Analysis of an age-structured epidemic model

Szemenyei Adrián László

December 5, 2024

1. Introduction
2. Presentation of our model
3. Analysis of the model
4. Literature review

The general age-dependent SIR model

$$\frac{\partial s(a,t)}{\partial t} + \frac{\partial s(a,t)}{\partial a} = -s(a,t)\lambda\big(a,i(.,t)\big) - \mu(a)s(a,t) \tag{1}$$

$$\frac{\partial i(a,t)}{\partial t} + \frac{\partial i(a,t)}{\partial a} = s(a,t)\lambda\big(a,i(.,t)\big) - \big(\mu(a)+\gamma(a)\big)i(a,t) \tag{2}$$

$$\frac{\partial r(a,t)}{\partial t} + \frac{\partial r(a,t)}{\partial a} = \gamma(a)i(a,t) - \mu(a)r(a,t) \tag{3}$$

where $s(a,t)$ is the density of susceptibles of age a at time t.
$i(a,t), r(a,t)$ are the infected and recovered subpopulations.

- Boundary conditions? what is $\mu(.)\,\lambda(.), \lambda(.)$?

Derivation of the equations

- Consider a cohort of individuals in an age interval $[a, a + \Delta a]$
- The number of susceptibles in that cohort is approx $s(a, t)\Delta a$
- after small time Δt: age $a \to a + \Delta t$, time $t \to t + \Delta t$
- number of individuals in this same cohort is $s(a + \Delta t, t + \Delta t)\Delta a$
- Change in the subpopulation by age-specific per-capita death rate $\mu(a)$ and getting infected
- The **balance law**:

$$s(a + \Delta t, t + \Delta t)\Delta a - s(a, t)\Delta a = -\mu(a)s(a, t)\Delta t\Delta a \quad (4)$$

$$- \text{age-spec incidence rate} s(a, t)\Delta t\Delta a \quad (5)$$

- dividing by $\Delta t\Delta a$ RHS:

$$\frac{s(a + \Delta t, t + \Delta t) - s(a, t + \Delta t)}{\Delta t} + \frac{s(a, t + \Delta t) - s(a, t)}{\Delta t}$$

- We suppose some regularity on $s(a, t)$ and take the limit $\Delta t \to 0$.

Derivation II.

- There is a **maximal age** $a\dagger$.
- No one survives the maximal age: $\lim_{a \to a\dagger} \mu(a) = \infty$.
- Boundary conditions:
  1. s(0,t) is the newborns at time t:

     $$s(0,t) = \int_0^{a\dagger} \beta(a)(s(a,t) + i(a,t) + r(a,t))da$$

     where $\beta(a)$ age-spec. per capita **birth rate**.
  2. Initial subpopulation (density)

     $$s(a,0) = s_0(a)$$

Age-dependent SIR model with vertical transmission

$$\frac{\partial s(a,t)}{\partial t} + \frac{\partial s(a,t)}{\partial a} = -s(a,t)\lambda\big(a,(i(.,t)\big) - \mu(a)s(a,t) \tag{6}$$

$$\frac{\partial i(a,t)}{\partial t} + \frac{\partial i(a,t)}{\partial a} = s(a,t)\lambda\big(a,(i(.,t)\big) - \big(\mu(a)+\gamma(a)\big)i(a,t) \tag{7}$$

$$\frac{\partial r(a,t)}{\partial t} + \frac{\partial i(a,t)}{\partial a} = \gamma(a)i(a,t) - \mu(a)r(a,t) \tag{8}$$

$$s(a,0) = s_0(a),\ i(a,0) = i_0(a),\ r(a,0) = r_0(a) \tag{9}$$

$$s(0,t) = \int_0^{a\dagger} \beta(a)\big(s(a,t) + r(a,t) + (1-q)i(a,t)\big)da \tag{10}$$

$$i(0,t) = q\int_0^{a\dagger} \beta(a)i(a,t)da \tag{11}$$

$$r(0,t) = 0, \tag{12}$$

what are $\gamma(a)$, q? what is λ?

