\|W^{n+1}\|_\infty \leq \max\{\|g_1'\|_\infty, \|g_3\|_\infty, \|g_3\|_\infty\}.$$

Consider the inequality in the interval $[S_r, S_l] = [m_1 K, m_2 K]$, where the mesh is uniform with step size $h = \min_i \{h_i\}$. Now, we get

$$\triangle t_n \leq \frac{h^2}{(1-\theta)} \left[ \left( S_{i-1/2}^2 \widehat{\sigma}_{i-1/2}^{2,n} + S_{i+1/2}^2 \widehat{\sigma}_{i+1/2}^{2,n} + h|r-q|S_i + h^2 q \right) \right]^{-1}.$$

Consequently:

$$\triangle t_n \leq \frac{h^2}{(1-\theta)\left(2(m_2 K)^2 \|\widehat{\sigma}^{2,n}\|_\infty + h|r-q|m_2 K + h^2 q\right)}. \qquad (5)$$

$K$ is a large number and $m_2 > 1$, therefore the restriction is quite severe.

Further we consider the unconditionally stable full implicit scheme ($\theta = 1$).

# Monotonicity

## Lemma

*(Monotonicity) If the restriction*

$$h_i \leq \frac{S_{i-1/2}^2}{S_i|r-q|} \min\left\{ \frac{1}{(1-\chi_{i+1}^-)} \frac{\partial(\widehat{\sigma}_{i+3/2}^{2,n+1} U_{i+3/2}^{n+1})}{\partial U_{i+3/2}^{n+1}}, \frac{1}{(1-\chi_i^+)} \frac{\partial(\widehat{\sigma}_{i-1/2}^{2,n+1} U_{i-1/2}^{n+1})}{\partial U_{i-1/2}^{n+1}}\right.$$

*is fulfilled, then the discretization, $\theta = 1$ is monotone.*

# Consistency and convergence

**Lemma**

*The discretization is consistent.*

Following the results in [G. Barles, Numerical Methods in Finance, Cambridge, 1997] for a second order non-linear PDE, from *consistency*, *stability* and *monotonicity* of the discretization follows the convergence to the *viscosity solution*.

**Theorem**

*The solution of the discrete scheme, $\theta = 1$, converges to viscosity solution as $(h, \triangle t) \to (0^+, 0^+)$, where $h = \max_{0 \le i \le M} h_i$ and $\triangle t = \max_{0 \le n \le N-1} \triangle t_n$.*

# Two-grid algorithms

Accelerate the efficiency and to improve the order of convergence.

$$\overline{\omega}_c = \{S_1 = 0,\ S_{i+1} = S_i + h_i^c,\ i = 1, \dots, M_c - 1,\ S_{M_c} = S_{\max}\},$$
$$\overline{\omega}_f = \{S_1 = 0,\ S_{i+1} = S_i + h_i^f,\ i = 1, \dots, M_f - 1,\ S_{M_f} = S_{\max}\},\ M_f \gg M_c.$$

**Two-grid Newton/Picard** (**TGNA**/ **TGPA**). At each time level $n = 0, 1, \dots$ we perform the two steps:

step 1. Set $W_c^{(0)} := W_c^n$ and compute $W_c^{n+1}$ Newton's/Picard's iterations on the coarse mesh $\overline{\omega}_c$.

step 2. Set $W_f^{(0)} := I(W_c^{n+1})$, where $I(W_c)$ is the interpolant of $W_c$ on the fine grid, perform *only one* Newton's/Picard's iteration on the fine mesh $\overline{\omega}_f$ and get $W_f^{n+1}$.

Table : One-grid and two-grid computations (TGPA), Leland model, butterfly option

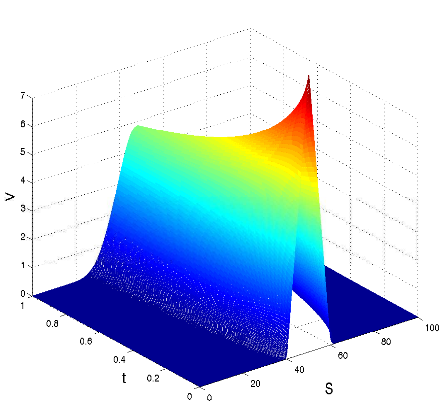
| One-grid, $\triangle t \sim \min_i h_i^2$ | | | | | Two-grid, $\triangle t \sim \min_i (h_i^f)^2$ | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $M$ | $-W(K,1)$ | diff | $CR$ | CPU | $M_c/M_f$ | $-W(K,1)$ | diff | $CR$ | CPU |
| 200 | 0.09476235 | | | 0.01 | | | | | |
| 400 | 0.09422126 | 5.4e-4 | | 0.03 | 200/400 | 0.0942273 | | | 0.08 |
| 800 | 0.09414325 | 7.8e-5 | **2.79** | 0.13 | | | | | |
| 1600 | 0.09413419 | 9.1e-6 | **3.10** | 0.71 | 400/1600 | 0.0941353 | 9.2e-5 | | 0.82 |
| 3200 | 0.09413285 | 1.3e-6 | **2.76** | 4.73 | | | | | |
| 6400 | 0.09413259 | 2.7e-7 | **2.31** | 36.28 | 800/6400 | 0.0941329 | 2.4e-6 | **5.24** | 32.08 |
| 12800 | 0.09413252 | 6.3e-8 | **2.09** | 285.56 | | | | | |
| 25600 | 0.09413251 | 1.5e-8 | **2.02** | 2294.13 | 1600/25600 | 0.0941326 | 2.8e-7 | **3.13** | 1637.29 |

Table : Passive Richardson extrapolation, one-grid and two-grid computations (TGPA), Leland model, butterfly option

| One-grid, $\triangle t \sim \min_i h_i$ | | | | | Two-grid, $\triangle t \sim \min_i h_i^f$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $M$ | $-\widetilde{W}(K,1)$ | diff | $CR$ | CPU | $M_c$ | $M_f$ | $-\widetilde{W}(K,1)$ | diff | $CR$ | CPU |
| 800 | 0.09401224 | | | 0.05 | | | | | | |
| 1600 | 0.09408530 | 7.31e-5 | | 0.14 | 400 | 1600 | 0.09408760 | | | 0.24 |
| 3200 | 0.09411725 | 3.19e-5 | **1.19** | 0.49 | | | | | | |
| 6400 | 0.09412799 | 1.07e-5 | **1.57** | 1.88 | 800 | 6400 | 0.09412831 | 5.52e-5 | | 1.26 |
| 12800 | 0.09413127 | 3.28e-6 | **1.71** | 6.98 | | | | | | |
| 25600 | 0.09413217 | 8.98e-7 | **1.87** | 23.26 | 1600 | 25600 | 0.09413223 | 3.92e-6 | **3.38** | 14.37 |
| 51200 | 0.09413242 | 2.46e-7 | **1.87** | 95.96 | | | | | | |
| 102400 | 0.09413248 | 6.42e-8 | **1.94** | 392.24 | 3200 | 102400 | 0.09413250 | 2.63e-7 | **3.90** | 249.60 |

