A gentle introduction to game theory

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Contents

Cooperative games

- Cooperative games vs. non cooperative games,
- Nash program,
- Transferable utility cooperative games (TU-games),
- Non transferable utility cooperative games (NTU-games).

TU-games

Definition

The nonempty finite set of the players is denoted by N.
A transferable utility (TU) game, henceforth TU-game v, with the set of players N, is a function $v\colon \mathcal{P}(N) \to \mathbb{R}$ such that $v(\emptyset) = 0$.
Let denote $\mathcal{G}^N$ the class of TU-games with player set N.

Example

- Let $N = \{1, 2, 3\}$,
- $v(\{1\}) = 0$, $v(\{2\}) = -1$ and $v(\{3\}) = 2$,
- $v(\{1, 2\}) = 1$, $v(\{1, 3\}) = 3$ and $v(\{2, 3\}) = 1$,
- $v(N) = v(\{1, 2, 3\}) = 4$.

Solution, value

Definition

A mapping ψ is a solution on the set of TU-games $A \subseteq \mathcal{G}^N$, if $\psi|: A \to \mathcal{P}(\mathbb{R}^N)$.

Definition

A ψ solution on the set of TU-games $A \subseteq \mathcal{G}^N$ is a value, if it is single valued, that is for all $v \in A$ it holds that $|\psi(v)| = 1$.

Core

Definition

Take a TU-game $v \in \mathcal{G}^N$. Then the core (Shapley, 1955) of the game v is defined as follows:

$$\text{core}\,(v) = \{x \in \mathbb{R}^N \colon x(N) = v(N) \text{ and } x(S) \geq v(S),\ S \subseteq N\}\,.$$

Example

Consider the following game: Let $T \subseteq N$, $T \neq \emptyset$, and for each $S \subseteq N$, let

$$u_T(S) \circeq \begin{cases} 1, & \text{if } T \subseteq S \\ 0 & \text{otherwise} \end{cases} .$$

Then the TU-game u_T is called unanimity game on coalition T.

$\mathrm{core}(u_T) = \mathrm{conv}\{x \in \mathbb{R}^N\colon$ there exists $i \in T$ such that $x_i = 1$ and $x_j = 0,\ j \neq i\}$.

Balanced set system

Definition

A sets system $\mathcal{B} \subseteq \mathcal{P}(N) \setminus \{\emptyset\}$ is a balanced set system, if there exists a vector $\lambda \in \mathbb{R}^{\mathcal{B}}_{++}$ such that

$$\sum_{S \in \mathcal{B}} \lambda_S \chi_S = \chi_N \,.$$

The weights $(\lambda_S)_{S \in \mathcal{N}}$ are called balancing weights.

Balanced games

Definition

A TU-game $v \in \mathcal{G}^N$ is balanced, if

$$\max_{\mathcal{B} \text{ is a balanced set system}} \sum_{S \in \mathcal{B}} \lambda_S v(S) \leq v(N)\,.$$

The Bondareva-Shapley Theorem

Theorem (Bondareva (1963), Shapley (1967))

A TU-game $v \in \mathcal{G}^N$ is balanced, if and only if, $\mathrm{core}(v) \neq \emptyset$, that is, if and only if the TU-game does have nonempty core.

Marginal contribution

Definition

Take a TU-game $v \in \mathcal{G}^N$. Then player $i \in N$ marginal contribution to coalition $S \subseteq N$ in the game v is $v_i'(S) = v(S \cup \{i\}) - v(S)$. Let v_i' denote the marginal contribution of player i in the TU-game v, that is, $v_i' = (v_i'(S))_{S \in \mathcal{P}(N)}$.

