Quasi-Newton variable preconditioning for non-symmetric nonlinear elliptic PDE systems

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Outline

Earlier results: symmetric case

Let H be a real Hilbert space and $F : H \to H$ a nonlinear operator. Abstract equation

$$F(u) = 0.$$

Suppose F' is Lipschitz continuous and $F'(u)$ is a self-adjoint operator for all $u \in H$.

Variable preconditioning: Quasi-Newton method

$$u_{n+1} = u_n - \frac{2}{m_n + M_n} D_n^{-1} F(u_n) \tag{1.1}$$

where D_n are SPD operators **spectrally equivalent** to $F'(u_n)$:

$$m_n \langle D_n h, h \rangle \le \langle F'(u_n) h, h \rangle \le M_n \langle D_n h, h \rangle.$$

Main idea: D_n suitable cheap approximations of the Jacobians.

Objectives

- To construct an appropriate preconditioner and prove linear convergence for the non-symmetric case. This will extend the previous results done in [4].
- Test our scheme on a nonlinear elliptic reaction-diffusion system.

Assumptions

Conditions for F:

- F is Gâteux differentiable such that F' is Lipschitz continuous:

$$\|F'(u) - F'(v)\| \leq L\|u - v\|. \tag{1.2}$$

- There exists $\lambda > 0$ such that

$$\lambda\|h\|^2 \leq \langle F'(u)h, h\rangle \quad (\forall u, h \in H). \tag{1.3}$$

- There exists $\Lambda > 0$ such that

$$\langle F'(u)h, v\rangle \leq \Lambda\|h\|\|v\| \quad (\forall u, h, v \in H). \tag{1.4}$$

Variable preconditioning iteration

Let $B_n : H \to H$ be a self-adjoint, uniformly positive operator and $\alpha_n > 0$:

$$u_{n+1} = u_n - \alpha_n B_n^{-1} F(u_n) \qquad (n \in \mathbf{N}), \tag{1.5}$$

Additional conditions for B_n:

- Common lower bound:

$$\lambda \|h\|^2 \leq \langle B_n h, h \rangle. \tag{1.6}$$

- There exists $M_n, m_n > 0$ such that

$$\langle F'(u_n)h, v \rangle \leq M_n \|h\|_{B_n} \|v\|_{B_n} \quad \text{and} \quad m_n \|h\|_{B_n}^2 \leq \langle F'(u_n)h, h \rangle \tag{1.7}$$

Preliminary results: contractivity estimates

Lemma 1.1

There exists $\alpha_n \geq 0$, $Q_n < 1$ *such that*

$$\left\| I - \alpha_n B_n^{-1} F'(u_n) \right\|_{B_n} \leq \; Q_n \,. \tag{1.8}$$

Moreover, Q_n *takes its smallest value* $Q_n = \sqrt{1 - \frac{m_n^2}{M_n^2}}$ *when* $\alpha_n = \frac{m_n}{M_n^2}$.

Corollary 1.1

$$\left\| I - \alpha_n F'(u_n) B_n^{-1} \right\|_{B_n^{-1}} \leq \; Q_n. \tag{1.9}$$

We define the norms

$$\|h\|_{u,\, symm} = \langle (F'(u)_{symm})^{-1} h, h \rangle^{1/2}, \quad \|h\|_* = \|h\|_{u^*,\, symm}.$$

Lemma 1.2

Let $u, h \in H$. Then

$$\lambda \|h\|^2_{u,symm} \le \|h\|^2 \le \Lambda \|h\|^2_{u,symm}. \tag{1.10}$$

Proposition 1.1

If $F(u^*) = 0$, then for any $u \in H$ there holds

$$\frac{1}{1+\mu(u)} \le \frac{\|h\|^2_{u^*,symm}}{\|h\|^2_{u,symm}} \le 1 + \mu(u), \tag{1.11}$$

where $\mu(u) = \frac{L\Lambda^{\frac{1}{2}}}{\lambda^2} \|F(u)\|_*$.

