On multiple solutions of nonlinear elliptic functional equations

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The main topics of this talk

It is well known that mathematical models of several applications are functional differential equations of one variable (e.g. delay equations). In the monograph by Jianhong Wu (Springer, 1996) semilinear evolutionary partial functional differential equations and applications are considered, where the book is based on the theory of semigroups and generators. In the monograph by A.L. Skubachevskii (Birkhäuser, 1997) linear elliptic functional differential equations (equations with nonlocal terms and nonlocal boundary conditions) and applications are considered. A nonlocal boundary value problem, arising in plasma theory, was considered by A.V. Bitsadze and A.A. Samarskii in 1969.

It turned out that the theory of pseudomonotone operators is useful to study nonlinear (quasilinear) partial functional differential equations (both stationary and evolutionary equations) and to prove existence of weak solutions (L. Simon, Application of Monotone type operators to Nonlinear PDEs, Budapest, 2013; M. Csirik, On pseudomonotone operators with functional dependence on unbounded domains, EJQTDE, 2016).

In the present talk first we shall consider weak solutions of the following elliptic functional differential equations:

$$-\sum_{j=1}^{n} D_j[a_j(x, u, Du; u)] + a_0(x, u, Du; u) = F(x), \quad x \in \Omega \tag{1}$$

(for simplicity) with homogeneous Dirichlet or Neumann boundary condition where $\Omega \subset \mathbb{R}^n$ is a bounded domain and ; u denotes nonlocal dependence on u.

By using the theory of pseudomonotone operators one can prove an existence theorem on weak solutions. After formulating the existence theorem, we shall investigate the number of solutions in certain particular cases and to prove existence of multiple solutions, based on fixed points of certain functions and operators, respectively.

Denote by $\Omega \subset \mathbb{R}^n$ a bounded domain, $1 < p < \infty$, $W^{1,p}(\Omega)$ the Sobolev space with the norm

$$\|u\| = \left[\int_\Omega \left(\sum_{j=1}^n |D_j u|^p + |u|^p \right) dx \right]^{1/p} .$$

Further, let $V \subset W^{1,p}(\Omega)$ be a closed linear subspace of $W^{1,p}(\Omega)$, $V^\star$ the dual space of V, the duality between $V^\star$ and V will be denoted by $\langle \cdot, \cdot \rangle$. First we formulate the assumptions of an existence and uniqueness theorem on weak solutions for nonlinear elliptic differential equations.

(A_1). The functions $a_j : \Omega \times \mathbb{R}^{n+1} \to \mathbb{R}$ $(j = 0, 1, \cdots, n)$ satisfy the Carathéodory conditions.
(A_2). There exist a constant C_1 and a function $k_1 \in L^q(\Omega)$ $(1/p + 1/q = 1)$ such that

$$|a_j(x, \xi)| \leq C_1[1 + |\xi|^{p-1}] + [k_1(u)](x),$$

$j = 0, 1, \cdots, n$, for a.e. $x \in \Omega$, each $\xi \in \mathbb{R}^{n+1}$.
(A_3). The inequality

$$\sum_{j=1}^{n} [a_j(x, \xi) - a_j(x, \xi^\star)](\xi_j - \xi_j^\star) \geq C_2|\xi - \xi^\star|^p$$

holds with come constant $C_2 > 0$.

Theorem

Assume (A_1) - (A_3). Then the operator $A : V \to V^\star$ defined by

$$\langle A(u), v\rangle = \int_\Omega \left[\sum_{j=1}^{n} a_j(x,u,Du)D_j v + a_0(x,u,Du)v \right] dx$$

is bounded, demicontinuous, uniformly monotone and coercive. Thus for any $F \in V^\star$ there exists a unique $u \in V$ satisfying $A(u) = F$ which is continuously depending on F.

(See, e.g., [5].) One can formulate and prove an existence theorem on nonlinear functional elliptic equations. (See, e.g., [5].) Further, in [5] there are examples satisfying the assumption of the existence theorem.