Incidence rate λ

In the literature the force of infection is

$$s(a,t)\lambda(a,(i(.,t)) = s(a,t)\int_0^{a\dagger}\kappa(a,\xi)i(\xi,t)d\xi$$

which is sometimes simplified into the separable case/proportional mixing case:

$$k(a,\xi) = k_1(a)k_2(\xi)$$

possibilities to generalize:

Case 1.

$$s(a,t)\lambda(a,(i(.,t)) = s(a,t)\kappa_1(a)g\Bigg(\int_0^{a\dagger}\kappa_2(a)i(\xi,t)d\xi\Bigg) \quad (13)$$

Case 2.

$$s(a,t)\lambda(a,(i(.,t)) = s(a,t)\int_0^{a\dagger}K(a,\xi)g\big(i(\xi,t)\big)d\xi, \quad (14)$$

where $g(.)$ is some function ($g = id$ case).

Remarks

2. Other way to heterogenize the population: Time-since Infection models

$$\frac{\mathrm{d}S(t)}{\mathrm{d}t} = \Lambda - S(t) \int_0^\infty \beta(\tau) i(\tau, t) d\tau - \mu S(t) \tag{15}$$

$$\frac{\partial i(\tau, t)}{\partial \tau} + \frac{\partial i(\tau, t)}{\partial t} = -\gamma(\tau) i(\tau, t) - \mu i(\tau, t) \tag{16}$$

$$i(0, t) = S(t) \int_0^\infty \beta(\tau) i(\tau, t) d\tau \tag{17}$$

$$\frac{\mathrm{d}R(t)}{\mathrm{d}t} = \int_0^\infty \gamma(\tau) i\tau, t) d\tau - \mu R(t) \tag{18}$$

Used tools are more similar to ODE case.

Remarks II.

2. The equations for the total population denoted by $p(a,t) := s(a,t) + i(a,t) + r(a,t)$ is

$$\frac{\partial p(a,t)}{\partial t} + \frac{\partial p(a,t)}{\partial a} = -\mu(a)p(a,t) \tag{19}$$

with boundary-values

$$p(a,0) = s_0(a) + i_0(a) + r_0(a) \tag{20}$$

$$p(0,t) = \int_0^{a\dagger} \beta(a)p(a,t)da \tag{21}$$

which is called the **(linear) Lotka-McKendrick model**. 3 possible cases: population size constant, converges to a stationary age-distr/ exponentially dies out/ explodes

Depending on

$$\int_0^{a\dagger} \beta(a)e^{-\int_0^a \mu(s)ds}da$$

The *Foerster McKendrick model* **without age dependence** in its parameters, by integration simplifies to the **Malthus population model**:

$$\frac{\mathrm{d}P(t)}{\mathrm{d}t} = \beta P(t) - \mu P(t)$$

i.e. the population grows exponentially. No competition for resources.
A model with competition for resources is the **Verhulst/Logistic model**:

$$\frac{\mathrm{d}P(t)}{\mathrm{d}t} = ((\beta - \mu) - \omega P(t))P(t)$$

The population growth depend on the size of the population.
In the age-dependent case, *the Curtin-MacCamy equations* model of this phenonema:

$$\frac{\partial p}{\partial t} + \frac{\partial p}{\partial a} = -\mu(a, P)p(a,t)$$

$$p(0,t) = B(t) = \int_0^A \beta(a, P)p(a,t)da$$

$$p(a,0) = p_0(a);\ P(t) := \int_0^A p(a,t)da$$

Back to our model

$$\frac{\partial s(a,t)}{\partial t} + \frac{\partial s(a,t)}{\partial a} = -s(a,t)\lambda\big(a,(i(.,t))\big) - \mu(a)s(a,t) \tag{22}$$

$$\frac{\partial i(a,t)}{\partial t} + \frac{\partial i(a,t)}{\partial a} = s(a,t)\lambda\big(a,(i(.,t))\big) - \big(\mu(a)+\gamma(a)\big)i(a,t) \tag{23}$$

$$\frac{\partial r(a,t)}{\partial t} + \frac{\partial i(a,t)}{\partial a} = \gamma(a)i(a,t) - \mu(a)r(a,t) \tag{24}$$

$$s(a,0) = s_0(a),\ i(a,0) = i_0(a),\ r(a,0) = r_0(a) \tag{25}$$

$$s(0,t) = \int_0^{a\dagger} \beta(a)\big(s(a,t) + r(a,t) + (1-q)i(a,t)\big)da \tag{26}$$

$$i(0,t) = q\int_0^{a\dagger} \beta(a)i(a,t)da \tag{27}$$

$$r(0,t) = 0, \tag{28}$$

with force of infection:

Case 1.

$$s(a,t)\lambda(a,(i(.,t)) = s(a,t)\kappa_1(a)g\Big(\int_0^{a\dagger}\kappa_2(a)i(\xi,t)d\xi\Big) \tag{29}$$

Case 2.

$$s(a,t)\lambda(a,(i(.,t)) = s(a,t)\int_0^{a\dagger}K(a,\xi)g\big(i(\xi,t)\big)d\xi, \tag{30}$$

Assumptions

(A0) $q \in [0,1]$

(A1) $\mu \in L^{\infty}_{loc,+}(0,a\dagger)$, $\int_0^{a\dagger} \mu(a)da = \infty$, with $0 < \underline{\mu} \leq \mu(a)$ a.e. where $\underline{\mu} :=$ essinf$_{a\in[0,a\dagger]}\mu(a)$

(A2) $\beta \in L^{\infty}_{+}(0,a\dagger)$, where $\beta(a) \leq \overline{\beta} :=$ essup$_{a\in[0,a\dagger]}\,\beta(a)$ a.e.

(A3) $\gamma,\theta \in W^{1,\infty}(0,a\dagger)$ where $0 \leq \gamma(a) \leq \overline{\gamma} :=$ essup$_{a\in[0,a\dagger]}\gamma(a)$ and $0 \leq \theta(a) \leq \overline{\theta} :=$ essup$_{a\in[0,a\dagger]}\theta(a)$

B $g:[0,\infty) \to [0,\infty)$ such that

(B1) $g(0) = 0$,

(B2) g is continuously differentiable

(B3) g is strictly monotone increasing for $x \geq 0$ and concave.

(C1) $\kappa_1,\kappa_2 \in L^{\infty}_{+}(0,a\dagger)$ or $K \in L^{\infty}_{+}((0,a\dagger)\times(0,a\dagger))$ such that $\kappa_1,\kappa_2 \not\equiv 0$ a.e. or $K_{\not\equiv 0}$ a.e., respectively.

Remarks

1. assumptions on g implies the (local) lipschitz cont. on bounded sets.
2. Thus $g(||i||_1) \le c(r)||i||_1$ $(\forall\, 0 < ||i|| \le r)$
3. Without age depenedence we get back the non-linear SIR
4. g is fairly general considering useful epidemic models

Questions

We search for solutions in the state space $\mathbf{X} := \left(L^1(0, a\dagger)\right)^3$ with norm $||(s, i, r)^T||_{\mathbf{X}} = ||s||_1 + ||i||_1 + ||r||_1$. We denote the positive cone of $\mathbf{X}$ as $\mathbf{X}_+$, which is $\geq 0\, a.e.$, which is a Banach-Lattice.

- Does the solution **uniquely exists**?
- Does it **exists globally**?
- Does it **stays non-negative** if init. conds are non-negative.

$\Longrightarrow$ Answers through **Semigroup theory**.

- questions considerg the equilibria

Semilinear ACPs[1]

Proposition

Let the ACP be

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = \hat{A}u + \hat{F}(u) \tag{31}$$
$$u(0) = x \in \mathbf{Y} \tag{32}$$

where $(\hat{A}, \mathrm{Dom}(\hat{A}))$ *is the infinitesimal generator of a* C_0 *semigroup* $\left(T(t)\right)_{t\geq 0}$ *on the Banach space* $\mathbf{Y}$. *Then*

1. *If* $\hat{F}$ *is locally Lipschitz continuous (on bounded sets), then for each* $x \in \mathbf{Y}$ *there exist a maximal interval of existence* $[0, T_x)$ *and a unique continuous function* $t \mapsto u(t)$ *from* $[0, T_x)$ *to* $\mathbf{Y}$ *such that it is a mild solution of the ACP, namely*

[1] from the book: *Theory of nonlinear age-dependent population dynamics* by Webb Glenn (1985),Proposition 4.16.