Theorem 1: Symmetric part preconditioning ($B_n = S_n$)

Let $u^* \in H$ be the unique solution of $F(u) = 0$. We define the iterative sequence

$$u_{n+1} = u_n - \alpha_n S_n^{-1} F(u_n), \quad (n \in \mathbb{N}) \tag{2.12}$$

where $S_n = F'(u_n)_{symm} := \frac{1}{2}\left(F'(u_n) + F'(u_n)^*\right)$ satisfies:

$$\langle F'(u_n)h, v\rangle \leq M_n \|h\|_{S_n} \|v\|_{S_n} \quad (\forall h, v \in H).$$

with $\alpha_n = \frac{1}{M_n^2}$ for some $1 < M_n < M$. Then, the iteration (2.12) converges locally to u^* and

$$\limsup \frac{\|F(u_{n+1})\|_*}{\|F(u_n)\|_*} \leq Q < 1, \tag{2.13}$$

where $Q := \limsup \sqrt{1 - \frac{1}{M_n^2}}$.

Sketch of proof

- Corollary 1.1:

$$\left\| I - \alpha_n F'(u_n) S_n^{-1} \right\|_{S_n^{-1}} \leq Q_n$$

yields contractivity in the S_n^{-1}-norms.

- Finally, Proposition 1.1

$$\frac{1}{1+\mu(u)} \leq \frac{\langle F'(u^*)^{-1}_{symm} h, h \rangle}{\langle F'(u)^{-1}_{symm} h, h \rangle} \leq 1 + \mu(u),$$

preserves it asymptotically for the $*-$norm:

$$\limsup \frac{\|F(u_{n+1})\|_*}{\|F(u_n)\|_*} \leq \limsup Q_n < 1$$

Remark

The asymptotic estimate

$$\limsup \frac{\|F(u_{n+1})\|_*}{\|F(u_n)\|_*} \leq Q < 1, \tag{2.14}$$

implies

$$\limsup \sqrt[n]{\|u_n - u^*\|} \leq Q. \tag{2.15}$$

Theorem 2: More general preconditioner B_n

Conditions for B_n:

- B_n is a self-adjoint, uniformly positive operator with common lower bound:

$$\lambda \|h\|^2 \leq \langle B_n h, h \rangle.$$

- There exists $M_n, m_n > 0$ such that

$$\langle F'(u_n)h, v \rangle \leq M_n \|h\|_{B_n} \|v\|_{B_n}, \quad \text{and} \quad m_n \|h\|_{B_n}^2 \leq \langle F'(u_n)h, h \rangle$$

- B_n is spectrally equivalent to S_n. More concretely, there exists $\epsilon > 0$ such that

$$\frac{1}{1+\epsilon} \langle S_n h, h \rangle \leq \langle B_n h, h \rangle \leq (1+\epsilon) \langle S_n h, h \rangle \tag{2.16}$$

- The sequence $Q_n = \sqrt{1 - \frac{m_n^2}{M_n^2}}$, $\quad Q := \limsup Q_n$ satisfies

$$\overline{Q} := Q(1+\epsilon) < 1. \tag{2.17}$$

Scheme

$$u_{n+1} = u_n - \alpha_n B_n^{-1} F(u_n) \tag{2.18}$$

where $\alpha_n = \frac{m_n}{M_n^2}$ and $B_n : H \to H$ is a self-adjoint operator.

Under these conditions, the iteration (2.18) converges locally to u^* and

$$\limsup \frac{\|F(u_{n+1})\|_*}{\|F(u_n)\|_*} \leq \overline{Q}. \tag{2.19}$$

Remark

The estimation

$$\limsup \frac{\|F(u_{n+1})\|_*}{\|F(u_n)\|_*} \leq \overline{Q} < 1, \tag{2.20}$$

implies

$$\limsup \sqrt[n]{\|u_n - u^*\|} \leq \overline{Q} < 1. \tag{2.21}$$

Example

- Nonlinear elliptic reaction diffusion system:

$$\begin{cases} -\mathrm{div}(K_i \nabla u_i) + \mathbf{w}_i \cdot \nabla u_i + f_i(x, u_1, \dots, u_\ell) & = g_i; \\ u_i|_{\partial\Omega} & = 0, \end{cases} \tag{3.22}$$

where

$$f_i(x, \xi_1, \dots, \xi_\ell) := R_i(x, \xi_1, \dots, \xi_\ell) + \frac{1}{\tau}\xi_i \tag{3.23}$$

and $\mathrm{div}\mathbf{w} = 0$.