Now consider particular cases for the functions a_j

$$a_j(x, \eta, \zeta; u) = \tilde{a}_j(x, \eta, \zeta, M(u)), \quad j = 0, 1, \cdots, n$$

where $M : V \to \mathbb{R}$ is a bounded, continuous (possibly nonlinear) operator and

$$\tilde{a}_j : \Omega \times \mathbb{R}^{n+1} \times \mathbb{R} \to \mathbb{R}$$

satisfy the Carathéodory conditions. In the case of first (homogeneous) boundary condition $V = W_0^{1,p}(\Omega)$ and for second boundary condition $V = W^{1,p}(\Omega)$.

Assume that the assumptions (A_1) - (A_3) hold for the operator A_λ such that for every $\lambda \in \mathbb{R}$ there exists a unique solution $u_\lambda \in V$ of

$$A_\lambda(u_\lambda) = F \quad (F \in V^\star)$$

where $A_\lambda : V \to V^\star$ is defined by

$$\langle A_\lambda(u), v \rangle = \int_\Omega \left[\sum_{j=1}^{n} \tilde{a}_j(x, u, Du, \lambda) D_j v + \tilde{a}_0(x, u, Du, \lambda) v \right] dx$$

Theorem

Define the function $g : \mathbb{R} \to \mathbb{R}$ by $g(\lambda) = M(u_\lambda)$. Then a function $u \in V$ is a solution of

$$\int_\Omega \left[\sum_{j=1}^n \tilde{a}_j(x, u, Du, M(u)) D_j v + \tilde{a}_0(x, u, Du, M(u)) v \right] dx = \langle F, v \rangle \tag{2}$$

if and only if $\lambda = M(u)$ satisfies $\lambda = g(\lambda)$.

Consider the following particular case

$$\tilde{a}_j(x, u, Du, M(u)) = b_j(x, u, Du) h(M(u)), \text{ i.e.}$$

$$\tilde{a}_j(x, u, Du, \lambda) = b_j(x, u, Du) h(\lambda),$$

$j = 1, \cdots, n$ and

$$\tilde{a}_0(x, u, Du, \lambda) = b_0(x, u, Du) h(\lambda) + \beta(x) l(\lambda),$$

with some continuous functions $h : \mathbb{R} \to \mathbb{R}^+$, $l : \mathbb{R} \to \mathbb{R}$ and $\beta \in L^q(\Omega)$. Then

$$A_\lambda(u) = F$$

can be written in the form

$$B(u) = \frac{F - l(\lambda)\beta}{h(\lambda)}$$

where $B(u)$ is defined by

$$\langle B(u), v \rangle = \int_\Omega \left[\sum_{j=1}^n b_j(x, u, Du) D_j v + b_0(x, u, Du) v \right], \quad u, v \in V \tag{3}$$

Theorem

Assume that operator $B : V \to V^\star$ satisfies the assumptions (A_1) - (A_3) then the unique solution of

$$A_\lambda(u) = F \text{ is}$$

$$u = u_\lambda = B^{-1}\left(\frac{F - l(\lambda)\beta}{h(\lambda)}\right)$$

and thus

$$g(\lambda) = M(u_\lambda) = M\left[B^{-1}\left(\frac{F - l(\lambda)\beta}{h(\lambda)}\right)\right].$$

Since $B^{-1} : V^\star \to V$ and $M : V \to \mathbb{R}$, l, h are continuous, $g : \mathbb{R} \to \mathbb{R}$ is a continuous function. Further, the number of solutions of problem (1) with homogeneous boundary condition equals to the number of real solutions of the equation $g(\lambda) = \lambda$.