Table : Active Richrdson extrapolation, one-grid and two-grid computations (TGPA), Leland model, butterfly option

| One-grid, $\triangle t \sim \min_i h_i$ | | | | | Two-grid, $\triangle t \sim \min_i h_i^f$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $M$ | $-\widetilde{W}(K,1)$ | diff | $CR$ | CPU | $M_c$ | $M_f$ | $-\widetilde{W}(K,1)$ | diff | $CR$ | CPU |
| 800 | 0.09402322 | | | 0.06 | | | | | | |
| 1600 | 0.09409481 | 7.16e-5 | **0.638** | 0.15 | 400 | 1600 | 0.09409508 | 1.74e-4 | | 0.20 |
| 3200 | 0.09412219 | 2.74e-5 | **1.387** | 0.50 | | | | | | |
| 6400 | 0.09413016 | 7.97e-6 | **1.780** | 1.90 | 800 | 6400 | 0.09413051 | 3.54e-5 | **2.299** | 1.22 |
| 12800 | 0.09413211 | 1.94e-6 | **2.037** | 7.01 | | | | | | |
| 25600 | 0.09413253 | 4.25e-7 | **2.192** | 23.86 | 1600 | 25600 | 0.09413260 | 2.08e-6 | **4.088** | 14.17 |



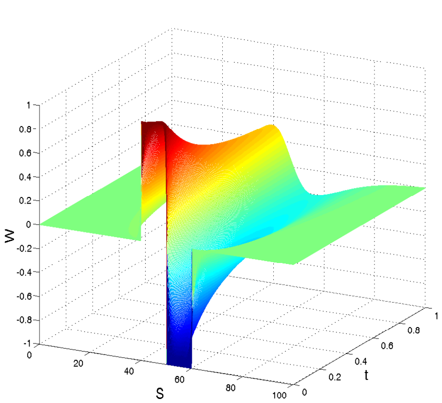




Figure : Evolution graphics for option value $(V)$, $t \in [\triangle t, 1]$ (*left*) and Greek Delta $(W)$, $t \in [0, 1]$ (*right*), obtained by TGPA with active Richardson extrapolation for Leland model, butterfly option, $M_c = 400$

# *Model problem*: Optimal Portfolio in an Exponential Utility Regime-Switching Model

Exponential utility function $U(y) = -\alpha e^{-\alpha y}$, $\alpha > 0$, $y \in \mathbb{R}$ [Valdez, Vargiolu, 2013].

Value functions $V^k(t, s, y)$, $k = 1, 2, \dots m$, given by

$$V^k = -\alpha e^{-\alpha r(T-t)(y - C^k(t,S))},$$

$S = (S_1, S_2, \dots, S_d) \subset \mathbb{R}^d$ - vector of stock prices $S_i \in [0, \infty)$, $i = 1, \dots, d$, $t \in [0, T]$ is a time variable and $C^k(t, s)$:

$$C^k_t + rSC^k_S + \frac{1}{2}\mathrm{tr}(\overline{S}\Sigma_k\Sigma_k^T\overline{S}C^k_{SS}) + \frac{e^{-r(T-t)}}{\alpha}\left[\sum_{j=1}^{m}(e^{-\alpha\phi(C^k - C^j)} - 1)\lambda^{kj} - \frac{1}{2}z_k^2\right] = rC^k,$$

$$C^k(T) = 0,$$

r is the risk-free interest rate, $\phi(t) := e^{r(T-t)}$ ,

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$$C^k(T) = 0,$$

$\Sigma_k\Sigma_k^T$ are locally Lipschitz and bounded,
$\Sigma_k = \Sigma_k(t, s) : [0, T] \times B \subset \mathbb{R}^d \to \mathbb{R}^{d \times d}$ is nonsingular for all $(t, s)$,

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$$C_t^k + rSC_S^k + \frac{1}{2}\mathrm{tr}(\overline{S}\Sigma_k\Sigma_k^T\overline{S}C_{SS}^k) + \frac{e^{-r(T-t)}}{\alpha}\left[\sum_{j=1}^{m}(e^{-\alpha\phi(C^k-C^j)}-1)\lambda^{kj} - \frac{1}{2}z_k^2\right] = rC^k,$$

$$C^k(T) = 0,$$

$z_k^2$ are defined by

$$z_k^2(t,s) := (\mu_k(t,s) - r\mathbf{1})^T \left(\Sigma_k(t,s)\Sigma_k^T(t,s)\right)^{-1}(\mu_k(t,s) - r\mathbf{1}).$$

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$$C^k_t + rSC^k_S + \frac{1}{2}\mathrm{tr}(\overline{S}\Sigma_k\Sigma_k^T\overline{S}C^k_{SS}) + \frac{e^{-r(T-t)}}{\alpha}\left[\sum_{j=1}^{m}(e^{-\alpha\phi(C^k - C^j)} - 1)\lambda^{kj} - \frac{1}{2}z_k^2\right] = rC^k,$$

$$C^k(T) = 0,$$

$\Sigma_k^{-1}\mu_k$ and $\lambda^{kj} : [0, T] \times B \to [0,\infty)$, $\lambda^{kj} \in C^1_b([0, T] \times B)$ are bounded on $\Omega_T = [0, T] \times B$, $B = (0,\infty)^d$ for all $k, j = 1, \dots, m$,

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$$C_t^k + rSC_S^k + \frac{1}{2}\mathrm{tr}(\overline{S}\Sigma_k\Sigma_k^T\overline{S}C_{SS}^k) + \frac{e^{-r(T-t)}}{\alpha}\left[\sum_{j=1}^{m}(e^{-\alpha\phi(C^k - C^j)} - 1)\lambda^{kj} - \frac{1}{2}z_k^2\right] = rC^k,$$

$$C^k(T) = 0,$$

$\overline{S} := \mathrm{diag}(S)$, $C_S^k$ is the gradient with respect to the vector $S$,

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Exponential utility function $U(y) = -\alpha e^{-\alpha y}$, $\alpha > 0$, $y \in \mathbb{R}$ [Valdez, Vargiolu, 2013].