Arises from the (implicit) time discretization of

$$\frac{\partial u_i}{\partial t} - \mathrm{div}(K_i \nabla u_i) + \mathbf{w}_i \cdot \nabla u_i + R_i(x, u_1, \dots, u_\ell) = \omega_i.$$

Assumptions

(a) Small time discretization step: there exists $\sigma_{min} > 0$ such that

$$\inf_{\substack{(x,\xi)\in\Omega\times\mathbf{R}^\ell \\ |\theta|=1}} R'_\xi(x,\xi)\theta\cdot\theta \geq -\sigma_{min} \qquad \text{and} \qquad \tau \leq \frac{1}{2\sigma_{min}},$$

(b) We also assume an upper counterpart of the above:

$$\sup_{\substack{(x,\xi)\in\Omega\times\mathbf{R}^\ell \\ |\theta|=|\eta|=1}} R'_\xi(x,\xi)\theta\cdot\eta = \sigma_{max} < \infty.$$

(c) Global Lipschitz continuity:

$$\|f'_\xi(x,\xi_1) - f'_\xi(x,\xi_2)\| \leq L_f|\xi_1 - \xi_2|. \tag{3.24}$$

Arising linear operators

- From the weak form:

$$\langle F'(\mathbf{u}_n)\mathbf{h},\mathbf{v}\rangle_{H_0^1(\Omega)} = \int_\Omega \Big(\mathbf{K}\nabla\mathbf{h}\cdot\nabla\mathbf{v} + (\mathbf{w}\cdot\nabla\mathbf{h})\cdot\mathbf{v} + R'_\xi(x,\mathbf{u}_n)\mathbf{h}\cdot\mathbf{v} + \frac{1}{\tau}\mathbf{h}\cdot\mathbf{v}\Big)$$

- Symmetric part:

$$\langle S_n\mathbf{h},\mathbf{v}\rangle_{H_0^1(\Omega)} = \int_\Omega \Big(\mathbf{K}\nabla\mathbf{h}\cdot\nabla\mathbf{v} + R'_\xi(x,\mathbf{u}_n)_{symm}\,\mathbf{h}\cdot\mathbf{v} + \frac{1}{\tau}\mathbf{h}\cdot\mathbf{v}\Big).$$

- Approximation of symmetric part:

$$\langle B_n\mathbf{h},\mathbf{v}\rangle_{H_0^1(\Omega)} = \int_\Omega \Big(\mathbf{K}\nabla\mathbf{h}\cdot\nabla\mathbf{v} + D_n\,\mathbf{h}\cdot\mathbf{v} + \frac{1}{\tau}\mathbf{h}\cdot\mathbf{v}\Big).$$

where $D_n := diag(R'_\xi(x,\mathbf{u}_n))$.

- Assumptions imply spectral equivalence:

$$-\sigma_{min} I \leq R'_{\xi}(x,\xi)_{symm} \leq \sigma_{max} I. \tag{3.25}$$

- We also proved:

$$\frac{1}{1+const.\cdot\tau} \leq \frac{\langle B_n \mathbf{h}, \mathbf{h}\rangle_{\mathrm{H}_0^1(\Omega)}}{\langle S_n \mathbf{h}, \mathbf{h}\rangle_{\mathrm{H}_0^1(\Omega)}} \leq 1 + const.\cdot\tau, \tag{3.26}$$

where $const = 2(\sigma_{min} + \sigma_{max})$.

Conclusions

- If $\tau \leq \frac{1}{2\sigma_{min}}$, then the iteration

$$\mathbf{u}_{n+1} = \mathbf{u}_n - \alpha_n B_n^{-1} F(\mathbf{u}_n)$$

  converges linearly to the weak solution of (3.22) according to Theorem 2.
- This scheme is cheaper to implement than the Newton Method. Auxiliary problems:

$$B_n \mathbf{p}_n = F(\mathbf{u}_n). \tag{4.27}$$

  These are discrete linear PDE systems that are independent scalar equations $\Longrightarrow$ they can be solved in parallel.
- Future work: numerical implementation and tests for air pollution systems.

References

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Thank you for your attention!