Proposition (cont)

1

$$u(t) = T(t)x + \int_0^t T(t-s)\hat{F}(u(s))ds \tag{33}$$

for all $t \in [0, T_x)$. *In addition the solution either exist globally, or blows up in finite time, i.e.* $T_x = \infty$ *or* $\limsup_{t \to T_x} \|u(t)\|_{\mathbf{Y}} = \infty$, *respectively.*

2 *There is a continuous dependence on the initial conditions, namely: if* $x \in \mathbf{Y}$ *and* $0 \le t < T_x$, *then exists* $C, \varepsilon > 0$ *such that if* $\hat{x} \in \mathbf{Y}$ *and* $\|x - \hat{x}\|_{\mathbf{Y} < \varepsilon}$ *then* $t < T_{\hat{x}}$ *and* $\|u(s) - \hat{u}(s)\| \le C\|x - \hat{x}\|_Y$ *for all* $0 \le s \le t$, *where* $\hat{u}(t)$ *is the mild solution of the ACP* (31)-(32) *with initial condition* $\hat{x}$.

1. *If* $\hat{F}$ *is continuously Fréchet differentiable, then for all* $x \in \mathrm{Dom}(\hat{A})$,$u(t)$ *is a classical solution, namely:* $u(t) \in \mathrm{Dom}(\hat{A})$ $\big(\forall t \in [0, T_x)\big)$ *and* $t \mapsto u(t)$ *continously differentiable and satisfies the ACP* (31)-(32) *for* $t \in [0, T_x)$.

Semilinear Abstract Cauchy Problem formulation

Denote $S = diag(-\frac{\mathrm{d}}{\mathrm{d}a}, -\frac{\mathrm{d}}{\mathrm{d}a}, -\frac{\mathrm{d}}{\mathrm{d}a})$ with domain $Dom(S) = (W^{1,1}(0, a\dagger))^3$, $M_\mu = diag(-\mu, -\mu, -\mu)$ with domain $Dom(M_\mu) = \{\psi \in \mathbf{X} \,|\, \mu\psi \in \mathbf{X}\}$ and

$$M_{rest} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\gamma & 0 \\ 0 & \gamma & 0 \end{pmatrix} \tag{34}$$

which is a bounded linear operator in $\mathbf{X}$ (i.e. $M_{rest} \in L(\mathbf{X})$). Denote

$$B(a) = \begin{pmatrix} \beta(a) & (1-q)\beta(a) & \beta(a) \\ 0 & q\beta(a) & 0 \\ 0 & 0 & 0 \end{pmatrix}; \; \mathbf{B}\psi = \int_0^{a\dagger} B(a)\psi(a)da \in L(\mathbf{X}, \mathbb{R}^3) \tag{35}$$

cont.

Finally, denote

$$A := S + M_\mu,\ Dom(A) = \{\psi \in Dom(S) \cap Dom(M_\mu) \,|\, \psi(0) = \mathbf{B}\psi\} \quad (36)$$

Then equation (22)-(28) can be rewritten as an ACP:

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = Au + M_{rest}u + F(u) \quad (37)$$

$$u(0) = u_0 \in \mathbf{X} \quad (38)$$

where the nonlinear part, $F(u)$ is with $u = (s,i,r)^T \in \mathbf{X}$

$$F((s,i,r)^T) = \begin{pmatrix} -\lambda(.,i)s \\ \lambda(.,i)s \\ 0 \end{pmatrix} \quad (39)$$

which maps from **X** to **X**.

Proposition

[1] *The linear operator A generates a C_0 operator semigroup in* $\mathbf{X}$, *denoted by* $\left(e^{tA}\right)_{t\geq 0}$ *such that*

$$\|e^{tA}\|_{L(\mathbf{X})} \leq e^{(\bar{\beta}-\underline{\mu})t} \quad (\forall t \geq 0).$$

Proposition (from bounded perturbation thm. and Mentzler matrix struct.)