Value functions $V^k(t, s, y)$, $k = 1, 2, \dots m$, given by

$$V^k = -\alpha e^{-\alpha r(T-t)(y - C^k(t,S))},$$

$S = (S_1, S_2, \dots, S_d) \subset \mathbb{R}^d$ - vector of stock prices $S_i \in [0, \infty)$, $i = 1, \dots, d$, $t \in [0, T]$ is a time variable and $C^k(t, s)$:

$$C_t^k + rSC_S^k + \frac{1}{2}\mathrm{tr}(\overline{S}\Sigma_k\Sigma_k^T\overline{S}C_{SS}^k) + \frac{e^{-r(T-t)}}{\alpha}\left[\sum_{j=1}^{m}(e^{-\alpha\phi(C^k - C^j)} - 1)\lambda^{kj} - \frac{1}{2}z_k^2\right] = rC^k,$$

$$C^k(T) = 0,$$

$C_{SS}^k$ is the Hessian matrix with entries $C_{S_iS_j}^k$, $i, j = 1, 2, \dots, d$,

# *Model problem*: Optimal Portfolio in an Exponential Utility Regime-Switching Model

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$$C^k_t + rSC^k_S + \frac{1}{2}\mathrm{tr}(\overline{S}\Sigma_k\Sigma_k^T\overline{S}C^k_{SS}) + \frac{e^{-r(T-t)}}{\alpha}\left[\sum_{j=1}^{m}(e^{-\alpha\phi(C^k - C^j)} - 1)\lambda^{kj} - \frac{1}{2}z_k^2\right] = rC^k,$$

$$C^k(T) = 0,$$

Existence of unique classical solution [Valdez, Vargiolu, 2013] $C(t, S) := (C^k(t, S))_{k=1,\dots,m} \in C_b^{1,2}([0, T] \times B, \mathbb{R}^m)$, Numerical methods for more simple regime-switching models in the case of exponential utility are constructed and analyzed in


- M.N. Koleva, W. Mudzimbabwe, L.G. Vulkov, *Numerical Algorithms*, 2016.
- W. Mudzimbabwe, L.G. Vulkov, *J. Comp. Appl. Math.*, 2016.
- M.N. Koleva, L.G. Vulkov, *J. Comp. Appl. Math.*, 2017.
- M.N. Koleva, L.G. Vulkov, *LNCS* 9374, Springer, 2015.
- M.N. Koleva, L.G. Vulkov, *J. Comp. Appl. Math.*, submitted.
- M.N. Koleva, L.G. Vulkov, *AIP CP*, accepted.

**Boundary conditions:**

$S_i \to 0,\ i = 1, 2, \dots, d$

$$C_t^k + rSC_S^k + \frac{1}{2}\mathrm{tr}(\overline{S}\Sigma_k\Sigma_k^T\overline{S}C_{SS}^k) + \frac{e^{-r(T-t)}}{\alpha}\left[\sum_{j=1}^{m}(e^{-\alpha\phi(C^k-C^j)} - 1)\lambda^{kj} - \frac{1}{2}z_k^2\right]$$

Change the variables $S_i^* = 1/S_i$, $S_i^* = 0$ $(S_i \to \infty)$, $i = 1, 2, \dots, d$

$$C_t^k - S^*(r\mathbf{1} - \mathcal{D}(\Sigma_k\Sigma_k^T))C_{S^*}^k + \frac{1}{2}\mathrm{tr}(\overline{S^*}\Sigma_k\Sigma_k^T\overline{S^*}C_{S^*S^*}^k)$$
$$+\frac{e^{-r(T-t)}}{\alpha}\left[\sum_{j=1}^{m}(e^{-\alpha\phi(C^k-C^j)} - 1)\lambda^{kj} - \frac{1}{2}z_k^2\right] = rC^k,$$

**1** - d-dimensional unit column vector, $\mathcal{D}(\Sigma_k\Sigma_k^T)$ is a d-dimensional column vector, generated from the main diagonal elements of $\Sigma_k\Sigma_k^T$

We solve the system with more general terminal condition $C^k(T,S) = C_0^k(S)$.

$\tau = T - t$, $x_i = \ln S_i \Rightarrow x_i \in (-\infty, \infty) \to D_i = (L_i^-, L_i^+)$, $L_i^- < 0$, $L_i^+ > 0$.

Define the function $\delta_i$, $i = 1, 2, \dots, d$, vector $\delta$ and diagonal matrix $\overline{\delta}$

$$\delta_i = \begin{cases} 0, & x_i = L_i^{\pm}, \\ 1, & \text{otherwise}, \end{cases} \qquad \delta_i^2 = \delta_i, \qquad \delta = (\delta_1, \delta_2, \dots, \delta_d), \qquad \overline{\delta} = \mathrm{diag}(\delta).$$

$$C_\tau^k - (r\mathbf{1} - \tfrac{1}{2}\mathcal{D}(\Sigma_k\Sigma_k^T))^T\overline{\delta}C_x^k - \tfrac{1}{2}\mathrm{tr}(\overline{\delta}\Sigma_k\Sigma_k^T\overline{\delta}C_{xx}^k) + rC^k = \frac{e^{-r\tau}}{\alpha}\left[\sum_{j=1}^m (e^{-\alpha\phi(C^k - C^j)} - 1)\lambda^{kj} - \tfrac{1}{2}z_k^2\right], \quad (\tau, x) \in Q_T,$$

where $Q_T = (0,T) \times D$, $D = \times_{i=1}^d D_i$, $\phi = \phi(\tau) = e^{r\tau}$, $\mu_k = \mu_k(\tau, e^x)$, $z_k = z_k(\tau, e^x)$, $\lambda^{kj} = \lambda^{kj}(\tau, e^x)$.

# 2D case

$\Sigma_k \Sigma_k^T = \{\sigma_{il}^k\}_{i,l=1}^{d,d} \geq 0$, denote $2\rho_{il}^k = \sigma_{il}^k + \sigma_{li}^k$ .

