RED REFINEMENTS AND ZHANG TETRAHEDRA

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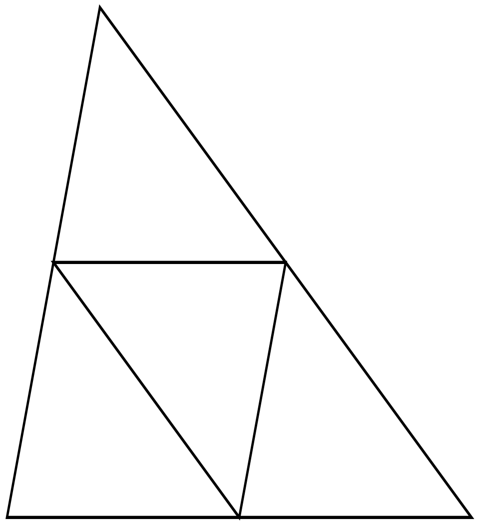
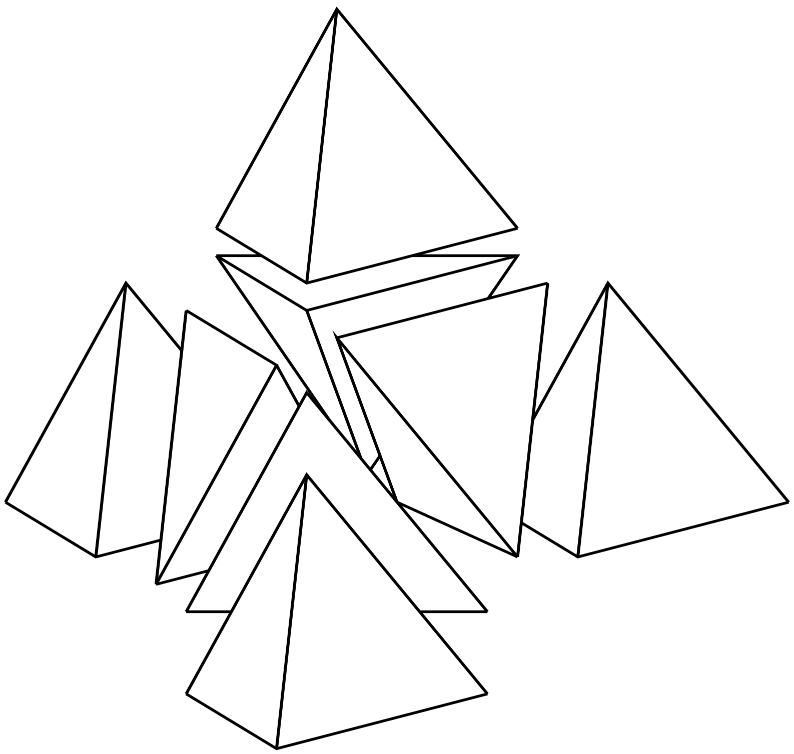
– based on joint collaboration with M. Křížek (Prague)

INTRODUCTION

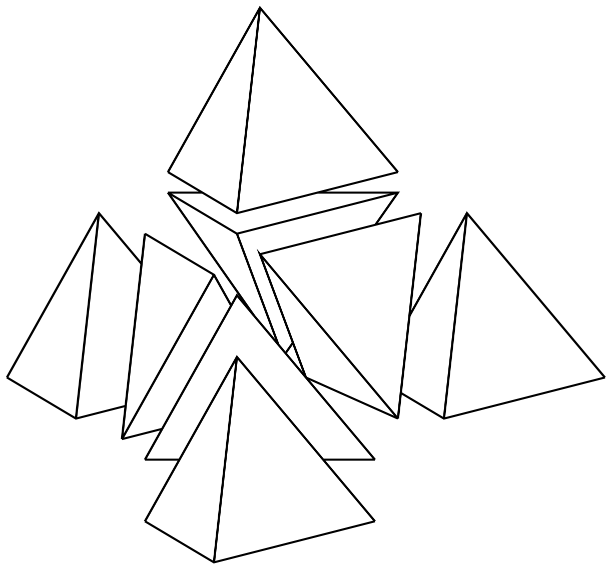
The red refinement is one of techniques widely used for simplicial mesh generation and adaptivity purposes in various applications.