Now consider two particular cases.
1. Assume that B, M are homogeneous in the sense

$$B^{-1}(\mu F) = \mu^{\frac{1}{p-1}} B^{-1}(F) \text{ for all } \mu \geq 0 \quad (p > 1),$$

$$M(\mu u) = \mu^{\sigma} M(u) \text{ for all } \mu \geq 0 \quad (\sigma \geq 0)$$

(M is nonnegative). Then

$$g(\lambda) = \frac{M\{B^{-1}[F - l(\lambda)\beta]\}}{h(\lambda)^{\frac{\sigma}{p-1}}}.$$

Theorem

Assume that l, β are arbitrary continuous functions and g is a positive continuous function such that $\lambda = g(\lambda)$ has exactly N roots ($N = 0, 1, \cdots, \infty$) then our boundary value problem (with 0 boundary condition) has exactly N solutions with

$$h(\lambda) = \left[\frac{M\{B^{-1}[F - l(\lambda)\beta]\}}{g(\lambda)}\right]^{\frac{p-1}{\sigma}}.$$

We have this particular case with $\beta = 0$ if e.g. B is defined by the p-Laplacian, i.e.

$$b_j(x, \eta, \zeta) = |\zeta|^{p-2}\zeta, \quad j = 1, \cdots, n, \quad b_0(x, \eta, \zeta) = c|\eta|^{p-2}\eta$$

$\eta \in \mathbb{R}$, $\zeta \in \mathbb{R}^n$ with some $c > 0$. (If $V = W_0^{1,p}$ then c may be 0, too.) Further,

$$M(u) = \int_\Omega \left[\sum_{j=1}^{n} a_j(x)|D_j u|^\sigma + a_0(x)|u|^\sigma \right] dx$$

where $a_j \in L^\infty(\Omega)$, $a_j > 0$, $0 < \sigma \le p$.

2. Assume that B and M are linear Then

$$g(\lambda) = \frac{M[B^{-1}(F)] - l(\lambda)M[B^{-1}(\beta)]}{h(\lambda)}.$$

Theorem

If g is a positive continuous function such that $\lambda = g(\lambda)$ has N roots ($N = 0, 1, \cdots, \infty$) then our boundary value problem has N solutions with

$$h(\lambda) = \frac{M[B^{-1}(F)] - l(\lambda)M[B^{-1}(\beta)]}{g(\lambda)}$$

and arbitrary continuous function l. Similarly, if $M[B^{-1}(\beta)] \neq 0$ and g is a continuous function such that $\lambda = g(\lambda)$ has N roots then our boundary value problem has N solutions with

$$l(\lambda) = \frac{-g(\lambda)h(\lambda) + M[B^{-1}(F)]}{M[B^{-1}(\beta)]}$$

In this case operator $M : W^{1,2}(\Omega) \to \mathbb{R}$ may have the form

$$Mu = \int \left[\sum_{j=1}^{n} a_j D_j u + a_0 u \right] + \int_{\partial\Omega} b_0 u d\sigma$$

where $a_j \in L^2(\Omega)$, $b_0 \in L^2(\partial\Omega)$.

Corollary

Consider the particular case $h(\lambda) = 1$, $l \in C^1(\mathbb{R})$, $M(B^{-1}\beta) \neq 0$. If $\inf_{\lambda \in \mathbb{R}} l'(\lambda) > 0$ or $\sup_{\lambda \in \mathbb{R}} l'(\lambda) < 0$ then for any $F \in V^\star$ the problem (2) has exactly one solution u. In this case the mapping $F \mapsto u$ is continuous since $h^{-1} : \mathbb{R} \to \mathbb{R}$ is continuous.

Corollary

Consider (for simplicity) the case $h(\lambda) = 1$, $M(B^{-1}\beta) \neq 0$. With fixed function l the number of solutions depends on F (on the value of $M(B^{-1}F)$).