$$
\begin{aligned}
& C_\tau^k - \delta_1 \left( r - \frac{1}{2}\sigma_{11}^k \right) C_{x_1}^k - \delta_2 \left( r - \frac{1}{2}\sigma_{22}^k \right) C_{x_2}^k \\
& -\frac{1}{2}\left( \delta_1 \sigma_{11}^k C_{x_1 x_1}^k + 2\delta_1\delta_2 \rho_{12}^k C_{x_1 x_2}^k + \delta_2 \sigma_{22}^k C_{x_2 x_2}^k \right) + rC^k \\
& = \frac{e^{-r\tau}}{\alpha} \left[ \left( \sum_{j=1}^m e^{-\alpha\phi(C^k - C^j)} - 1 \right) \lambda^{kj} - \frac{1}{2} z_k^2 \right], \quad (\tau, x) \in (0, T] \times \overline{D},
\end{aligned}
$$

$$
C^k(0, x) = C_0^k(e^x), \quad x \in D.
$$

## Theorem

*Suppose that the functions* $C^k \in C(\overline{Q}_T) \cap C^{3,1}(Q_T)$, *satisfy in* $\overline{Q}_T$ *the problem* (21) *and assume that*

$$C^k(0, x) = C_0^k(x) \leq 0, \quad k = 1, \ldots, m.$$

*Then* $C^k(x) \leq 0$ *for* $(x, t) \in \overline{Q}_T$

# Numerical Method

$$(C^k)^{+}_{x_i x_s} = \frac{1}{2}[C^k_{x_i x_s} + C^k_{\overline{x}_i \overline{x}_s}], \ (C^k)^{-}_{x_i x_s} = \frac{1}{2}[C^k_{\overline{x}_i x_s} + C^k_{x_i \overline{x}_s}], \quad i, s = 1, 2, \dots, d.$$

$$A^k_i \frac{\partial C^k}{\partial x_i}(x_{[j]}) \simeq A^k_i \frac{\mathcal{C}^k_{[j]+e_i/2} - \mathcal{C}^k_{[j]-e_i/2}}{h_i} = A^k_i \mathcal{C}^k_{\mathring{x}_{i_{[j]}}}, \ A^k_i = r - \frac{1}{2}\sigma^k_{ii}, \ i = 1, 2, \dots, \tag{6}$$

To approximate $\mathcal{C}^k_{[j]\pm e_i/2}$, we apply van Leer flux limiter technique [A. Gerisch et al., 2001] in each space direction.

Using gradient ratios

$$\Theta^k_{[j]+e_i/2} = \frac{C^k_{x_{i[j]}}}{C^k_{\overline{x}_{i[j]}}}, \tag{7}$$

we define van Leer flux limiter [A. Gerisch et al., 2001, R.J. LeVeque, 1992]

$$\Phi(\Theta^k) = \frac{|\Theta^k| + \Theta^k}{1 + |\Theta^k|}. \tag{8}$$

$\Phi(\Theta^k)$ is Lipschitz continuous, continuously differentiable, $\Theta^k \neq 0$,

$$\Phi(\Theta^k) = 0, \quad \text{if} \ \ \Theta^k \leq 0 \ \ \text{and} \ \ \Phi(\Theta^k) \leq 2\min\{1, \Theta^k\}. \tag{9}$$

$$\mathcal{C}^k_{[j]+e_i/2} = C^k_{[j]} + \frac{1}{2}\Phi(\Theta^k_{[j]+e_i/2})(C^k_{[j]} - C^k_{[j]-e_i}). \tag{10}$$

$$\mathcal{C}^k_{[j]+e_i/2} = C^k_{[j]+e_i} + \frac{1}{2}\Phi((\Theta^k_{[j]+3e_i/2})^{-1})(C^k_{[j]+e_i} - C^k_{[j]+e_{i+1}}). \tag{11}$$

Using $\Phi(\Theta) = \Theta\Phi(\Theta^{-1})$ [D. Kusmin, S. Turek, 2004], we approximate $A^k_i C^k_{x_i}$ at point $(\tau, x_{[j]})$, in dependence of the sign of $A^k = (A^k)^+ - (A^k)^-$:

$$A^k_i \frac{\partial C^k}{\partial x_i} \simeq (A^k_i)^+(\Lambda^k_i)^+ C^k_{x_i} - (A^k_i)^-(\Lambda^k_i)^- C^k_{\overline{x}_i}, \tag{12}$$

$$(\Lambda^k_i)^+ = 1 + \tfrac{1}{2}\Phi((\Theta^k_{[j]+e_i/2})^{-1}) - \tfrac{1}{2}\Phi(\Theta^k_{[j]+3e_i/2}), \tag{13}$$

$$(\Lambda^k_i)^- = 1 + \tfrac{1}{2}\Phi(\Theta^k_{[j]+e_i/2}) - \tfrac{1}{2}\Phi((\Theta^k_{[j]-e_i/2})^{-1}), \tag{14}$$

where in view of (8), (9) we have

$$0 \le (\Lambda^k_i)^- \le 2 \quad \text{and} \quad 0 \le (\Lambda^k_i)^+ \le 2. \tag{15}$$

# Weighted ($\theta_1, \theta_2, \theta_3 \in [0,1]$) FD approximation

$$
\begin{aligned}
&C_t^k - \sum_{i=1}^{d} \delta_i \left[\theta_1 (A_i^k)^+ (\widehat{\Lambda}_i^k)^+ \widehat{C}_{x_i}^k - \theta_1 (A_i^k)^- (\widehat{\Lambda}_i^k)^- \widehat{C}_{\overline{x}_i}^k\right] - \frac{\theta_2}{2} \mathrm{tr}(\overline{\delta}\Sigma_k \Sigma_k^T \overline{\delta} \widehat{\mathbf{C}}_{xx}^k) + \theta_2 r \widehat{C}^k \\
&= \sum_{i=1}^{d} \delta_i \left[(1-\theta_1)(A_i^k)^+ (\Lambda_i^k)^+ C_{x_i}^k - (1-\theta_1)(A_i^k)^- (\Lambda_i^k)^- C_{\overline{x}_i}^k\right] \\
&+ \frac{1-\theta_2}{2} \mathrm{tr}(\overline{\delta}\Sigma_k \Sigma_k^T \overline{\delta} \mathbf{C}_{xx}^k) - (1-\theta_2) r C^k \\
&+ \theta_3 \frac{e^{-r\tau^{n+1}}}{\alpha} \left[\sum_{j=1}^{m} (e^{-\alpha \widehat{\phi}(\widehat{C}^k - \widehat{C}^j)} - 1)\widehat{\lambda}^{kj} - \frac{1}{2}\widehat{z}_k^2\right] \\
&+ (1-\theta_3) \frac{e^{-r\tau^{n}}}{\alpha} \left[\sum_{j=1}^{m} (e^{-\alpha \phi(C^k - C^j)} - 1)\lambda^{kj} - \frac{1}{2} z_k^2\right], \quad (\tau^{n+1}, x_{[j]}) \in Q_T^h, \\
&C^k(0, x_{[j]}) = C_0^k(e^{x_{[j]}}), \quad x_{[j]} \in \overline{\omega}_h.
\end{aligned}
$$