NONUNIQUENESS

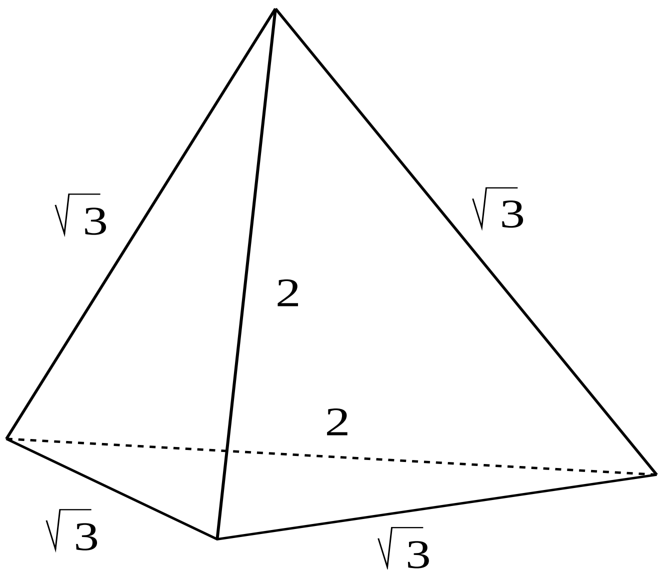
However, this technique is not uniquely defined in three and higher dimensions. In particular, in the case of tetrahedral partitions, inside each tetrahedron, on each refinement level, we have three different possibilities for dividing the tetrahedron.



Nevertheless, for any strategy selected all produced tetrahedral partitions stay face-to-face, i.e. conforming.

NO SIMILARITY PROPERTY

Similarity property holds only for the Sommerville tetrahedron (its two dihedral angles are right, the other four are 60°).



S. Zhang. Successive subdivisions of tetrahedra and multigrid methods on tetrahedral meshes, Houston J. Math. 21 (1995)

S. Korotov, M. Křížek. Red refinements of simplices into congruent subsimplices, Comput. Math. Appl. 67 (2014)

Non-uniqueness in the selection of diagonals and absence of similarity properties in most of the cases in three (and higher) dimensions makes an analysis of this refinement technique hard and therefore not so many (mathematical) results on this topic exist in the literature though the first results were obtained already in 1982:

M. Křížek. An equilibrium finite element method in three-dimensional elasticity, Apl. Mat. 27 (1982)

In what follows we present most important recent results on red refinements of tetrahedral partitions.

PRELIMINARIES

• We deal with face-to-face tetrahedral partitions of a bounded polyhedral domain $\Omega \subset \mathbf{R}^3$. They are denoted by $\mathcal{T}_h$, where h is the so-called discretization parameter defined as $h = \max_{T \in \mathcal{T}_h} h_T$ with

$$h_T = \operatorname{diam} T \tag{1}$$

for a tetrahedron $T \in \mathcal{T}_h$.

• We consider families $\mathcal{F} = \{\mathcal{T}_h\}_{h \to 0}$ of tetrahedral partitions.

• The radius r_T of the inscribed ball of the tetrahedron T is often called the inradius of T, and it can be computed as

$$r_T = \frac{3 \operatorname{vol}_3 T}{\operatorname{vol}_2 \partial T}. \tag{2}$$

Definition: A family $\mathcal{F} = \{\mathcal{T}_h\}_{h\to 0}$ of partitions into tetrahedra is said to be regular if there exists a constant $\kappa > 0$ such that for any $\mathcal{T}_h \in \mathcal{F}$ and any $T \in \mathcal{T}_h$ we have

$$\kappa h_T \leq r_T. \tag{3}$$

- **In fact, regularity means that tetrahedra cannot shrink.**

- Regularity property (for meshes) is used in many FEM convergence proofs for elliptic and parabolic problems.

