Gaussian curvature of piecewise flat manifolds

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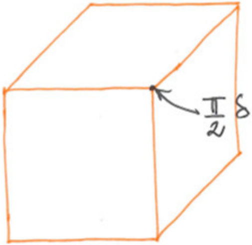
Outline

- Two words on curvature and Einstein equations.
- Non-linear Regge calculus: definition of curvature.
- Linearized Regge calculus in 3D.

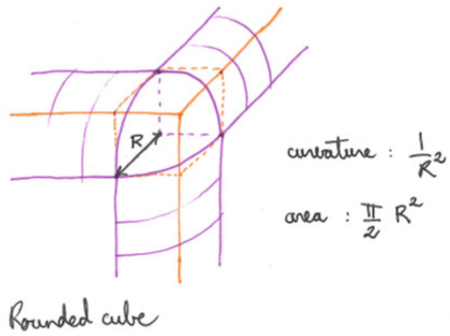


Gaussian curvature of cube

Curvature of the surface of the cube

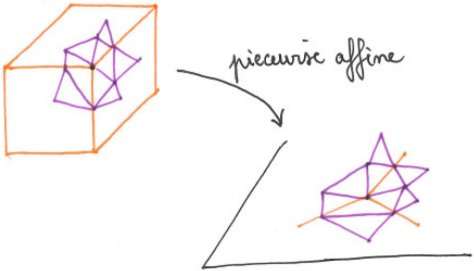
- Gauss-Bonnet theorem:
  Integral of densitized Gaussian curvature is
  the Euler-Poincaré characteristic times 2π.
- Constraints for a hypothetical curvature:
  – integral should be $4\pi = 2\pi(8 - 12 + 6)$,
  – no curvature on faces or edges.
  – symmetries of the cube: 8 equivalent vertices.

A first justification of the curvature of the cube



Works for convex surfaces... not intrinsic (cf. theorema egregium).

Another strategy: flattening



Flattening a part of the cube

Smoothe metric by convolution and prove convergence.

Riemannian geometry

- In coordinates a metric has the form:

$$g : \mathbb{R}^n \to \mathbb{R}^{n\times n}_{\mathrm{Sym}} \tag{1}$$

  Positive definite or signature $(-+++)$ for GR.
- Curvatures have the form:

$$R(g) = L_g(\partial^2 g) + Q_g(\partial g), \tag{2}$$

  with L_g linear and Q_g quadratic.
- Highly undefined for discontinuous metrics...
  Discontinuous times derivatives and products of Diracs.
  Cancellations for partial continuity? (tangential-tangential)

Einstein equations

- Unknown is a Lorentzian metric on a manifold.
- Vacuum: Einstein curvature is zero.
- Diffeomorphism invariance:
  Gauge fixing and constraint preservation.
- Einstein-Hilbert action:

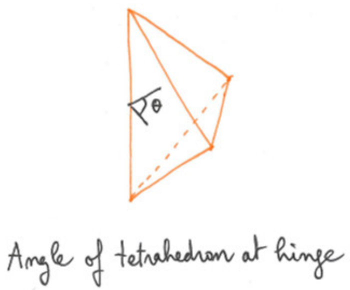
$$\mathcal{A}(g) = \int \kappa(g)\mu_g. \tag{3}$$

- 2D: all metrics are solutions,
  3D: only flat space (locally up to diffeomorphism),
  4D: interesting.

Regge calculus

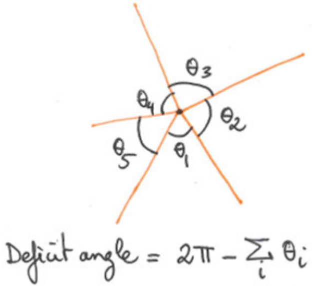
- REGGE, T.: *General relativity without coordinates*; Nuovo Cimento (10), Vol. 19, p. 558–571, 1961.
- Manifold represented by simplicial complex.
- Piecewise constant metric determined by edge lengths. Equivalently: TT continuous.
- Combinatorial action: from edge lengths sum over hinges of deficit angles times area.
- Discrete variational principle: Find critical points of combinatorial action on discrete space.