$$
\mathbf{C}_{xx} = \begin{bmatrix}
C_{\overline{x}_1 x_1} & \varrho_{12}^+ C_{x_1 x_2}^+ & \dots & \varrho_{1d}^+ C_{x_1 x_d}^+ \\
\varrho_{12}^+ C_{x_2 x_1}^+ & C_{\overline{x}_2 x_2} & \dots & \varrho_{2d}^+ C_{x_2 x_d}^+ \\
\dots & \dots & \dots & \dots \\
\varrho_{1d}^+ C_{x_d x_1}^+ & \varrho_{2d}^+ C_{x_d x_2}^+ & \dots & C_{\overline{x}_d x_d}
\end{bmatrix}
-
\begin{bmatrix}
0 & \varrho_{12}^- C_{x_1 x_2}^- & \dots & \varrho_{1d}^- C_{x_1 x_d}^- \\
\varrho_{12}^- C_{x_2 x_1}^- & 0 & \dots & \varrho_{2d}^- C_{x_2 x_d}^- \\
\dots & \dots & \dots & \dots \\
\varrho_{1d}^- C_{x_d x_1}^- & \varrho_{2d}^- C_{x_d x_2}^- & \dots & 0
\end{bmatrix},
$$

and $\varrho_{il}^{\pm} = \rho_{il}^{\pm} / |\rho_{il}|$.

We use the following Newton-like linearization for the nonlinear term

$$e^{-\alpha\widehat{\phi}\widehat{v}^k} = e^{-\alpha\widehat{\phi}v^k} - \alpha\widehat{\phi}e^{-\alpha\widehat{\phi}v^k}(\widehat{v}^k - v^k) + \mathcal{O}[(\triangle\tau)^2], \quad v^k := C^k - C^j. \quad (16)$$

In the case of two asset ($d = 2$):

$$C_t^k - \theta_1[\delta_1(A_1^k)^+(\widehat{\Lambda}_1^k)^+\widehat{C}_{x_1}^k - \delta_1(A_1^k)^-(\widehat{\Lambda}_1^k)^-\widehat{C}_{\overline{x}_1}^k + \delta_2(A_2^k)^+(\widehat{\Lambda}_2^k)^+\widehat{C}_{x_2}^k - \delta_2(A_2^k)^-(\widehat{\Lambda}_2^k)^-\widehat{C}_{\overline{x}_2}^k]$$
$$+r\theta_2\widehat{C}^k - \frac{1}{2}\theta_2\left(\delta_1\sigma_{11}^k\widehat{C}_{\overline{x}_1x_1}^k + 2\delta_1\delta_2(\rho_{12}^k)^+(\widehat{C}^k)_{x_1x_2}^+ - 2\delta_1\delta_2(\rho_{12}^k)^-(\widehat{C}^k)_{x_1x_2}^- + \delta_2\sigma_{22}^k\widehat{C}_{\overline{x}_2x_2}^k\right)$$
$$+\theta_3\sum_{j=1}^m e^{-\alpha\widehat{\phi}(C^k-C^j)}(\widehat{C}^k - \widehat{C}^j)\widehat{\lambda}^{kj}$$
$$= (1-\theta_1)[\delta_1(A_1^k)^+(\Lambda_1^k)^+C_{x_1}^k - \delta_1(A_1^k)^-(\Lambda_1^k)^-C_{\overline{x}_1}^k + \delta_2(A_2^k)^+(\Lambda_2^k)^+C_{x_2}^k - \delta_2(A_2^k)^-(\Lambda_2^k)^-C_{\overline{x}_2}^k]$$
$$+\frac{1}{2}(1-\theta_2)\left(\delta_1\sigma_{11}^kC_{\overline{x}_1x_1}^k + 2\delta_1\delta_2(\rho_{12}^k)^+(C^k)_{x_1x_2}^+ - 2\delta_1\delta_2(\rho_{12}^k)^-(C^k)_{x_1x_2}^- + \delta_2\sigma_{22}^kC_{\overline{x}_2x_2}^k\right)$$
$$-r(1-\theta_2)C^k + \mathcal{F}_2^k(C^1, C^2, \dots, C^m), \quad k = 1, 2,$$

$$\mathcal{F}_m^k = (1-\theta_3)\frac{e^{-r\tau^n}}{\alpha}\left[\sum_{j=1}^m(e^{-\alpha\phi(C^k-C^j)} - 1)\lambda^{kj} - \frac{1}{2}z_k^2\right]$$
$$+\theta_3\frac{e^{-r\tau^{n+1}}}{\alpha}\left[\sum_{j=1}^m[(1+\alpha\widehat{\phi}(C^k - C^j))e^{-\alpha\widehat{\phi}(C^k-C^j)} - 1]\widehat{\lambda}^{kj} - \frac{1}{2}\widehat{z}_k^2\right].$$

# Numerical analysis

## Lemma

*If the following conditions are fulfilled*

$$\frac{|\rho_{12}^k|}{\sigma_{22}^k} \leq \frac{h_1}{h_2} \leq \frac{\sigma_{11}^k}{|\rho_{12}^k|}, \qquad (17)$$

*then, the coefficient matrix* $\mathbf{M}$ *is an M-matrix.*

## Remark

*Note that* $0 < |\rho_{12}^k|/\sigma_{22}^k < 1$, $\sigma_{11}^k/|\rho_{12}^k| > 1$ *and therefore the condition* (17) *is not restrictive. For example, if* $h_1 = h_2$, *then* (17) *is always satisfied.*

# Numerical analysis

## Lemma

*If the following conditions are fulfilled*

$$\frac{|\rho_{12}^k|}{\sigma_{22}^k} \leq \frac{h_1}{h_2} \leq \frac{\sigma_{11}^k}{|\rho_{12}^k|}, \tag{17}$$

*then, the coefficient matrix* $\mathbf{M}$ *is an* $M$*-matrix.*

## Remark

*For the multi-dimensional problem, the restrictions* (17) *become:*

$$\frac{|\rho_{il}^k|}{\sigma_{ll}^k} \leq \frac{h_i}{h_l} \leq \frac{\sigma_{il}^k}{|\rho_{il}^k|}, \quad i, l = 1, 2, \ldots, d, \quad k = 1, 2, \ldots, m. \tag{18}$$

## Theorem (Negativity preserving)

*Let the conditions of Lemma 1 are fulfilled,* $C_0^k \le 0$, $k = 1, 2$, $[z_1^2, z_2^2] \neq [0, 0]$ *and*

$$\triangle\tau^n \le \frac{h_1 h_2}{2(1-\theta_1)(|A_1^k|h_2+|A_2^k|h_1)+(1-\theta_2)(2\sigma_{11}^k\sigma_{22}^k/\rho_{12}^k+rh_1h_2)+h_1h_2\|\mathcal{N}^k(\check{C}^1,\check{C}^2)\|},$$

$$\mathcal{N}^k(C^1, C^2) = \frac{e^{-r\tau^n}}{\alpha}\left[\sum_{\substack{j=1\\ j\neq k}}^{2}\left(e^{-\alpha\widehat{\phi}(C^k-C^j)^-} - 1\right)^+ \widetilde{\lambda}^{kj} - \widetilde{z}_k^2\right]^+ \oslash (\check{C}^k)^-_, \quad k = 1, 2,$$

*where* $\widetilde{\lambda}_J^{kj} = \max\{\lambda_J^{kj}, \widehat{\lambda}_J^{kj}\}$, $\widetilde{z}_{k_J}^2 = \min\{z_{k_J}^2, \widehat{z}_{k_J}^2\}$. *Then the numerical scheme is negativity preserving.*

$\|\cdot\|$ - maximal discrete norm, $\oslash$ - the element-wise division of vectors.