- There are some other (equivalent) definitions of regularity (say, the minimum angle condition).

J. Brandts, S. Korotov, M. Křížek. Simplicial partitions with applications to the finite element method. Springer, 2020.

Definition: Let T be an arbitrary tetrahedron. Then the ratio

$$\sigma_T = \frac{h_T}{r_T} \tag{4}$$

is called a measure of the degeneracy of T.

• **In fact, regularity means that tetrahedra cannot degenerate. Really, then $\sigma_T \leq \kappa^{-1}$ for all tetrahedra.**

• **But for shrinking (i.e. degenerating) tetrahedra parameter σ_T attains large values.**

Definition: A family $\mathcal{F}$ of partitions is said to be degenerating if for every positive integer $n \in \mathbf{N}$ there exist a partition $\mathcal{T}_h \in \mathcal{F}$ and a tetrahedron $T \in \mathcal{T}_h$ whose measure of degeneracy satisfies $\sigma_T > n$.

Definition: We say that the family $\mathcal{F}$ satisfies the maximum angle condition if there exists a constant $\gamma_0 < \pi$ such that for any tetrahedron $T \in \mathcal{T}_h$ and any $\mathcal{T}_h \in \mathcal{F}$ we have

$$\gamma_T \leq \gamma_0 \tag{5}$$

and

$$\varphi_T \leq \gamma_0, \tag{6}$$

where γ_T is the maximum angle of all triangular faces of the tetrahedron T and φ_T is the maximum dihedral angle between faces of T.

- **The maximum angle condition is weaker than the regularity requirement, but still allows to prove FEM convergence.**

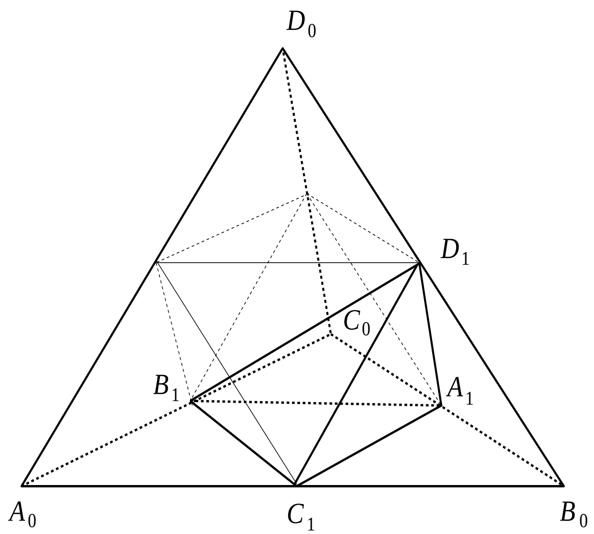
- **Some of degenerating families can still satisfy the maximum angle condition (e.g. those consisting of shrinking path tetrahedra).**

AROUND REGULAR TETRAHEDRON

Let $A_0B_0C_0D_0$ be the regular tetrahedron whose edges have length 1. For $k \in \{0, 1, 2, \dots\}$ define the following midpoints

$$A_{k+1} = \frac{1}{2}(B_k + C_k), \quad B_{k+1} = \frac{1}{2}(A_k + C_k), \quad C_{k+1} = \frac{1}{2}(A_k + B_k), \tag{7}$$

and D_{k+1} will be one of the midpoints of A_kD_k or B_kD_k or C_kD_k (it depends on the diagonal choice strategy).



Then the lenghts of all six edges of the tetrahedra $A_k B_k C_k D_k$ multiplied by the scaling factor 2^k can be divided into the following three groups:

a) If we always choose the shortest diagonal, then we get the periodic sequence

$$(1,1,1,1,1,1),\ (1,1,1,1,1,\sqrt{2}),\ (1,1,1,1,1,1),\ (1,1,1,1,1,\sqrt{2}),\dots$$

b) If we always choose the second-longest diagonal, we obtain

$$(1,1,1,1,1,1),(1,1,1,1,1,\sqrt{2}),(1,1,1,1,\sqrt{2},\sqrt{3}),(1,1,1,1,\sqrt{3},\sqrt{3}),$$
$$(1,1,1,1,\sqrt{2},\sqrt{3}),\dots$$

Since the third term is the same as the fifth term, this sequence is also periodic starting from its third term.