Angles inside simplices



Angles on codim 2 faces, between codim 1 faces.
Determined by edgelengths.

Deficit angle at a hinge



Example: deficit angle at vertex of cube is $2\pi - 3 \cdot \pi/2 = \pi/2$.

Regge's discrete equations

- Combinatorial Regge action, given edge lengths:
  sum, over hinges, of deficit angles times area.
- Relation to Einstein-Hilbert action?
- Local interpretation: Regge curvature defined by:

$$\sum_{\text{hinges}: H} (2\pi - \sum \theta)_H \delta_H. \tag{4}$$

- Also important: relation of discrete solutions to exact ones...

Nice properties of 2D Regge calculus

- Gauss-Bonnet relates the curvature of a smooth metric to the Euler-Poincaré characteristic of simplicial complex:

$$\int \kappa(g)\mu_g = 2\pi\chi = 2\pi(\# V - \# E + \# F). \tag{5}$$

- Analogue in Regge (using $2\# E = 3\# F$):

$$\int \sum_V \text{deficit}_V \delta_V = \sum_V (2\pi - \sum \theta) \tag{6}$$

$$= 2\pi \# V - \pi \# F = 2\pi(\# V - \# E + \# F) \tag{7}$$

- Regge calculus defines a local curvature, supported where reasonable, with right global property.

Is Regge calculus conforming?

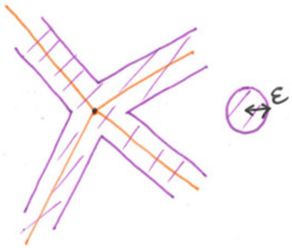
- Positive metric $g : \mathbb{R}^n \to \mathbb{R}^{n\times n}_{\mathrm{Sym}}$ of Regge type: piecewise constant and TT-continuous on interfaces.
- Regularize it by convolution:

$$g_\epsilon = g * \phi_\epsilon. \tag{8}$$

- Claim: curvatures converge to Regge curvature for $\epsilon \to 0$:

$$\kappa(g_\epsilon)\mu_{g_\epsilon} \to (2\pi - \sum \theta)\delta, \tag{9}$$

in the sense of measures.



Smoothing a piecewise constant metric

Proof : 2-D

- Claim: Fix ϵ, then $\kappa_(g_\epsilon)\mu_{g_\epsilon}$ has bounded support and:

$$\int \kappa(g_\epsilon)\mu_{g_\epsilon} = 2\pi - \sum \theta. \tag{10}$$

- In orthonormal frame, connection 1-form A of g_ϵ:

$$\mathrm{d}A = Rie = J\kappa\mu, \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{11}$$

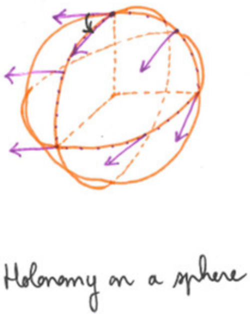
  Integrate, use Stokes:

$$\int_{\partial T} A = J \int_T \kappa\mu. \tag{12}$$

- Take exponential, recognize a parallell transport on LHS.

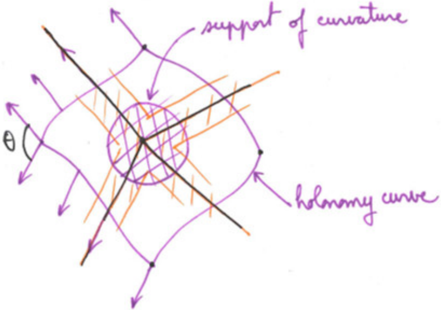
Sailing around on the globe

Parallel transport: solve $\dot{Q} + AQ = 0$ along a path.