This statement can be illustrated by the following bifurcation result. Let $F \in V^\star$ be fixed and consider μF instead of F with some parameter $\mu \in \mathbb{R}$. Then equation $\lambda = g(\lambda)$ has the form

$$\lambda = \mu M(B^{-1}F) - l(\lambda)M(B^{-1}\beta). \tag{4}$$

Let

$$l(\lambda) = -\frac{\lambda}{M(B^{-1}\beta)} + \sin\lambda,$$

thus (4) can be written as

$$0 = \mu M(B^{-1}F) - M(B^{-1}\beta)\sin\lambda. \tag{5}$$

Consequently, for

$$\mu = \mu_0 = \frac{M(B^{-1}\beta)}{M(B^{-1}F)}$$

equation (4) has infinitely many λ roots and for $\mu > \mu_0$ it has no roots.
It is not difficult to show that if

$$I(\lambda) = -\frac{\lambda}{M(B^{-1}\beta)} + \sin\lambda + \frac{1}{\lambda},$$

then for $\mu = \mu_0$ the equation (4) has no roots but for $0 < \mu/\mu_0 < 1$ it has infinitely many roots.

Now consider equations (1) containing nonlinear and nonlocal operators of the form

$$B(u) = F(u) \tag{6}$$

where B is given by (3) and $F : V \to V^\star$ is a given nonlinear operator. Clearly, $u \in V$ satisfies (6) iff

$$u = B^{-1}[F(u)] = G(u) \tag{7}$$

where $G : V \to V$ is a given operator, i.e. u is a fixed point of G. Then

$$F(u) = B[G(u)]. \tag{8}$$

Now we shall consider particular cases for G.

1.

$$[G(u)](x) = [K(u)](x) = \int_\Omega \mathcal{K}(x,y)u(y)dy \tag{9}$$

where $\mathcal{K} \in L^2(\Omega \times \Omega)$, $u \in V \subset W^{1,2}(\Omega)$ and B is a linear strongly elliptic differential operator. The equation (7) has k (finite) $u \in V$ solutions if 1 is an eigenvalue of G with multiplicity k. If $\mathcal{K}$ is sufficiently smooth with respect to x then by (8)

$$[F(u)](x) = \int_\Omega B_x[\mathcal{K}(x,y)]u(y)dy$$

and (6) has the form

$$[B(u)](x) = \int_\Omega B_x[\mathcal{K}(x,y)]u(y)dy. \tag{10}$$

Theorem

If 1 is an eigenvalue of G with multiplicity k then (10) has k solutions.

2.

$$G(u) = Ku + h(P(u))g \tag{11}$$

where K is given by (9), $P : V \to \mathbb{R}$ is a linear continuous functional, $h : \mathbb{R} \to \mathbb{R}$ is a continuous function and $g \in V$. Assume that $\mathcal{K}$ is the function before and 1 is not an eigenvalue of the operator K. Then u satisfies (11) iff

$$u = h(P(u))[I - K]^{-1}(g) \tag{12}$$

and (6) has the form

$$B(u) = \int_\Omega B_x[\mathcal{K}(x,y)]u(y)dy + h(P(u))Bg. \tag{13}$$

Let $u_\lambda = h(\lambda)[I - K]^{-1}(g)$ then

$$P(u_\lambda) = h(\lambda)P([I - K]^{-1}(g)).$$

Theorem

u is a solution of (13) iff $u = h(\lambda)[I - K]^{-1}(g)$ where λ is a root of the equation

$$\lambda = h(\lambda)P([I - K]^{-1}(g)). \tag{14}$$

Thus the number of solutions of (13) equals the number of solutions of equation (14).

Now we remind the definition of weak solutions of initial-boundary value value problems for nonlinear parabolic differential equations

$$D_t u - \sum_{j=1}^{n} D_j[a_j(t, x, u, Du; u)] + a_0(t, x, u, Du; u) = F \qquad (15)$$

(for simplicity) with homogeneous initial and boundary condition. Denote by $L^p(0, T; V)$ the Banach space of functions $u : (0, T) \to V$ ($V \subset W^{1,p}(\Omega)$ is a closed linear subspace) with the norm

$$\| u \| = \left[\int_0^T \| u(t) \|_V^p \, dt \right]^{1/p} \qquad (1 < p < \infty).$$

The dual space of $L^p(0, T; V)$ is $L^q(0, T; V^\star)$ where $1/p + 1/q = 1$.

