Investigation of continuous and discrete population dynamics models

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Contents

Population dynamics

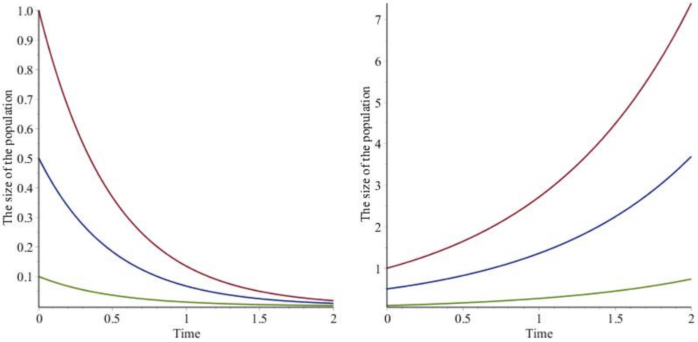
- Describes the dynamics of biological populations.
- Studies the biological and environmental processes (birth rates, death rates, migration).
- Gives a forecast and strategies for their prevention.

The Malthusian Growth Model

$$\frac{dP(t)}{dt} = kP(t) \tag{1}$$

- $P(t)$ denotes the population at time t
- k is a constant (depends on the birth and death rate, migration..)
- the solution is $P(t) = P_0 e^{kt}$
- the equilibrium point of the system is $P(t) = 0$

The Malthusian Growth Model



(a) $k < 0$ (b) $k > 0$

Figure: The solutions of the Malthusian Growth Model.

Logistic Population Model

$$\frac{dP(t)}{dt} = r\left(1 - \frac{P(t)}{K}\right)P(t) \tag{2}$$

- $P(t)$ denotes the population at time t
- r is a constant
- K is the carrying capacity
- the equilibrium points are $P(t) = 0$ and $P(t) = K$

Logistic Population Model

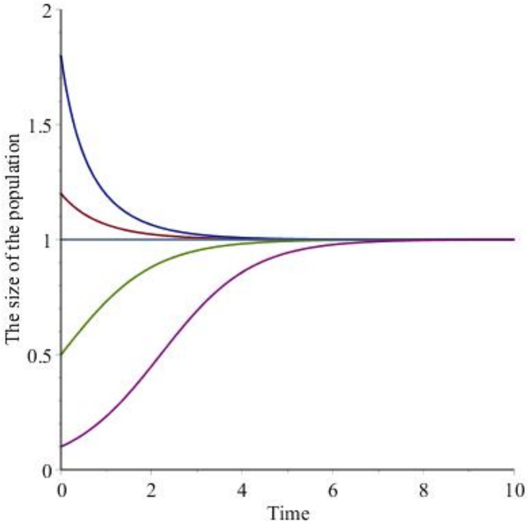


Figure: The solutions of the logistic population model.

Harvesting types

$$\frac{dP(t)}{dt} = f(P(t)) - H(P(t)) \tag{3}$$

- constant harvesting: $H(P(t)) = H$
- linear harvesting: $H(P(t)) = cP(t)$
- Holling type II harvesting: $H(P(t)) = \frac{c_1 P(t)}{c_2 + P(t)}$

Harvesting types

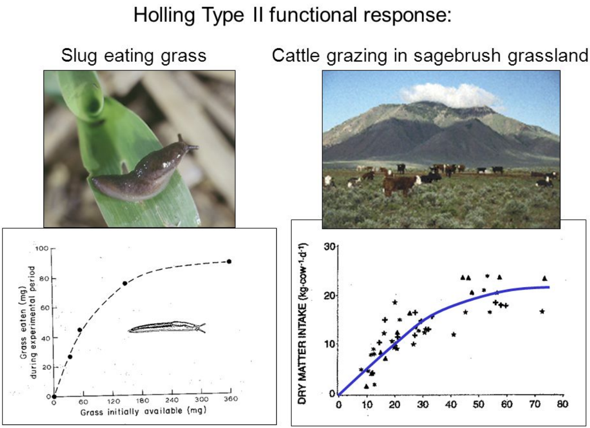


Figure: Holling Type II functional response.