After one loop the vector gets turned by $\pi/2$...

Sailing around on a rough plane



Holonomy around hinge

Proof : 2-D continued

- Parallell transport around ∂T (SO(2) abelian).

$$\mathrm{Hol}(A, \partial T) = \exp(\int_{\partial T} A) \tag{13}$$

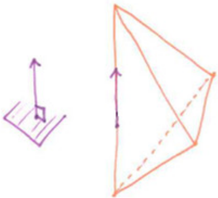
- One vector is easy to parallell transport.
  Determines LHS as rotation matrix by angle $-\sum\theta$.

$$-\sum\theta = \int_T \kappa\mu + 2\pi k, \quad k \in \mathbb{Z}. \tag{14}$$

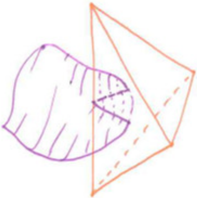
- Continuity argument gives $k = -1$.
- g_ϵ related to $g_{\epsilon'}$ by pullback and scaling.
  Densitized curvatures related just by pullback.

Proof : N-D

- Idea: find 2-dimensional manifold, transverse to hinge, on which to apply previous techniques.
  – Linear: difficult to relate curvature of restriction to curvature, and deficit angles inside to deficit angles outside.
  – Orthogonal to hinge in each sector: polyhedral, nonsmooth.
- Fix ϵ, smoothe the metric. At each point consider the orthogonal of the hinge, with respect to the smoothed metric: 2-dimensional linear space.



An integrable distribution



A 2-dimensional transverse manifold

Proof : N-D

- This is an integrable distribution so Frobenius theorem applies and defines a 2-dimensional smooth manifold.
  Essential ingredient: $\nabla f = 0$ for f in hinge.
- Integral of densitized curvature on 2-manifold is deficit angle.
- Conclude by pointwise growth estimates as $\epsilon \to 0$.

3D Elasticity complex

$$\begin{array}{ccccccc}
C^\infty(S,\mathbb{V}) & \xrightarrow{\text{def}} & C^\infty(S,\mathbb{S}) & \xrightarrow{\text{curl}\,\mathrm{T}\,\text{curl}} & C^\infty(S,\mathbb{S}) & \xrightarrow{\text{div}} & C^\infty(S,\mathbb{V}) \\
\downarrow I_h^0 & & \downarrow I_h^1 & & \downarrow I_h^2 & & \downarrow I_h^3 \\
X_h^0 & \xrightarrow{\text{def}} & X_h^1 & \xrightarrow{\text{curl}\,\mathrm{T}\,\text{curl}} & X_h^2 & \xrightarrow{\text{div}} & X_h^3
\end{array} \tag{15}$$

- Notations: $S \subset \mathbb{R}^3$, $\mathbb{V} = \mathbb{R}^3$, $\mathbb{S} = \mathbb{R}^{3\times 3}_{\mathrm{Sym}}$.
- Arnold, Falk and Winther 2006.
- Good FE spaces X_h^2, X_h^3 presuppose good FE spaces X_h^1. Conforming polynomials require high order (Argyris...)
- Related to de Rham complex by BGG construction.

Elasticity complex: relativity

$$\begin{array}{ccccccc}
C^\infty(S,\mathbb{V}) & \xrightarrow{\operatorname{def}} & C^\infty(S,\mathbb{S}) & \xrightarrow{\operatorname{curl} \operatorname{T} \operatorname{curl}} & C^\infty(S,\mathbb{S}) & \xrightarrow{\operatorname{div}} & C^\infty(S,\mathbb{V}) \\
\downarrow I_h^0 & & \downarrow I_h^1 & & \downarrow I_h^2 & & \downarrow I_h^3 \\
X_h^0 & \xrightarrow{\operatorname{def}} & {\color{red}X_h^1} & \xrightarrow{\operatorname{curl} \operatorname{T} \operatorname{curl}} & X_h^2 & \xrightarrow{\operatorname{div}} & X_h^3
\end{array} \tag{16}$$

- SHC 2011. Lower regularity for X_h^2 than elasticity.
- Complex encodes:
  – 1: linearized diffeomorphism invariance,
  – 2: linearized Bianchi identity (div free energy momentum).