$(A + M_{rest}, Dom(A))$ *generates a positive C_0 operator semigroup in* $\mathbf{X}$, *denoted by* $\left(e^{t(A+M_{rest})}\right)_{t\geq 0}$ *such that*

$$\|e^{t(A+M_{rest})}\|_{L(\mathbf{X})} \leq e^{(\bar{\beta}-\underline{\mu})t} \quad (\forall t \geq 0). \tag{40}$$

Proposition

F, *defined in* (39) *is locally Lipschitz continuous on* $\mathbf{X}$.

[1] very similar to *Solvability of Age-Structured Epidemiological Models(...)*,

Proposition

F in (37) defined as (39) in Case 1. is also continuously Fréchet differentiable.

Proposition

The ACP (37)-(38) is positivity preserving, namely if $x \in \mathbf{X}_+$, then its solution $u(t) \in \mathbf{X}_+$ for all $t \in [0, T_x$.

Rewrite the ACP (37)-(38) as

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = \big(A + M_{rest} - \hat{\kappa} I\big)u + \big(\hat{\kappa} I + F\big)(u) \tag{41}$$

$$u(0) = x \in \mathbf{X}_+, \tag{42}$$

where $I \in L(\mathbf{X})$ is the identity operator on $\mathbf{X}$ and $\hat{\kappa} \geq 0$ will have to be determined later. The mild-solution for (41)-(42) is

$$u(t) = e^{-\hat{\kappa} t} e^{(A+M_{rest})t} x + \int_0^t e^{\hat{\kappa}(t-s)} e^{(A+M_{rest})(t-s)} (\hat{\kappa} I + F)\big(u(s)\big) ds \tag{43}$$

for $0 \leq t < T_x$.

Let $\overline{B}(r) := \{y \in \mathbf{X} \mid \|x\|_{\mathbf{X}} \leq r\}$ and $x \in \mathbf{X}_+ \cap \overline{B}(r)$. Then if we show that

$$(\hat{\kappa} I + F(\mathbf{X}_+ \cap \overline{B}(r)) \subset \mathbf{X}_+. \quad (44)$$

for some $\hat{\kappa} \geq 0$, which depends on r, then we are done, since the positivity of the mild solution (43) follows from the positivity of its Picard iterates. from[1]

Proposition

For any $x \in \mathbf{X}_+$ the unique mild solution of (37)-(38) *in Case 1. exists for all $t \in [0, \infty)$.*

[1] *from Solvability of Age-Structured Epidemiological Models with Intracohort Transmission*, by Banasiak, Massoukou

Questions considering the equilibria

Extra assumptions:

- Stationary population case (already at the initial time) $p(a,t) = p_\infty(a)$
- No vertical transmission (right now)

Usual trick:

$$x(a,t) := \frac{s(a,t)}{p_\infty(a)}, \qquad y(a,t) := \frac{i(a,t)}{p_\infty(a)}, \qquad z(a,t) := \frac{r(a,t)}{p_\infty(a)} \tag{45}$$

$$\frac{\partial x(a,t)}{\partial t} + \frac{\partial x(a,t)}{\partial a} = -x(a,t)\lambda\big(a,(y(.,t))\big) \tag{46}$$

$$\frac{\partial y(a,t)}{\partial t} + \frac{\partial y(a,t)}{\partial a} = x(a,t)\lambda\big(a,(y(.,t))\big) - \gamma(a)y(a,t) \tag{47}$$

$$\frac{\partial z(a,t)}{\partial t} + \frac{\partial z(a,t)}{\partial a} = \gamma(a)y(a,t) \tag{48}$$

$$\lambda(a,(y(.,t)) = \kappa_1(a)g\left(\int_0^{a\dagger} \kappa_2(a)p_\infty(\xi)y(\xi,t)d\xi\right) \tag{49}$$

Question of equilibria simplifies to a fixed-point problem $\hat{\lambda} = g\left(\hat{\lambda} \int_0^{a\dagger} x^*(a)h(a)da\right)$, where T is a linear majorant.

Proposition

If $T < 1$, then the only stationary solution/equilibrium is the trivial solution, i.e. $\hat{\lambda} = 0$.