Shapley value

Definition (Shapley (1953))

Let $v \in \mathcal{G}^N$ and

$$p^i_{Sh}(S) = \begin{cases} \dfrac{|S|!(|N \setminus S| - 1)!}{|N|!}, & \text{if } i \notin S \\ 0 & \text{otherwise} \end{cases}.$$

Then $\phi_i(v)$, the Shapley value of player i in the TU-game v is the p^i_{Sh} expected value of v'_i. In other words,

$$\phi_i(v) = \sum_{S \subseteq N} v'_i(S)\; p^i_{Sh}(S)\,.$$

Example I

Let $N = \{1, 2, 3\}$, and let $v \in \mathcal{G}^N$ be as follows:

$$\begin{array}{lll} v(\{1\}) = 1, & v(\{2\}) = 0, & v(\{3\}) = 2, \\ v(\{1,2\}) = 2, & v(\{1,3\}) = 2, & v(\{2,3\}) = 3, \\ v(\{1,2,3\}) = 5. & & \end{array}$$

Orders:

$$\begin{array}{cccccc} 1 & 1 & 2 & 2 & 3 & 3 \\ 2 & 3 & 1 & 3 & 1 & 2 \\ 3 & 2 & 3 & 1 & 2 & 1 \end{array}$$

Marginal contributions:

$$\begin{array}{cccccc} 1 & 1 & 2 & 2 & 0 & 2 \\ 1 & 3 & 0 & 0 & 3 & 1 \\ 3 & 1 & 3 & 3 & 2 & 2 \end{array}$$

Example II

The Shapley values:

$$\phi_1(v) = \frac{1}{6}\big(1 + 1 + 2 + 2 + 0 + 2\big) = \frac{8}{6},$$
$$\phi_2(v) = \frac{1}{6}\big(1 + 3 + 0 + 0 + 3 + 1\big) = \frac{8}{6},$$
$$\phi_3(v) = \frac{1}{6}\big(3 + 1 + 3 + 3 + 2 + 2\big) = \frac{14}{6}.$$

Equivalence of players

Definition

Players $i, j \in N$ are equivalent in a TU-game $v \in \mathcal{G}^N$, $i \sim^v j$, if for all $S \subseteq N \setminus \{i, j\}$ it holds that $v_i'(S) = v_j'(S)$.

Axioms I

A value ψ on the set of games $A \subseteq \mathcal{G}^N$ is / satisfies

- Pareto optimal (PO), if for each $v \in A$, $\sum_{i \in N} \psi_i(v) = v(N)$,
- Null-player Property (NP), if for all $v \in A$, $i \in N$, $v_i' = 0$ implies $\psi_i(v) = 0$,
- Equal Treatment Property (ETP), if for all $v \in A$, $i, j \in N$, $i \sim^v j$ implies $\psi_i(v) = \psi_j(v)$,
- Additive (ADD), if for all $v, w \in A$ such that $v + w \in A$, $\psi(v + w) = \psi(v) + \psi(w)$,

Shapley's axiomatization of the Shapley value

Theorem (Shapley (1953))

A value ψ defined on the class of TU-games $\mathcal{G}^N$ meets axioms PO, NP, ETP and ADD if and only if it is the Shapley value.

A tuple $\Gamma = (N, \{A_i\}_{i \in N}, \{f_i\}_{i \in N})$ is a noncooperative game in normal form, if

- N is the nonempty players set,
- A_i is the nonempty set of Player i's actions,
- $f_i \colon A \to \mathbb{R}$ is the payoff function of Player i,

where $A = \times_{i \in N} A_i$.

The game of chicken

		Player 2	
		Swerve	*Straight*
Player 1	*Swerve*	(6,6)	(2,7)
	Straight	(7,2)	(0,0)

The prisoner's dilemma

		Prisoner 2	
		Cooperate	*Defeat*
Prisoner 1	*Cooperate*	(-2,-2)	(-10,-1)
	Defeat	(-1,-10)	(-5,-5)

Nash equilibrium

Definition

An action profile $a^* \in A$ is a Nash equilibrium of the game $(N, \{A_i\}_{i \in N}, \{f_i\}_{i \in N})$, if for all $i \in N$ and $a_i \in A_i$ we have

$$f_i(a_i, a^*_{-i}) \leq f_i(a^*)\,.$$

Mixed extension

Definition

The mixed extension of a game $(N, \{A_i\}_{i \in N}, \{f_i\}_{i \in N})$ is a game $(N, \{\Delta(A_i)\}_{i \in N}, \{\hat{f}_i\}_{i \in N})$, where $\Delta(X)$ is the set of probability distributions over set X, and $\hat{f}_i(\hat{a}) = \int f_i \, \mathrm{d}\hat{a}$, $\hat{a} \in \times_{i \in N} \Delta(A_i)$.