Weak solutions of (15) with zero initial and boundary condition is a function $u \in L^p(0, T; V)$ satisfying $D_t u \in L^q(0, T; V^\star)$ and

$$D_t u + A(u) = F, \quad u(0) = 0$$

where $F \in L^q(0, T; V^\star)$ is a given function,

$$\langle [A(u)](t), v \rangle = \int_\Omega \left[\sum_{j=1}^n a_j(t, x, u, Du; u) D_j v + a_0(t, x, u, Du; u) v \right] dx \tag{16}$$

for all $v \in V$, almost all $t \in [0, T]$. (For $p \geq 2$, $u \in L^p(0, T; V)$ and $D_t u \in L^q(0, T; V^\star)$ imply $u \in C([0, T]; L^2(\Omega))$. By using the theory of pseudomonotone operators, one can formulate and prove an existence theorem on the above problem.

Now we shall consider certain particular cases for this problem when existence of multiple solutions will be proved. Consider first the case of this problem without nonlocal terms

$$D_t u + \tilde{A}(u) = F, \quad u(0) = 0 \tag{17}$$

where

$$\tilde{A}(u) = -\sum_{j=1}^{n} D_j[a_j(t, x, u, Du)] + a_0(t, x, u, Du),$$

i.e.

$$\langle [\tilde{A}(u)](t), v\rangle = \int_\Omega \left[\sum_{j=1}^{n} a_j(t, x, u, Du) D_j v + a_0(t, x, u, Du) v \right] dx \tag{18}$$

(C_1). The functions $a_j : (0, T) \times \Omega \times \mathbb{R}^{n+1} \to \mathbb{R}$ $(j = 0, 1, \cdots, n)$ satisfy the Carathéodory conditions.
(C_2). There exist a constant C_1 and a function $k_1 \in L^q((0, T) \times \Omega)$ $(1/p + 1/q = 1)$ such that

$$|a_j(t, x, \xi)| \leq C_1[1 + |\xi|^{p-1}] + [k_1(t, u)](x),$$

$j = 0, 1, \cdots, n$, for a.e. $(t, x) \in (0, T) \times \Omega$, each $\xi \in \mathbb{R}^{n+1}$.
(C_3). The inequality

$$\sum_{j=1}^{n} [a_j(t, x, \xi) - a_j(t, x, \xi^\star)](\xi_j - \xi_j^\star) \geq C_2|\xi - \xi^\star|^p$$

holds with come constant $C_2 > 0$.

Theorem

Assume (C1) - (C3). Then for any $F \in L^q(0, T; V^\star)$ there exists a unique $u \in L^p(0, T; V)$ weak solution of (17) which depends on F continuously.

First consider a semilinear parabolic functional equation of the form

$$\tilde{B}u := D_t u + Au = D_t u - \sum_{j,k=1}^{n} D_j[a_{jk}(x) D_k u] + a_0(x)u = k(Mu)F_1 + F_2 \tag{19}$$

(i.e. the elliptic operator A in (16) is linear), where $M : L^2(0,T;V) \to \mathbb{R}$ is a given linear continuous functional, $V \subset W^{1,2}(\Omega)$, $k : \mathbb{R} \to \mathbb{R}$ is a given continuous function, $F_1, F_2 \in L^2(0,T;V^\star)$. Further, $a_{jk}, a_0 \in L^\infty(\Omega)$, $a_{jk} = a_{kj}$ and the functions a_{jk} satisfy the uniform ellipticity condition

$$c_1|\xi|^2 \leq \sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k + a_0(x)\xi_0^2 \leq c_2|\xi|^2$$

for all $\xi = (\xi_0, \xi_1, ..., \xi_n) \in \mathbb{R}^{n+1}$, $x \in \Omega$ with some positive constants c_1, c_2.