If $T > 1$, then there is a unique positive equilibria.

where

$$T := g'(0)\left(\int_0^{a\dagger} h(a)da \right) \tag{50}$$

and

$$h(a) = \kappa_1(a) \int_a^{a\dagger} p_\infty(\eta) exp\left(- \int_\eta^a \gamma(\xi)d\xi \right) d\eta. \tag{51}$$

Stability of the equilibria

Possible tools:

- Lyapunov functional for global stability
- Local stability through perturbation and cont. dep. on initial conditions
- Other tools like persistence theory etc.

Proposition

If $T < 1$, then the disease-free equilibrium is **locally** *assymptotically stable, while for $T > 1$ it is unstable.*

We search for solutions in the form of

$$x_1(a,t) = H_1(a)e^{\rho} \text{ etc.}$$

where $x_1(a,t)$ is a perturbation of the equilibria $(x^*(a), \ldots)$. The question is the sign of ρ.

Fixed-point problem.

Global stability for the SIS model[1]

Monotone dynamical systems approach:

E_+ be its positive cone. Let $z(t)$ be a population vector that takes a value in a closed convex subset $C \subset E_+$. Suppose that the dynamics of the population vector $z(t)$ are written as a **semilinear Cauchy problem**:

$$\frac{\mathrm{d}z(t)}{\mathrm{d}t} = Az(t) + F(z(t)), \quad t > 0, \quad z(0) = z_0$$

We assume:

- A is a generator of a positive C_0 semigroup $\{e^{tA}\}_{t \geq 0}$ on E that satisfies $e^{tA}(C) \subset C$
- F is cont. Fréchet differentiable
- there exist $\alpha > 0$:
    - $(I - \alpha A)^{-1}(C) \subset C$; $(I + \alpha F)(C) \subset C$
    - (monotonicity of Resolvent)$(I - \alpha A)^{-1}\varphi \geq (I - \alpha A)^{-1}\psi \quad (\forall \varphi \geq \psi \in C)$
    - (monotonicity of F)$(I + \alpha F)\varphi \geq (I + \alpha F)\psi \quad (\forall \varphi \geq \psi \in C)$
    - (concavity of F) $\xi(I + \alpha F)\varphi \leq (I + \alpha F)\xi\varphi \quad (\forall \varphi \in C)\,(\forall \xi \in (0,1))$

[1] Busenberg

Existence of mild solution

One can rewrite the ACP as:

$$\frac{\mathrm{d}}{\mathrm{d}t} z(t) = \left(A - \frac{1}{\alpha}\right) z(t) + \frac{1}{\alpha}(I + \alpha F) z(t), \quad t > 0, \quad z(0) = z_0,$$

with its mild solution

$$z(t) = \mathrm{e}^{-\frac{1}{\alpha}t} \mathrm{e}^{tA} z_0 + \frac{1}{\alpha} \int_0^t \mathrm{e}^{-\frac{1}{\alpha}(t-\sigma)} \mathrm{e}^{(t-\sigma)A} (I + \alpha F) z(\sigma) \mathrm{d}\sigma.$$

The classical iterative procedure with the above assumptions gives the existence of the mild solution.

Under the above assumptions the mild solution $z(t) = U(t)z_0$ satisfies the following monotonicity and concavity:

$U(t)(C) \subset C$ and $U(t)\varphi \leq U(t)\psi$ for all $\varphi, \psi \in C$ such that $\varphi \leq \psi$,
$\xi U(t)\varphi \leq U(t)\xi\varphi$ for all $\varphi \in C$ and $\xi \in (0, 1)$.

Existence and stability of equilibria

Let z^* denote an equilibrium. Then, we have

$$\left(A - \frac{1}{\alpha}I\right)z^* + \frac{1}{\alpha}(I + \alpha F)z^* = 0.$$

Because $-(A - (1/\alpha)I)$ is positively invertible, we have the fixed point equation for z^* :

$$z^* = -\frac{1}{\alpha}\left(A - \frac{1}{\alpha}I\right)^{-1}(I + \alpha F)z^* = (I - \alpha A)^{-1}(I + \alpha F)z^* =: \Phi\left(z^*\right),$$

where Φ is a positive nonlinear operator preserving the invariance of the subset C. If Φ has a positive fixed point, it gives a positive equilibrium. Define the Fréchet derivative at the origin:

$$K_\alpha := \Phi'[0] := K_\alpha = (I - \alpha A)^{-1}\left(I + \alpha F'[0]\right),$$

where $F'[0]$ is the Fréchet derivative of the operator F at the origin.