Minimax theorem

Theorem (von Neumann (1928))

Take a matrix game A. Then

$$\max_{x\in\Delta^n}\min_{y\in\Delta^m} x^\top Ay = \min_{y\in\Delta^m}\max_{x\in\Delta^n} x^\top Ay\,,$$

where Δ^d is the $d-1$-dimensional unit simplex, and A is an $n\times m$ matrix.
Moreover, (x^*, y^*) for which the equality holds is a solution of the mixed extension of the matrix game, and $v=\max_{x\in\Delta^n}\min_{y\in\Delta^m} x^\top Ay = \min_{y\in\Delta^m}\max_{x\in\Delta^n} x^\top Ay$ is called the value the matrix game A.

The existence of Nash equilibrium

Theorem (Nash (1950, 1951))

The mixed extension of a finite game in normal form has a Nash equilibrium.

1st version of Rationalizability I

Given a finite game in normal form $(N, \{A_i\}_{i \in N}, \{f_i\}_{i \in N})$. An action of Player i $a_i \in A_i$ is rationalizable, if there exists

- a collection of sets $\{\{X_j^t\}_{j \in N}\}^{t \in \mathbb{N}}$ with $X_j^t \subseteq A_j$ for all j and t,
- a belief of player i $p_i^1 \in \Delta(A_{-i})$,
- for all $j \in N$, $t \in \mathbb{N}$ and $a_j \in X_j^t$ we have a belief of player j $p_j^{t+1}(a_j)$ such that its support is a subset of X_{-j}^t

such that

- a_i is a best response to the belief of player i p_i^1,
- $X_i^1 = \emptyset$ and for each $j \in N \setminus \{i\}$ the set X_j^1 is the set of all $a_j' \in A_j$ such that there is some a_{-i} in the support of p_i^1 for which $a_j' = (a_{-i})_j$,

1st version of Rationalizability II

- for every player $j \in N$ and every $t \geq 1$ every action $a_j \in X_j^t$ is a best response to the belief of player j $p_j^{t+1}(a_j)$,
- for each $t \geq 2$ and each $j \in N$ the set X_j^t is the set of all $a_j' \in A_j$ such that there is some player $k \in N \setminus \{j\}$, some action $a_k \in X_k^{t-1}$, and some a_{-k} in the support of $p_k^t(a_k)$ for which $a_j' = (a_{-k})_j$.

2nd version of Rationalizability

Definition

Given a finite game in normal form $(N, \{A_i\}_{i \in N}, \{f_i\}_{i \in N})$. An action of Player i $a_i \in A_i$ is rationalizable, if for each $j \in N$ there is a set $Z_j \subseteq A_j$ such that

- $a_i \in Z_i$,
- every action $a_j \in Z_j$ is a best response to a belief of player j $p_j(a_j)$ such that its support is a subset of Z_{-j}.

The equivalence of the two versions

Theorem

The two versions of rationalizability are equivalent.

Game of chicken

Consider the game of chicken

		Player 2	
		Swerve	*Straight*
Player 1	*Swerve*	(6,6)	(2,7)
	Straight	(7,2)	(0,0)

Then action Swerve is rationalizable for both players.