We observe that, in cases a) and b) one produces sequences of nondegenerating tetrahedra only.

c) If we always choose the longest diagonal, we find

$$(1,1,1,1,1,1),(1,1,1,1,1,\sqrt{2}),(1,1,1,1,\sqrt{2},\sqrt{3}),(1,1,1,\sqrt{2},\sqrt{3},\sqrt{5}),$$
$$(1,1,1,\sqrt{3},\sqrt{5},\sqrt{7}),(1,1,1,\sqrt{5},\sqrt{7},\sqrt{11}),\ldots \tag{8}$$

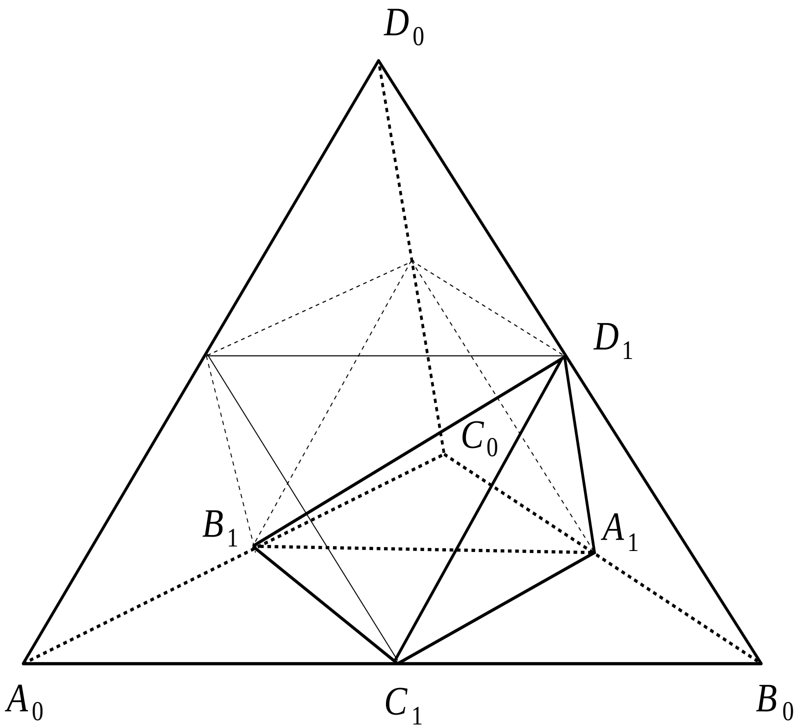
All terms in this sequence are different and the measure of degeneracy grows to ∞ as $k \to \infty$ (shown in the table later on).

In this case, the point D_{k+1} is actually selected as follows:

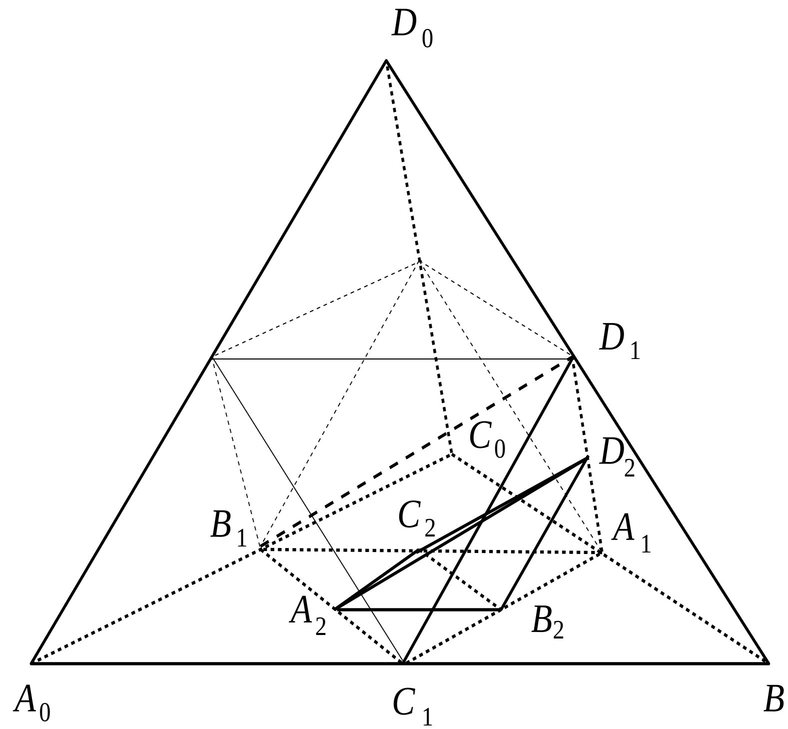
$$D_{k+1} = \frac{1}{2}(A_k + D_k) \quad \text{if } k \text{ is odd,} \quad \text{or} \quad D_{k+1} = \frac{1}{2}(B_k + D_k) \quad \text{if } k \text{ is even.} \tag{9}$$

In the case c), all $A_k B_k C_k D_k$, $k = 1, 2, \ldots$, are called the Zhang tetrahedra. (They are called in this way also under any translation, rotation, reflection, and scaling.)

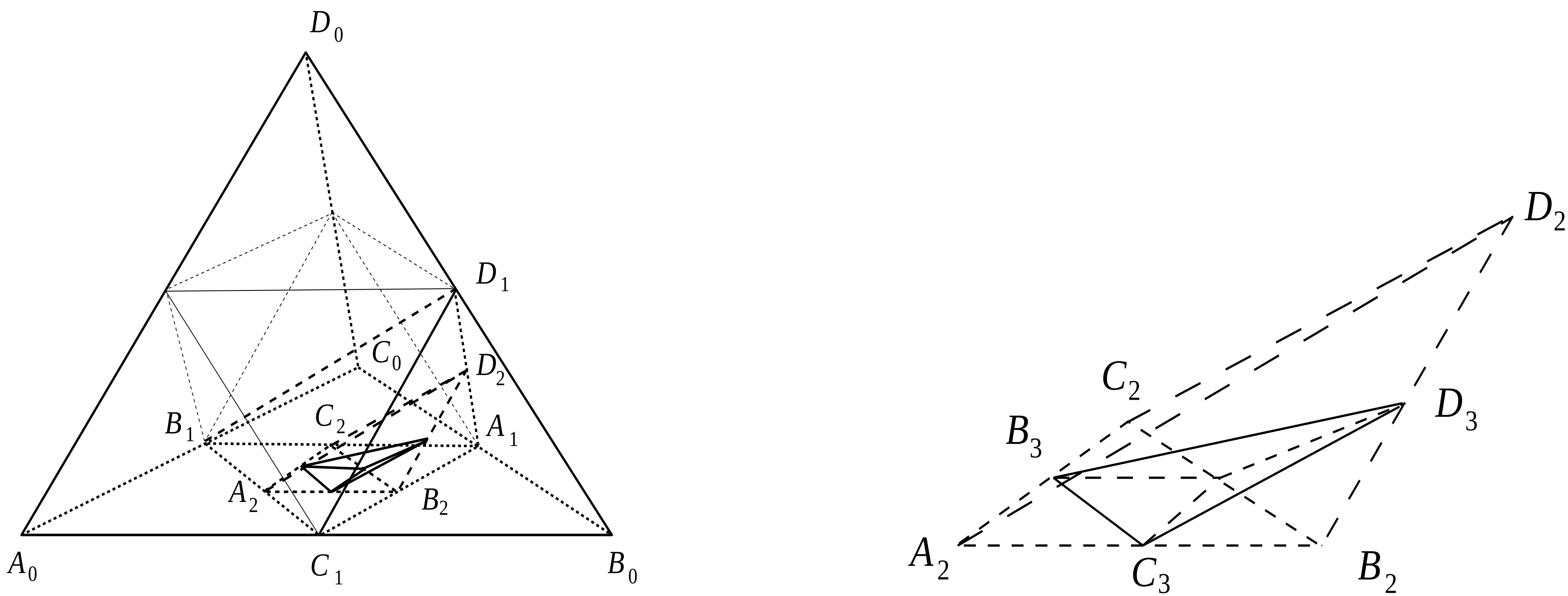
S. Zhang. Successive subdivisions of tetrahedra and multigrid methods on tetrahedral meshes, Houston J. Math. 21 (1995)