## Remark

*In the multi-dimensional case for an arbitrary number m, the restriction for k=1,2,...,m, is*

$$\triangle\tau^n \leq \frac{\prod_{i=1}^{d} h_i}{2(1-\theta_1)\sum_{i=1}^{d}\Big(|A_i^k|\prod_{\substack{l=1\\ l\neq i}}^{d} h_l\Big)+(1-\theta_2)\Big(d\max_{1\leq i,l\leq d}\{\sigma_{ii}^k\sigma_{ll}^k/(\rho_{il}^k h_i h_l)\}+r\Big)\prod_{i=1}^{d} h_i+\prod_{i=1}^{d} h_i\|\mathcal{N}^k(\cdot)\|},$$

$$\mathcal{N}^k(\check{C}^1,\dots,\check{C}^m)=\frac{e^{-r\tau^n}}{\alpha}\left[\sum_{j=1}^m\left(e^{-\alpha\widehat{\phi}(\check{C}^k-\check{C}^j)^-}-1\right)^+\widetilde{\lambda}^{kj}-\widetilde{z}_k^2\right]^+\oslash(\check{C}^k)^-.$$

### Theorem (Convergence)

*Let the conditions of Lemma 1 are fulfilled and*

$$\triangle\tau^n \leq \frac{h_1 h_2}{2(1-\theta_1)(|A_1^k|h_2 + |A_2^k|h_1) + (1-\theta_2)(2\sigma_{11}^k\sigma_{22}^k/\rho_{12}^k + rh_1h_2)}. \tag{19}$$

*Suppose that there exist a classical solution* $C^k(\tau, x) \in C^{2,4}(Q_T)$, $k = 1, 2$ *of* (**??**), *where* $C^k$ *is monotone with respect to each space variable, then*

$$\|\widehat{\mathbf{C}} - [\widehat{\mathbf{C}}]\| \leq \mathcal{O}\left(\left(\frac{1}{2} - \theta\right)\triangle\tau + (\triangle\tau)^2 + |h|^2\right). \tag{20}$$

# Multi-dimensional case

## Remark

*The same convergence result can be established for and $m > 2$, if instead of* (19) *we require*

$$\triangle\tau^n \leq \prod_{i=1}^{d} h_i \left[ 2(1-\theta_1) \sum_{i=1}^{d} \left( |A_i^k| \prod_{\substack{l=1 \\ l \neq i}}^{d} h_l \right) + (1-\theta_2) \left( d \max_{1 \leq i,l \leq d} \{ \sigma_{ii}^k \sigma_{ll}^k / (\rho_{il}^k h_i h_l) \} + r \right) \prod_{i=1}^{d} h_i \right]^{-1}.$$

# Numerical implementation

• Picard-Newton iteration process. $\Lambda$ is computed at old iteration, while for the exponential non-linearity we apply Newton iterations.

• Decoupling. At each iteration $p$, the discrete equations from the system are solved separately and consecutively.

Find $(C^k)^{(p+1)}$ from the $k$-th discrete equation. In $e^{-\alpha\widehat{\phi}(C^k - C^j)}$, we set $(C^j)^{(p+1)} := (C^j)^{(p)}$ for $j > k$, while for $(C^j)^{(p+1)}$, $j < k$, we use the updated values of the solution.

# Computational results: $m = 2$, $d = 2$

Table : $E_h^k$, $CR^k$, iter and CPU time for Crank-Nicolson scheme *with decoupling*, $\triangle\tau = h$, Example 2

| N | $E_h^1$ | $CR^1$ | $E_h^2$ | $CR^2$ | iter | CPU |
|---|---|---|---|---|---|---|
| 41 | 1.701e-3 | | 2.371e-3 | | 3.350 | 1.368 |
| 81 | 4.238e-4 | 2.0054 | 5.877e-4 | 2.0123 | 3.000 | 11.149 |
| 161 | 1.056e-4 | 2.0051 | 1.459e-4 | 2.0097 | 2.875 | 83.228 |
| 321 | 2.63e-5 | 2.0039 | 3.631e-5 | 2.0067 | 2.794 | 741.501 |
| 641 | 6.57e-6 | 2.0026 | 9.051e-6 | 2.0044 | 2.672 | 9262.832 |

Table : $E_h^k$, $CR^k$ and CPU time for implicit-explicit scheme, *Richardson extrapolation*, $\triangle\tau = h$, Example 2

| N | $E_h^1$ | $CR^1$ | $E_h^2$ | $CR^2$ | CPU |
|---|---|---|---|---|---|
| 41 | 2.557e-3 | | 1.387e-3 | | 1.012 |
| 81 | 6.486e-4 | 1.9789 | 3.508e-4 | 1.9832 | 9.105 |
| 161 | 1.632e-4 | 1.9903 | 8.824e-5 | 1.9911 | 73.727 |
| 321 | 4.095e-5 | 1.9952 | 2.213e-5 | 1.9952 | 692.560 |
| 641 | 1.025e-5 | 1.9976 | 5.542e-6 | 1.9976 | 7631.590 |

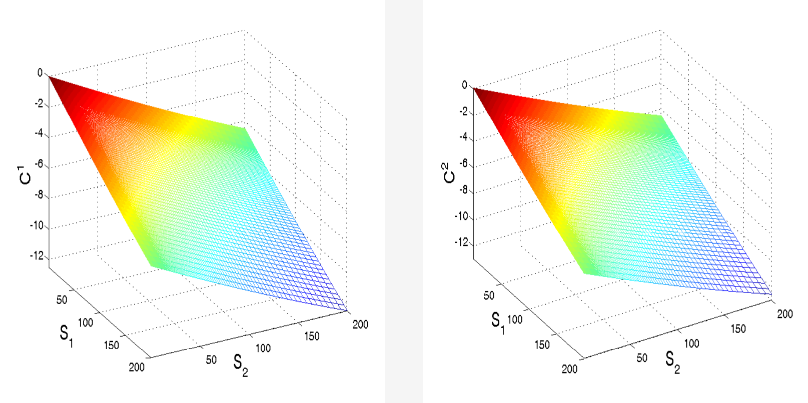


Figure : Numerical solutions $C^1$ (*left*) and $C^2$ (*right*), $N = 321$, $T = 10$ and $\triangle\tau = h$, $1/200 \leq S_1 \leq 200$, $1/200 \leq S_2 \leq 200$, IMEXS+R, Example 2

# *Model problem*: Optimal Portfolio in a Power Utility Regime-Switching Model

Power exponential utility function $U(y) = y^\gamma/\gamma$, $\gamma < 1$, $\gamma \neq 0$, $y \in \mathbb{R}$, [Valdez, Vargiolu, 2013].