Easter Island

(a) Moai on the Easter Island.

(b) Rat on the Easter Island.

Figure

Invasive Species Model

Basener W., Brooks B., Radin M., Wiandt T. (2008a) Rat instigated human population collapse on Easter Island. Nonlinear Dynamics, Psychology and Life Science, 12, 227-240.

$$\frac{dP(t)}{dt} = aP(t)\left(1 - \frac{P(t)}{T(t)}\right), \tag{4}$$

$$\frac{dR(t)}{dt} = cR(t)\left(1 - \frac{R(t)}{T(t)}\right), \tag{5}$$

$$\frac{dT(t)}{dt} = \frac{b}{1 + fR(t)}T(t)\left(1 - \frac{T(t)}{M}\right) - hP(t). \tag{6}$$

- P, R, T people, rat and tree populations
- a, b, c growth rates
- f, h the effect of the rats and humans

Equilibrium points and stability

The equilibrium points of the system:

$$\mathcal{P}_1(0, 0, M), \tag{7}$$

$$\mathcal{P}_2(0, M, M), \tag{8}$$

$$\mathcal{P}_3\left(\frac{(b-h)M}{b}, 0, \frac{(b-h)M}{b}\right), \tag{9}$$

$$\mathcal{P}_4\left(\frac{(b-h)M}{b+fhM}, \frac{(b-h)M}{b+fhM}, \frac{(b-h)M}{b+fhM}\right). \tag{10}$$

The points $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ are unstable, the point $\mathcal{P}_4$ is stable.

Invasive Species Model

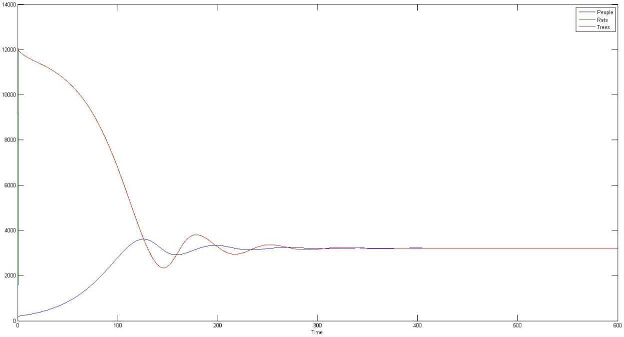


Figure: Equilibrium point $\mathcal{P}_4$.

Further analysis of the system

In the following we investigate the stability of different biological systems by the following steps:

- develop the invasive species model
- determine the equilibrium point
- linearising about an equilibrium point, determine the characteristic equation
- investigate the roots of the characteristic equation
- get conditions for the stability

Further analysis of the system - Linear harvesting

We decrease the amount of the rats in the following form

$$\frac{dP(t)}{dt} = aP(t)\left(1 - \frac{P(t)}{T(t)}\right), \tag{11}$$

$$\frac{dR(t)}{dt} = cR(t)\left(1 - \frac{R(t)}{T(t)}\right) - gR(t), \tag{12}$$

$$\frac{dT(t)}{dt} = \frac{b}{1 + fR(t)}T(t)\left(1 - \frac{T(t)}{M}\right) - hP(t). \tag{13}$$

Equilibrium points and stability

The equilibrium points of the system:

$$\mathcal{P}_1(0, M, 0), \tag{14}$$

$$\mathcal{P}_2(0, M, M\left(1-\frac{g}{c}\right), \tag{15}$$

$$\mathcal{P}_3\left(\frac{(b-h)M}{b}, \frac{(b-h)M}{b}, 0\right), \tag{16}$$

$$\mathcal{P}_4\left(E, E, E\left(1-\frac{g}{c}\right)\right), \quad E = \frac{cM(b-h)}{cb+fhM(c-g)}. \tag{17}$$

The points $\mathcal{P}_1, \mathcal{P}_2$ are unstable, the points $\mathcal{P}_4$ and $\mathcal{P}_3$ are stable.