D_0
D_1
C_0
B_1
A_1
A_0
C_1
B_0



D_0
D_1
C_0
D_2
C_2
B_1
A_1
A_2
B_2
A_0
C_1
B_0


D_0
D_1
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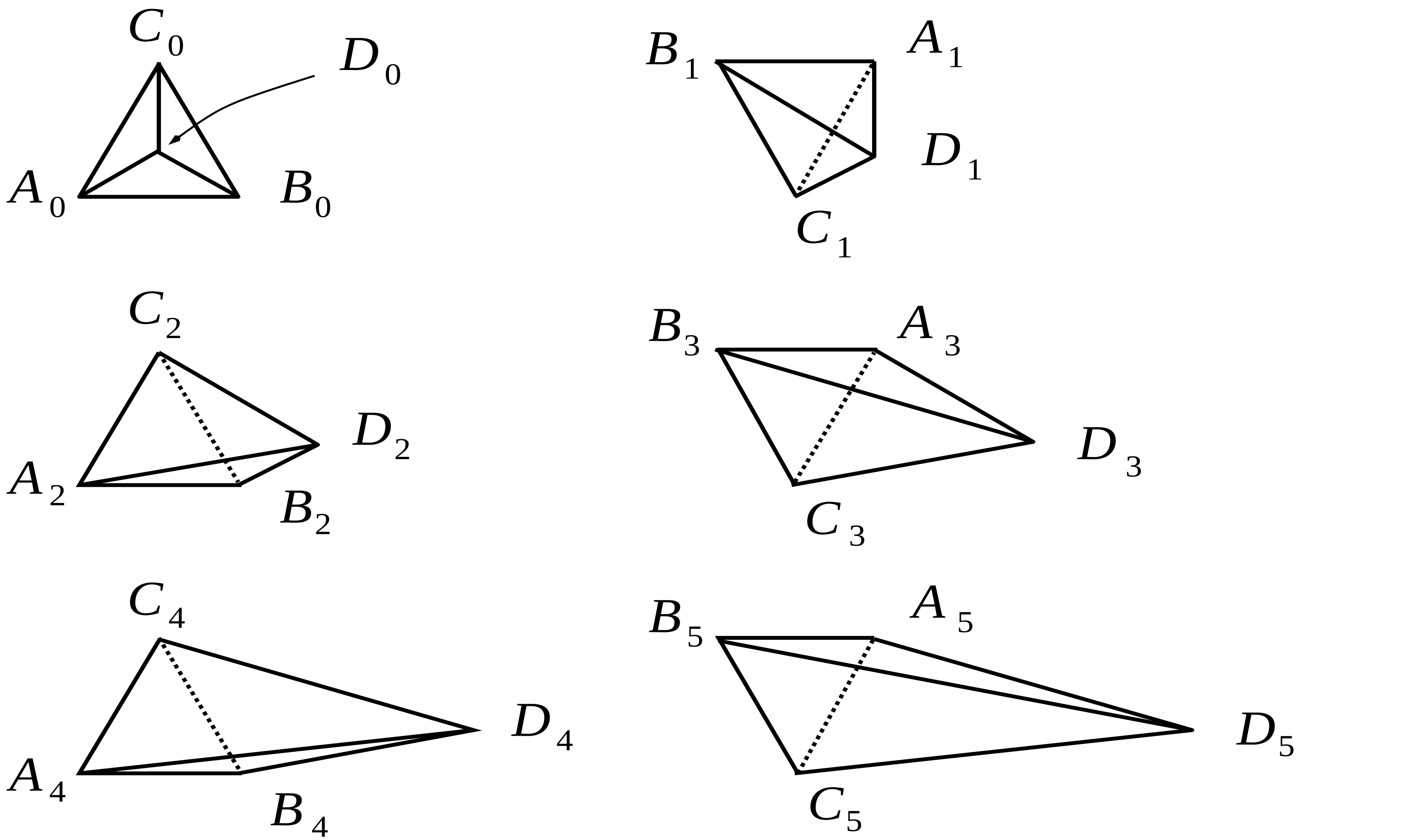
A_2
C_3
B_2