Value functions $V^k(t,s,y)$, $k = 1, 2, \dots m$, given by $V^k = (ye^{c^k})^\gamma/\gamma$,

$s = (s_1, s_2, \dots, s_d) \subset \mathbb{R}^d$ - vector of stock prices $s_i \in [0,\infty)$, $i = 1, \dots, d$, $t \in [0, T]$ is a time variable and $c^k(t,s)$:

$$c_t^k + \frac{1}{2}\mathrm{tr}(\bar{s}\Sigma_k\Sigma_k^T\bar{s}c_{ss}^k) + (\mu_k + \frac{\gamma}{1-\gamma}(\mu_k - r\mathbf{1}))\bar{s}c_s^k + \frac{1}{\gamma}\sum_{j=1}^{m}(e^{-\gamma(c^k - c^j)} - 1)\lambda^{kj}$$

$$+\frac{1}{2(1-\gamma)}z_k^2 + \frac{\gamma}{2(1-\gamma)}\|\bar{s}c_s^k\Sigma_k\|^2 + r = 0,$$

$$c^k(T) = 0.$$

If $\mu_k(t)$, $\Sigma_k(t)$ and $\lambda^{kj}(t)$ - system of ODE and in [Valdez, Vargiolu, 2013] is proved existence of unique classical local solution $c^k \in C_b^{1,2}([0,T], \mathbb{R}^m)$.

Consider $d = 1$ , $x = \ln s$, $\tau = T - t$.

The new space interval $x \in (-\infty, \infty) \to [-L, L]$, $L > 0$.

$$c_\tau^k - \frac{1}{2}(\sigma^k)^2 c_{xx}^k - \left(\mu_k + \frac{\gamma}{1-\gamma}(\mu_k - r) - \frac{1}{2}\sigma^k\right) c_x^k - \frac{1}{\gamma}\sum_{j=1}^{m}(e^{-\gamma(c^k - c^j)} - 1)\lambda^{kj}$$

$$-\frac{1}{2(1-\gamma)} z_k^2 - \frac{\gamma}{2(1-\gamma)}(\sigma^k c_x^k)^2 - r = 0, \ (\tau, x) \in Q_T, \qquad (21)$$

$$c_t^k - \frac{1}{\gamma}\sum_{j=1}^{m}(e^{-\gamma(c^k - c^j)} - 1)\lambda^{kj} - \frac{1}{2(1-\gamma)} z_k^2 - r = 0, \ x = \{-L, L\}, \ \tau \in (0, T],$$

$$c^k(0, x) = 0, \quad x \in [-L, L].$$

$$Q_T = (0, T] \times (-L, L)$$

## Theorem

*Let* $\mathfrak{m} = \min_k m_k$, $m_k \leq z_k^2(\tau, x)$, $(\tau, x) \in \overline{Q}_T$ *and suppose that the functions* $c^k \in C(\overline{Q}_T) \cap C^{1,3}(Q_T)$ *satisfy in* $Q_T$ *the problem* (21). *Assume that*

$$c^k(0, x) \geq 0, \quad k = 1, 2, \ldots, m.$$

*Then,*

$$c^k(\tau, x) \geq \left( \frac{\mathfrak{m}}{2(1-\gamma)} + r \right) \tau, \quad k = 1, 2, \ldots, m.$$

# Numerical Method

We construct implicit-explicit numerical scheme, taking into account the sign of $A^k$, $C^k_{\mathring{x}}$ and $\gamma$

$$C^k_{t_i} - \frac{1}{2}(\sigma^k)^2 \widehat{C}^k_{\overline{x}x_i} = (A^k_i)^+(\Lambda^k_i)^+ C^k_{x_i} - (A^k_i)^-(\Lambda^k_i)^- C^k_{\overline{x}_i} + \frac{1}{\gamma}\sum_{j=1}^{m}(e^{-\gamma(C^k_i - \widetilde{C}^j_i)} - 1)$$

$$+\frac{\gamma^+}{2(1-\gamma^+)}(\sigma^k_i C^k_{\mathring{x}_i})^2 + \frac{\gamma^-(\sigma^k_i)^2}{2(1+\gamma^-)} \underbrace{\left((C^k_{\mathring{x}_i})^-(\Lambda^k_i)^+ C^k_{x_i} - (C^k_{\mathring{x}_i})^+(\Lambda^k_i)^- C^k_{\overline{x}_i}\right)}_{C_x C_x} + r,$$

$$C^k_{t_i} = \frac{1}{\gamma}\sum_{j=1}^{m}(e^{-\gamma(C^k_i - \widetilde{C}^j_i)} - 1)\lambda^{kj}_i + \frac{1}{2(1-\gamma)}\widehat{z}^2_{k_i} + r, \quad i = \{1, N\},$$

$$C^k_i = 0, \quad \tau = 0, \quad i = 1, 2, \dots, N,$$

where $\widetilde{C}^j = C^j$ or $\widetilde{C}^j = \widehat{C}^j$, depending (as will become clear below) on $k$.