It is well known that in this case for all $F \in L^2(0,T;V^\star)$ there exists a unique weak solution u of

$$D_t u - \sum_{j,k=1}^{n} D_j[a_{jk}(x) D_k u] + a_0(x) u = F, \tag{20}$$

denoted by $u = \tilde{B}^{-1} F$ where $\tilde{B}^{-1} : L^2(0,T;V^\star) \to L^2(0,T;V)$ is a linear continuous operator. Consequently, $u \in L^2(0,T;V)$ is a weak solution of (19) if and only if

$$u = k(Mu)\tilde{B}^{-1}F_1 + \tilde{B}^{-1}F_2. \tag{21}$$

This equality implies that

$$Mu = k(Mu)M(\tilde{B}^{-1}F_1) + M(\tilde{B}^{-1}F_2). \tag{22}$$

It is not difficult to prove

Theorem

A function $u \in L^2(0,T;V)$ is a weak solution of (19) if and only if $\lambda = Mu$ satisfies the equation

$$\lambda = k(\lambda)M(\tilde{B}^{-1}F_1) + M(\tilde{B}^{-1}F_2). \tag{23}$$

and

$$u = k(\lambda)\tilde{B}^{-1}F_1 + \tilde{B}^{-1}F_2. \tag{24}$$

Remark

The linear continuous functional $M : L^2(0,T;V) \to \mathbb{R}$ may have the form

$$Mu = \int_0^T \int_\Omega \left[K_0(t,x)u(t,x) + \sum_{j=1}^n K_j(t,x)D_j u(t,x) \right] dtdx \tag{25}$$

Corollary

The number of weak solutions u of (19) (with homogeneous initial-boundary condition) equals the number of solutions λ of equation (23).
Further, assuming $M(\tilde{B}^{-1}F_1) \neq 0$, for arbitrary $N = 0, 1, ..., \infty$ we can construct continuous functions $k : \mathbb{R} \to \mathbb{R}$ such that the initial-boundary value problem (19) has exactly N weak solutions, as follows. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function having N zeros and define function k by the formula

$$k(\lambda) = \frac{g(\lambda) + \lambda - M(\tilde{B}^{-1}F_2)}{M(\tilde{B}^{-1}F_1)}. \tag{26}$$

Then, clearly, equation (23) has N solutions. The number of solutions of (19), with fixed function k depends on F_2 (on the value of $M(\tilde{B}^{-1}F_2)$). It is not difficult to construct examples on bifurcation results on the number of solutions depending on F_2.

Now consider partial functional equations of the form

$$\tilde{B}u = D_t u + Au = C(u) \tag{27}$$

where A is a uniformly elliptic linear differential operator (see (19) or (20)) and $C : L^2(0, T; V) \to L^2(0, T; V^\star)$ is a given (possibly nonlinear) operator. Clearly, $u \in V$ satisfies (27) if and only if

$$u = \tilde{B}^{-1}[C(u)] =: G(u) \tag{28}$$

where $G : L^2(0, T; V) \to L^2(0, T; V$ is a given (possibly nonlinear) operator, i.e. u is a fixed point of G. Then

$$C(u) = \tilde{B}[G(u)] \tag{29}$$

Now we consider three particular cases for G

1. The operator G is defined by

$$[G(u)](t,x) = (Lu)(t,x) + F(t,x) = \tag{30}$$

$$\int_0^T \int_\Omega K(t,\tau,x,y)u(\tau,y)d\tau dy + F(t,x)$$

where $K \in L^2([0,T] \times [0,T] \times \Omega \times \Omega)$, $u \in L^2((0,T) \times \Omega)$.