Further analysis of the system - Delay differential equation

We develop the model with a dealy differential equation:

$$\frac{dP(t)}{dt} = aP(t)\left(1 - \frac{P(t)}{T(t)}\right), \tag{18}$$

$$\frac{dR(t)}{dt} = cR(t)\left(1 - \frac{R(t)}{T(t)}\right), \tag{19}$$

$$\frac{dT(t)}{dt} = \frac{b}{1 + f\,R(t-\tau)}\,T(t-\tau)\left(1 - \frac{T(t-\tau)}{M}\right) - hP(t). \tag{20}$$

Further analysis of the system - Delay differential equation

The equilibrium points $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ are unstable. The characteristic equation is at the equilibrium point $\mathcal{P}_4$:

$$\lambda^3 + \lambda^2(a+c) + \lambda(ac+ah) - \lambda^2 e^{-\lambda\tau} F_2 + \lambda e^{-\lambda\tau}(-cF_1 - cF_2 -$$
$$+ e^{-\lambda\tau}(-acF_1 - acF_2) + ahc = 0. \tag{21}$$

where F_1 and F_2 denote

$$F_1 = \frac{bfF(F-M)}{(1+fF)^2 M}, \tag{22}$$

$$F_2 = \frac{b(M-2F)}{M(1+fF)}. \tag{23}$$

Further analysis of the system - Delay differential equation

We use the Rouché theorem to investigate the stability. The equilibrium points $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ are unstable and the equilibrium point $\mathcal{P}_4$ is stable.

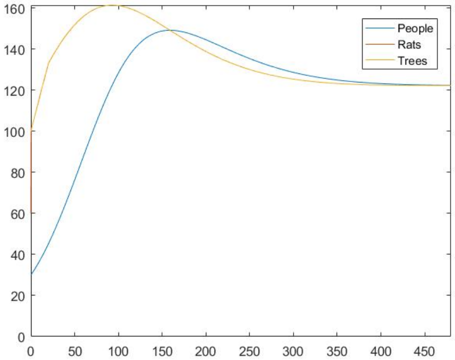


Figure: The numerical solution of the delay model.

Biological model - Lotka-Volterra equations

$$\begin{aligned} \frac{dN}{dt} &= aN - bNP \\ \frac{dP}{dt} &= cNP - dP \end{aligned} \tag{24}$$

- $N(t)$: the prey population,
- $P(t)$: predator population,
- a, b, c, d positive parameters, show the connections between the species.

Equilibrium points and stability

The system (24) has two equilibrium points. The point $\mathcal{P}_1 = (0, 0)$ is unstable, the point is an $\mathcal{P}_2 = \left(\frac{d}{c}, \frac{a}{b}\right)$ centre.

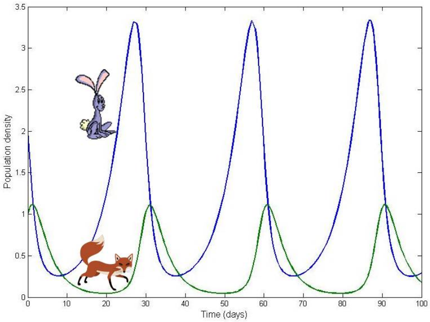


Figure: The prey and predator populations.

Equilibrium points and stability

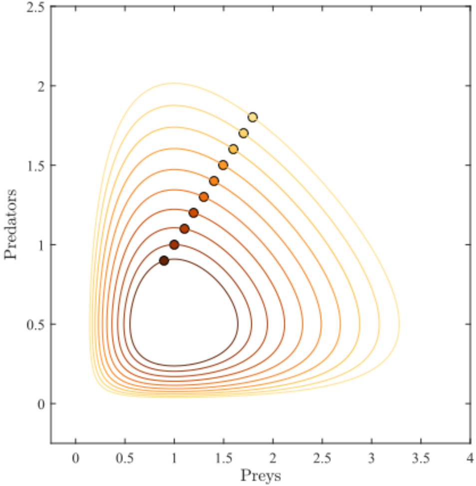


Figure: Closed curves around the equilibrium points.