Stability of equilibria

We can expect the spectral radius $\Phi'[0]$ to determine the existence and stability of the endemic and disease-free equilibrium. For this, the following is useful:
Lemma: The sign of $r(K_\alpha) - 1$ is independent of $\alpha > 0$ and coincides with the sign of $R_0 := r(F'[0](-A)^{-1}) - 1$. which can be interpreted as the asymptotic exponential growth rate of infective population.

The main theorem:

- for $R_0 < 1$ the DFE 0 is globally attractive in C.
- for $R_0 > 1$ the system has a unique equilibrium $i^* \in (D(A) \cap C) - \{0\}$ which is globally attractive in $C - \{0\}$.

Remarks:

- By the above theorem periodic solutions do not exist
- Local stability of the equilibria (if I haven't said it yet)

To show that for $r(K_\alpha) > 1$ there is at least one endemic equilibrium, one can show that K_α is

- $E_+ - E_+$ dense in E (for Riesz spaces $E_+ - E_+ = E$ since $x = x_+ - x_-$)
- positive operator
- bounded
- compact (by the Kolmogorov-Fréchet thm.)

Thus one can use the Krein-Rutman theorem i.e. $r(K_\alpha)$ is an eigenvalue of K_α associated with a positive eigenvector $\varphi \in C \subset E_+$. For this eigenvalue showing that for $0 < \xi$ small enough

$$\Phi(\xi\varphi)(a) \geq \xi\varphi(a)$$

Thus $\varphi_n = \Phi^n(\xi\varphi)$ converges to a nontrivial fixed point.
For the uniqueness, suppose that we do not have $u_\infty \leq v_\infty$, we show that **they can be compared**, then show that they equal also by order relations.

For the convergence of the equilibria:

$$\frac{\mathrm{d}i(t)}{\mathrm{d}t} \leq (A + F'[0])i(t), \qquad t > 0, i(0) \in C$$

where

$$F'[0](\varphi)(a) = \lambda[a|\varphi]\varphi(a) - \gamma(a)\varphi(a)$$

The spectral bound $\omega(A + F'[0])$ gives the Malthusian parameter of infective population (we won't prove) and

$$sign(R_0 - 1) = sign(\omega(A + F'[0])) = sign(r(K_\alpha) - 1)$$

thus for $r(K_\alpha) < 1$ the global stability of the trivial equilibrium follows.

For the $r(K_\alpha) > 1$, if one shows that:

- the endemic equilibria i^* is eventually positive, that is there exists $\xi \in (0,1)$ and $t^* > 0$ such that

  $$\xi i^* \leq U(t^*) i_0$$

  provided that $i_0 \in C - \{0\}$
  Which only means, that the solution is comparable with the steady state for one time instance.
- there exist a maximal point of C denoted by $\hat{i}$ (which in our case is $\hat{i} \equiv 1$ a.e.)

From the monotonic and concave properties of the operator:

$$\xi i^* = \xi U(t) i^* \leq U(t) \xi i^* \leq U(t) U(t^*) i_0 \leq U(t) \hat{i} \leq \hat{i}.$$

Hence, we can construct a nondecreasing sequence $\{U(t)^n \xi i^*\}_{n=0}^{+\infty}$ and a nonincreasing sequence $\left\{U(t)^n \hat{i}\right\}_{n=0}^{+\infty}$, both of which are bounded and converge to the unique i^*. Consequently, $U(t) U\left(t^*\right) i_0 = U\left(t + t^*\right) i_0$ also converges to i^* as $t \to +\infty$.

Above theorems can be used for the finite difference discretization.
for $K(.,)$ we get fixed point problems for operators and functions

Köszönöm a figyelmet!
Thank you for your attention!