Repeated games I

Consider a finite normal form game $\Gamma = \{N, \{A_i\}_{i \in N}, \{f_i\}_{i \in N}\}$ as a base game (stage game). Suppose that the game is played repeatedly $T \in \mathbb{N} \cup \{\infty\}$ times (we consider only discrete time models). Then the repeated game (supergame or iterated game) is a normal form game, where

- N is the players set,
- The history at t is $h_t = (a^0, \dots, a^{t-1})$, if $t \geq 1$, and $h_0 = \{\emptyset\}$, the set of histories is denoted by H,
- The strategy of player i is as follows: $s_i \colon H \to A_i$, hence at each stage t the strategy depends only on h_t.
- Binary relation $\succsim_i$ is the preference of Player i over $\times_{t=0}^{T} A$ such that (weak separability) for all $a, a' \in A$, $f_i(a) \geq f_i(a')$ we have the following for all t:

Repeated games II

$$\left(a^0, \ldots, a^{t-1}, a, a^{t+1}, \ldots, a^T\right) \succsim_i \left(a^0, \ldots, a^{t-1}, a', a^{t+1}, \ldots, a^T\right).$$

A repeated game is a finitely repeated game, if $T < \infty$, otherwise it is an infinitely repeated game.

Preference specifications

Definition

- Discounting: there exists a discount factor $\delta > 0$ such that $a \succsim_i b$, if $\sum_{j=0}^{T} \delta^j (f_i(a^j) - f_i(b^j)) \geq 0$ for all $i \in N$.
- Limit of means: $a \succ_i b$, if $\liminf_{t \to T} \sum_{j=0}^{t} (f_i(a^j) - f_i(b^j))/t > 0$ for all $i \in N$.
- Overtaking: $a \succ_i b$, if $\liminf_{t \to T} \sum_{j=0}^{t} (f_i(a^j) - f_i(b^j)) > 0$ for all $i \in N$.

An example I

Let the base game be as follows

		Player 2	
		B	J
Player 1	F	(3,3)	(4,2)
	A	(2,4)	(5,5)

This game has two pure Nash equilibria (F, B) and (A, J). Let $T = 1$, therefore the base game is played two times (one repetition). Then it is easy to see that in the repeated game each player has 32 strategies (at the second stage there are 4 histories, so $2^4 \times 2$).

1. It is easy to see that the strategy when Player 1 plays F in both games, and Player 2 plays B isn both games is a Nash equilibrium in the repeated game.

An example II

2. Consider the following strategy profile:
   Player 1: at stage 0 she plays A, at stage 1 she plays A, if the outcome of the game at stage 0 is (A, J) and she plays F otherwise.
   Player 2: at stage 0 he plays J, at stage 1 he plays J, if the outcome of the game at stage 0 is (A, J) and she plays B otherwise.

   This strategy profile is a Nash equilibrium in the repeated game, since, if Player 1 wants to deviate she does not do at stage 1 because (A, J) is a Nash equilibrium in the base game. She does not deviate at stage 0 either because it leads to outcome (F, B) at stage 1 which means worse payoff (3) than 5.

Strategies

Definition

- A strategy is stationer, if at each stage the same action is played,
- A strategy is Markovian, if at each stage t the played action does not depend on the elements of the history at the stage older than $t - 1$,
- A strategy is a trigger strategy, if it initially cooperates but punishes the opponent, if a certain level of defection (i.e., the trigger) is observed.

Stationer subgame perfect Nash equilibrium

Theorem

Suppose that $T < \infty$ and $a \in A$ is the only Nash equilibrium in the base game. Then the stationer strategy profile applying a at each stage is the unique subgame perfect Nash equilibrium of the repeated game.

Stationer subgame perfect Nash equilibrium revisited

Theorem

Suppose that $T < \infty$ and $a \in A$ is a Nash equilibrium in the base game. Then the stationer strategy profile applying a at each stage is a subgame perfect Nash equilibrium of the repeated game.

Nash folk theorem for the limit of means criterion

Theorem

For every $\varepsilon > 0$ *every enforceable payoff profile of* $\Gamma = (N, \{A_i\}_{i \in N}, \{f_i\}_{i \in N})$ *such that it is in the convex hull of* $\{f(a)\}_{a \in A}$ *is* ε*-close to a Nash equilibrium payoff profile of the limit of means infinitely repeated game of* Γ*.*

Thank you for your attention!

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