We consider the the following

$$\frac{dy}{dt} = f(y(t)) \tag{25}$$

ODE system, where $y = (y_1, y_2, \ldots, y_d)^T$ and $f(y(t)) = (f_1(y(t), \ldots, f_d(y(t)))^T$. We approximate the solution in the discrete points $t_0, t_1, t_2, \ldots$. We denote $\tau = \Delta t = t_i - t_{i-1}$ the mesh size.

- Explicit Euler method:

$$y^{n+1} = y^n + \tau f(y^n) \tag{26}$$

- Implicit Euler method:

$$y^{n+1} = y^n + \tau f(y^{n+1}) \tag{27}$$

Explicit Euler method

Click!

Implicit Euler method

Click!

Geometric integrators

Geometric integrators: such numerical methods, which preserve the geometric properties of the solution.

- First integral - preservation of the energy
- Phase space volume
- Symmetry
- Symplectic structure
- Poisson system

Hamiltonian system

$$\begin{aligned}\frac{dp}{dt} &= -\frac{\partial H(p,q)}{\partial q} \\ \frac{dq}{dt} &= \frac{\partial H(p,q)}{\partial p}\end{aligned} \tag{28}$$

- The flow of the system is $y(t) = \Phi_t(y(0))$.
- The flow of the Hamiltonian system is symplectic: $\Phi'_t(y)^T J \Phi'_t(y) = J$.

Symplectic numerical methods

- Numerical flow: Φ_τ, $y_n \to y_{n+1}$
- The numerical method has symplectic structure: $\Phi'_\tau(y)^T J \Phi'_\tau(y) = J$.

Poisson systems

$$\dot{y}(t) = B(y)\nabla H(y) \tag{29}$$

- The flow of the system: $y(t) = \Phi_t(y(0))$
- The flow of the Poisson system is a Poisson map: $\Phi_t'(y)^T B(y)\Phi_t'(y) = B(\Phi_t(y))$

Numerical solution with Poisson structure

- Numerical flow: $\Phi_\tau, \quad y_n \to y_{n+1}$
- The numerical method has Poisson structure: $\Phi_\tau'(y)^T B(y)\Phi_\tau'(y) = B(\Phi_\tau(y))$

Symplectic Euler method

$$\begin{aligned} N^{n+1} &= N^n + \tau(aN^{n+1} - bN^{n+1}P^n) \\ P^{n+1} &= P^n + \tau(cN^{n+1}P^n - dP^n) \end{aligned} \tag{30}$$

Click!

Operator splitting method

We consider the

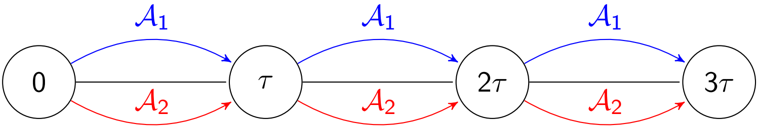
$$\frac{du(t)}{dt} = \mathcal{A}(u(t)) \qquad u(t_0) = u_0, \quad t \in [0, T] \tag{31}$$

Cauchy-problem. We separate the problem into two or more parts:

$$\frac{du_1(t)}{dt} = \mathcal{A}_1(u_1(t)) \quad \text{and} \quad \frac{du_2(t)}{dt} = \mathcal{A}_2(u_2(t)) \tag{32}$$

ahol $\mathcal{A}(u(t)) = \mathcal{A}_1(u_1(t)) + \mathcal{A}_2(u_2(t))$.

Operator splitting method



Geometric integrators for the Lotka-Volterra equations

- composition of different numerical methods
- different resolution of the function H
- preserving the symplectic and Poisson structure

The composition of the explicit and symplectic Euler method.

Thank you!