Theorem

If K and F are sufficiently smooth and "good" then the solution $u \in L^2((0,T)\times\Omega)$ of (28) with the operator (30) belongs to $L^2(0,T;V)$, $D_t u$ belongs to $L^2(0,T;V^\star)$, $u(0)=0$,

$$(Cu)(t,x) = \int_0^T \int_\Omega [D_t K(t,\tau,x,y) + A_x K(t,\tau,x,y)] u(\tau,y) d\tau dy +$$

$$D_t F(t,x) + A_x F(t,x)$$

and the equation (27) has the form

$$(Bu)(t,x) = \int_0^T \int_\Omega [D_t K(t,\tau,x,y) + A_x K(t,\tau,x,y)] u(\tau,y) d\tau dy + D_t F(t,x) + A_x F(t,x). \tag{31}$$

($A_x K(t, \tau, x, y)$ denotes the differential operator applied to $x \to K(t, \tau, x, y)$.)
Further, if 1 is an eigenvalue of the linear integral operator L with multiplicity N then (31) may have N linearly independent solutions.
The proof is similar to the previous ones.

Remark

Similarly to the problems in the previous section, the value of solutions u of (19) in some time t, are connected with the values of u for $t \in [0, T]$.

Now consider operators G of the form

$$G(u) = Lu + h(Pu)F + H \tag{32}$$

where operator L is defined in (30) and its kernel has the same smoothness property, $P : L^2(0, T; V) \to \mathbb{R}$ is a linear continuous functional, $h : \mathbb{R} \to \mathbb{R}$ is a given continuous function and $F, H \in L^2(0, T; V)$, $D_t F, D_t H \in L^2(0, T; V)$. Here assume that 1 is not an eigenvalue of the integral operator $L : L^2((0, T) \times \Omega) \to L^2((0, T) \times \Omega)$.
It is not difficult to prove

Theorem

In this case equation (27) has the form

$$\tilde{B}u = \int_0^T \int_\Omega [D_t K(t,\tau,x,y) + A_x K(t,\tau,x,y)]u(\tau,y)\,d\tau dy + \quad (33)$$

$$h(Pu)\tilde{B}F + \tilde{B}H.$$

Further, u is a weak solution of (33) if and only if $u = h(\lambda)P[(I-L)^{-1}F] + (I-L)^{-1}H$ where λ is a root of the equation

$$\lambda = h(\lambda)P[(I-L)^{-1}F] + P[(I-L)^{-1}H]. \quad (34)$$

Thus the number of solutions of (33) equals the number of the roots of (34).

Corollary

If $P[(I-L)^{-1}F] \neq 0$ then for arbitrary N $(= 0, 1, ..., \infty)$ we can construct h such that (33) has N solutions, in the following way. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous functions having N zeros. Then (33) has N solutions if

$$h(\lambda) = \frac{g(\lambda) + \lambda - P[(I-L)^{-1}H]}{P[(I-L)^{-1}F]}.$$

Remark

The linear functional $P : L^2(0, T; V) \to \mathbb{R}$ may have the form (25).

Remark

For fixed functions h, F the number of solutions of (33) depends on H by (34). It may happen that the number of solutions of the problem with μF (where μ is a real parameter) is 0 *for $\mu > \mu_0$ and is some N $(= 1, 2, ..., \infty)$ for $\mu = \mu_0$. (See Corollary in the elliptic case.)*
Further, assuming that for the function φ defined by

$$\varphi(\lambda) = \lambda - h(\lambda)P[(I - L)^{-1}F]$$

we have

$$\inf_{\lambda \in \mathbb{R}} \varphi'(\lambda) > 0 \text{ or } \sup_{\lambda \in \mathbb{R}} \varphi'(\lambda) < 0$$

then for any (sufficiently smooth) H the equation (33) has exactly one solution.

Further problems: Parabolic functional equations with nonlocal terms of Volterra type.
Exitence of multiple solutions for $t \in (0, \infty)$.
Qualitative properties of these solutions as $t \to \infty$.

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