# Preserving the qualitative properties

$$\beta^k = [\beta_1^k, \beta_2^k, \dots, \beta_N^k] \,\forall\, n{+}1, \;\; \beta_i^k = \begin{cases} 0, & C_i^k = \alpha, \quad \alpha = \left(\frac{\mathfrak{m}}{2(1-\gamma)} + r\right)\tau, \\ 1, & \text{otherwise.} \end{cases}$$

## Theorem

*Let* $\gamma > 0$. *Then at each lime level* $C^k \geq \left(\frac{\mathfrak{m}}{2(1-\gamma)} + r\right)\tau\mathbf{1}$, *if*

$$\triangle\tau^n \leq \frac{\gamma h}{2\gamma\|\beta^k.A^k\| + h\|\mathcal{N}^{kj}\|}, \quad n > 0,$$

$$\mathcal{N}_i^{kj} = \begin{cases} \frac{1}{C_i^k-\alpha}\left(\sum_{j=1}^{k-1}(1 - e^{-\gamma(C_i^k - \widehat{C}_i^j)^+})\lambda_i^{kj} + \sum_{j=k+1}^{m}(1 - e^{-\gamma(C_i^k - C_i^j)^+})\lambda_i^{kj}\right), & C_i^k > \alpha, \\ 0, & \textit{otherwise}, \end{cases}, i$$

## Theorem

*Let* $\gamma < 0$. *Then, if*

$$\triangle\tau^n \le \frac{\gamma^-(1+\gamma^-)h}{2\gamma^-(1+\gamma^-)\|\beta^k.A^k\| + (\gamma^-\sigma^k)^2\|\beta^k.C^k_{\mathring{x}}\| + h(1+\gamma^-)\|\mathcal{P}^{kj}\|}, \quad n > 0,$$

$$\mathcal{P}^{kj}_i = \begin{cases} \frac{1}{C^k_i-\alpha}\left(\sum_{j=1}^{k-1}(e^{\gamma^-(C^k_i-\widehat{C}^j_i)^+}-1)\lambda^{kj}_i + \sum_{j=k+1}^{m}(e^{\gamma^-(C^k_i-C^j_i)^+}-1)\lambda^{kj}_i\right), & C^k_i > \alpha, \\ 0, & \textit{otherwise}, \end{cases}, i$$

*for the numerical solution of* (22) *at each time level we have* $C^k \ge \left(\frac{\mathfrak{m}}{2(1-\gamma)} + r\right)\tau\mathbf{1}$.

# Numerical simulations

Test problem: System (21), $m = 3$.

$$r = 0.01, \quad \lambda^{kj} = 0.5(k+j)(T-t)/(s+1),$$
$$(\sigma^1)^2 = -0.1 + 0.6(0.35(s+0.02) + \sqrt{(s+0.02)^2 + 0.1})$$
$$[J.Gatheraletal., QuantitativeFinance, 2011],$$
$$(\sigma^2)^2 = 0.005, \quad (\sigma^3)^2 = 0.001, \quad \mu_k = 0.1(T-t)\sqrt{s}/k, \quad T = 1.$$

Exact solution test. Errors ($E^k$) in maximal discrete norm (denoted by $\|\cdot\|_h$) and convergence rate ($CR^k$)

$$E_h^k = \|C^k - [c^k]\|_h, \quad CR^k = \log_2 \frac{E_{2h}^k}{E_h^k}.$$

Table : $E_h^k$ and $CR^k$, $\gamma = 0.5$

| $N$ | $E_h^1$ | $CR^k$ | $E_h^2$ | $CR^2$ | $E_h^3$ | $CR^3$ |
|---|---|---|---|---|---|---|
| 81 | 1.45252e-3 | | 1.76734e-3 | | 1.76308e-3 | |
| 161 | 3.63335e-4 | 1.9992 | 4.44850e-4 | 1.9902 | 4.62349e-4 | 1.9310 |
| 321 | 9.08369e-5 | 2.0000 | 1.09657e-4 | 2.0203 | 1.17081e-4 | 1.9815 |
| 641 | 2.27103e-5 | 1.9999 | 2.68653e-5 | 2.0292 | 2.86694e-5 | 2.0299 |
| 1281 | 5.67785e-6 | 1.9999 | 6.59837e-6 | 2.0256 | 6.91900e-6 | 2.0509 |

Table : $E_h^k$ and $CR^k$, $\gamma = -3$

| $N$ | $E_h^1$ | $CR^k$ | $E_h^2$ | $CR^2$ | $E_h^3$ | $CR^3$ |
|---|---|---|---|---|---|---|
| 81 | 1.11939e-3 | | 5.94821e-4 | | 9.51074e-4 | |
| 161 | 2.80440e-4 | 1.9970 | 1.48471e-4 | 2.0023 | 2.37922e-4 | 1.9991 |
| 321 | 7.01799e-5 | 1.9986 | 3.71169e-5 | 2.0000 | 5.94893e-5 | 1.9998 |
| 641 | 1.75551e-5 | 1.9992 | 9.27886e-6 | 2.0001 | 1.48737e-5 | 1.9999 |
| 1281 | 4.39017e-6 | 1.9995 | 2.31971e-6 | 2.0000 | 3.71863e-6 | 1.9999 |

# Qualitative behavior of the solution

$L = \ln 200$ and restore the solution in the original variables $(t, s)$.

At each space-time grid node from $\overline{\omega}_h \times \omega_\tau$, this solution satisfies the inequality $C^k \geq \left(\frac{\mathfrak{m}}{2(1-\gamma)} + r\right)(T - \tau)$.

The time step is chosen in agrement with restrictions in Theorems 1,2

Let

$$Dif^k = C^k - \left(\frac{\mathfrak{m}}{2(1-\gamma)} + r\right)(T - \tau)\mathbf{1}$$

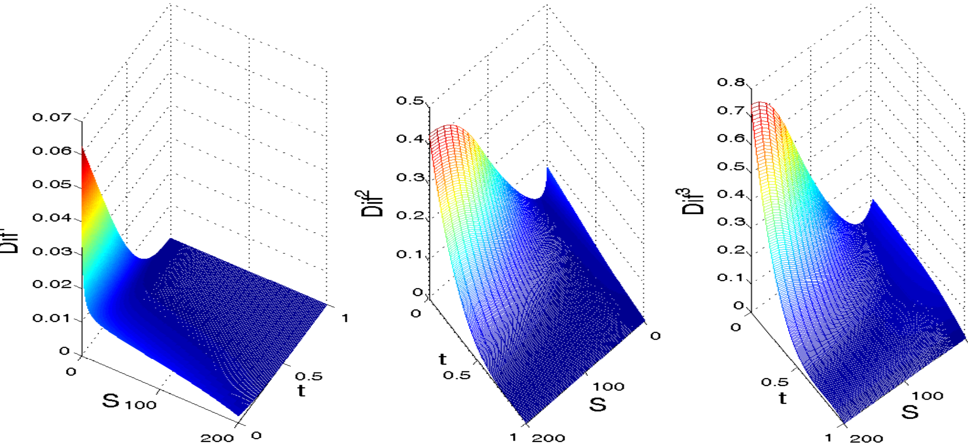


Figure : $Dif^1$ (*left*), $Dif^2$ (*center*), $Dif^3$ (*right*), $\gamma = 0.5$, $N = 161$