Elasticity complex: Regge

$$
\begin{array}{ccccccc}
C^\infty(S,\mathbb{V}) & \xrightarrow{\text{def}} & C^\infty(S,\mathbb{S}) & \xrightarrow{\text{curl}\,\mathrm{T}\,\text{curl}} & C^\infty(S,\mathbb{S}) & \xrightarrow{\text{div}} & C^\infty(S,\mathbb{V}) \\
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\end{array}
\tag{17}
$$

- X_h^0: Continuous piecewise affine vectorfields.
- X_h^1: TT-continuous piecewise constant metrics (Regge).
- X_h^2: Dirac deltas on edges: $\tau_e \otimes \tau_e \delta_e$.
- X_h^3: Dirac deltas on vertices: $\mathbb{V}\delta_v$.

Is Regge calculus conforming? (bis)

- The Einstein-Hilbert action and the Regge action have the same second variation (linearization) around flat, namely $u \mapsto \langle \operatorname{curl} \mathrm{T} \operatorname{curl} u, u \rangle$.
- *Proof:* combinatorial formula for second variation in Regge calculus matches combinatorial formula for $\operatorname{curl} \mathrm{T} \operatorname{curl}$ (computed in the sense of distributions).
- Ok to define parallell transport by hand.

Can Regge calculus be justified for linearized GR?

- Linearized GR is a wave equation with curl T curl in space. But constraints involving trace and divergence.
- Convergent eigenvalue problem for curl T curl in RC.
- Proof inspired by Maxwell eigenvalue problem:
  – Boffi, D.: *Finite element approximation of eigenvalue problems*; Acta Numer., Vol. 19, p. 1–120, 2010.
  – SHC and Winther, R.: *On variational eigenvalue approximation of semidefinite operators*; IMA J. Numer. Anal., Vol. 33, No. 1, p. 164–189, 2013.

Problems and ideas of the proof

- Mollified interpolators gives stable commuting projections.
  Guarantees nice kernel and nice orthogonal of the kernel.
- Non-conforming method (critical case):
  L^2 metrics with curl T curl in H^{-1} do not contain Regge!
- Non semi-definite, need for inf-sup.
  Proved combinatorially for a weak norm.

Prescribed densitized scalar curvature

- Funny identity:

$$(\kappa\mu)(\exp(2u)g_0) = (\Delta_0 u)\mu_0. \tag{18}$$

- Discrete conformal tranformations:

$$\mathcal{C}(u) : g_{ij} \mapsto \exp(u_i + u_j)g_{ij}. \tag{19}$$

- Given f find u such that:

$$(\kappa\mu)(\mathcal{C}(u, g_0)) = \sum f_i \delta_i. \tag{20}$$

- Linearize around $u = 0$ gives: Find u such that for all v

$$\int \operatorname{grad} u \cdot \operatorname{grad} v = \sum f_i v_i. \tag{21}$$

Laplace equation with P^1 elements (for g_0).

References

- SHC: *On the linearization of Regge calculus*; Numer. Math., Vol. 119, No. 4, p. 613–640, Springer, 2011.
- SHC: *Exact formulas for the approximation of connections and curvature*; arXiv:1307.3376.
- D. N. Arnold, R. S. Falk, R. Winther: *Differential complexes and stability of finite element methods. II. The elasticity complex*; Compatible spatial discretizations, IMA Vol. Math. Appl., Vol. 142, p. 47–67, Springer, 2006.
- D. N. Arnold, R. Winther: *Notes on linearized Einstein equations*; Unpublished notes. (Elasticity related to linearized Einstein equations)