C_0
D_0
A_0
B_0
B_1
A_1
D_1
C_1
C_2
D_2
A_2
B_2
B_3
A_3
D_3
C_3
C_4
D_4
A_4
B_4
B_5
A_5
D_5
C_5

Edge	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
A_kB_k	**70.53**	54.74	70.53	54.74	70.53	54.74
A_kC_k	**70.53**	**109.47**	35.26	**144.74**	22.00	**158.00**
B_kC_k	**70.53**	54.74	**125.26**	29.50	**150.50**	19.47
A_kD_k	**70.53**	54.74	90.00	45.00	97.61	46.51
B_kD_k	**70.53**	90.00	54.74	107.55	58.52	114.09
C_kD_k	**70.53**	54.74	45.00	31.48	25.94	20.51
r_k	0.204	0.189	0.171	0.143	0.127	0.109
σ_k	4.899	7.464	10.156	15.617	20.840	29.046

Dihedral angles in degrees $^\circ$ at particular edges of the Zhang tetrahedra $A_kB_kC_kD_k$, their inradii r_k multiplied by the scaling factor 2^k, and the corresponding measures of degeneracy σ_k.

ARBITRARY TETRAHEDRON

Now let T be an arbitrary tetrahedron. Consider a linear affine mapping from the regular reference tetrahedron $A_0B_0C_0D_0$ to T. Red refinements of T can be then defined via this mapping.

This idea was used to construct regular partitions by the red-type refinements in the works by Křížek, Ong, Zhang, Bey.

MAIN RESULTS

Theorem: There exists only one type of tetrahedron T (up to similarity) whose red refinement produces eight congruent subtetrahedra similar to T. It is the Sommerville tetrahedron.

The next result immediately follows from the fact that the four "exterior" subtetrahedra arising from the red refinement algorithm are similar to the original tetrahedron.

Theorem: The maximum (minimum) dihedral angles between faces and also the maximum (minimum) angles in all triangular faces of all tetrahedra $T \in \mathcal{T}_h \in \mathcal{F}$ generated by the red-type refinements form nondecreasing (nonincreasing) sequences as $h \to 0$.

Theorem: For any selection of diagonals we always produce a family of partitions with $h \to 0$.

Theorem: The measure of degeneracy of the Zhang tetrahedra tends to ∞ when $k \to \infty$ and their maximum dihedral angle tends to 180°, and the maximum angle between edges tends to 180° as well.

- **We have to be careful while using the red refinements !**

S. Korotov, M. Křížek. On degenerating tetrahedra resulting from red refinements of tetrahedal partitions, Num. Anal. Appl., 14 (2021)

WHAT ABOUT HIGHER DIMENSIONS ?

Theorem: The red refinement of an acute simplex in three and higher dimensions never yields subsimplices that would be all mutually congruent.

Its proof for tetrahedra follows immediately if we notice that for any choice of diagonal for red refinement we have four subtetrahedra sharing this diagonal. Therefore, four adjacent dihedral angles sum up 2π, and at least one of them is not acute, meaning that the associated subtetrahedron is not acute, and therefore not congruent to the four corner subtetrahedra.

S. Korotov, M. Křížek. Red refinements of simplices into congruent subsimplices, Comput. Math. Appl. 67 (2014)

THANK YOU FOR YOUR ATTENTION !