Epidemic models with spatial dependence

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Mathematical models

Dealing with the integral - using Taylor series

Dealing with the integral - using numerical integration

Numerical solutions

Conclusions, future work

The original Kermack-McKendrick model

$$\begin{cases} \dfrac{dS(t)}{dt} = -aS(t)I(t), \\ \dfrac{dI(t)}{dt} = aS(t)I(t) - bI(t), \\ \dfrac{dR(t)}{dt} = bI(t), \end{cases} \tag{1}$$

$S(t)$ - the number of susceptible people (healthy, but can be ill)

$I(t)$ - the number of ill people

$R(t)$ - the number of recovered people

The size of the population is constant (births = natural deaths).

Extend the model with spatial dependence

Previously: an infectious person only infects at a certain point.

Extension: let us describe the infection with a function $F(x, x', y, y')$:

$$F(x, x', y, y') = \begin{cases} f_1(x')f_2(y'), & (x', y') \in B_\delta((x, y)) \\ 0 & \text{otherwise.} \end{cases}$$

where $B_\delta((x, y))$ denotes the δ radius ball with center at (x, y).

The extended model

Let us consider a domain $\Omega \in \mathbb{R}^2$ in which the propagation of the illness takes place.

From now on, $S(t, x, y)$ denotes the density of the susceptible people at time t at a point $(x, y) \in \mathbb{R}^2$.

The first equation in extended form:

$$\frac{\partial S(t,x,y)}{\partial t} = \\ = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x,x',y,y') I(t,x',y')dx'dy' \cdot S(t,x,y).$$

By the definition of $F(x, x', y, y')$:

$$\frac{\partial S(t,x,y)}{\partial t} = \\ = -\int_{-\delta_1}^{\delta_1}\int_{-\delta_2}^{\delta_2} f_1(|u_1|)f_2(|u_2|)I(t,x+u_1,y+u_2)du_1du_2 \cdot S(t,x,y).$$

The extended model

$$
\begin{cases}
\dfrac{\partial S(t,x,y)}{\partial t} = \\
= -\displaystyle\int_{-\delta_1}^{\delta_1}\int_{-\delta_2}^{\delta_2} f_1(|u_1|)f_2(|u_2|)I(t,x+u_1,y+u_2)du_1du_2 \cdot S(t,x,y) \\
\\
\dfrac{\partial I(t,x,y)}{\partial t} = \\
= \displaystyle\int_{-\delta_1}^{\delta_1}\int_{-\delta_2}^{\delta_2} f_1(|u_1|)f_2(|u_2|)I(t,x+u_1,y+u_2)du_1du_2 \cdot S(t,x,y) - \\
\qquad - bI(t,x,y) \\
\\
\dfrac{\partial R(t,x,y)}{dt} = bI(t,x,y)
\end{cases}
\tag{2}
$$

How to handle the integral?

This is a system of integro-differential equations, which we would like to solve numerically.

How to deal with the integral?

Two methods:

- Use **Taylor** series to expand the integrant.
- Use **numerical integration**, e.g. trapezoid rule.

Properties we would like to preserve

C_1 : the numbers of the individuals in classes S, I and R are nonnegative

C_2 : the size of the whole population is constant, i.e. $\int_\Omega S(t,x,y) + I(t,x,y) + R(t,x,y)\mathrm{d}x\mathrm{d}y = Constant$ for every t

C_3 : the size of the population of S is non-increasing in time

C_4 : the size of the population of R is non-decreasing in time

Using Taylor expansion

Main idea: let us approximate $I(t, x + u_1, y + u_2)$ using the Taylor expansion:

$$I(t,x + u_1, y + u_2) \approx$$

$$\approx I(t,x,y) + u_1 \frac{\partial}{\partial x} I(t,x,y) + u_2 \frac{\partial}{\partial y} I(t,x,y) +$$

$$+ \frac{u_1^2}{2!} \frac{\partial^2}{\partial x^2} I(t,x,y) + \frac{u_2^2}{2!} \frac{\partial^2}{\partial y^2} I(t,x,y) + u_1 u_2 \frac{\partial^2}{\partial x \partial y} I(t,x,y).$$

Using Taylor expansion

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x,x',y,y')I(t,x',y')dx'dy' \approx$$

$$\approx I(t,x,y)\int_{-\delta_1}^{\delta_1}\int_{-\delta_2}^{\delta_2} f_1(|u_1|)f_2(|u_2|)du_1du_2 +$$

$$+\frac{1}{2}\frac{\partial^2}{\partial x^2}I(t,x,y)\int_{-\delta_1}^{\delta_1}\int_{-\delta_2}^{\delta_2} u_1^2 f_1(|u_1|)f_2(|u_2|)du_1du_2 +$$

$$+\frac{1}{2}\frac{\partial^2}{\partial y^2}I(t,x,y)\int_{-\delta_1}^{\delta_1}\int_{-\delta_2}^{\delta_2} u_2^2 f_1(|u_1|)f_2(|u_2|)du_1du_2$$

Using Taylor expansion

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x,x',y,y')I(t,x',y')dx'dy' \approx$$

$$\approx I(t,x,y)\int_{-\delta_1}^{\delta_1}\int_{-\delta_2}^{\delta_2} f_1(|u_1|)f_2(|u_2|)du_1du_2+$$

$$+\frac{1}{2}\frac{\partial^2}{\partial x^2}I(t,x,y)\int_{-\delta_1}^{\delta_1}\int_{-\delta_2}^{\delta_2} u_1^2 f_1(|u_1|)f_2(|u_2|)du_1du_2+$$

$$+\frac{1}{2}\frac{\partial^2}{\partial y^2}I(t,x,y)\int_{-\delta_1}^{\delta_1}\int_{-\delta_2}^{\delta_2} u_2^2 f_1(|u_1|)f_2(|u_2|)du_1du_2$$

Using Taylor expansion

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x,x',y,y')I(t,x',y')dx'dy' \approx$$
$$\approx I(t,x,y)\theta +$$
$$+\frac{1}{2}\frac{\partial^2}{\partial x^2}I(t,x,y)\phi_1 +$$
$$+\frac{1}{2}\frac{\partial^2}{\partial y^2}I(t,x,y)\phi_2$$

The new equation

This way our equation takes the form:

$$\begin{cases} \dfrac{\partial S}{\partial t} = \\ = -S(t,x,y)\left(\theta I(t,x,y) + \phi_1 \dfrac{\partial^2 I(t,x,y)}{\partial x^2} + \phi_2 \dfrac{\partial^2 I(t,x,y)}{\partial y^2}\right), \\ \dfrac{\partial I}{\partial t} = \\ = S(t,x,y)\left(\theta I(t,x,y) + \phi_1 \dfrac{\partial^2 I(t,x,y)}{\partial x^2} + \phi_2 \dfrac{\partial^2 I(t,x,y)}{\partial y^2}\right) - \\ \qquad - bI(t,x,y), \\ \dfrac{\partial R}{\partial t} = bI(t,x,y). \end{cases} \tag{3}$$

The new equation - in a more simpler form

If we use the notation

$$J(t,x,y) := \left(\theta I(t,x,y) + \phi_1 \frac{\partial^2 I(t,x,y)}{\partial x^2} + \phi_2 \frac{\partial^2 I(t,x,y)}{\partial y^2}\right)$$

then the equation reduces to:

$$\begin{cases} \dfrac{\partial S}{\partial t} = -S(t,x,y)J(t,x,y), \\ \dfrac{\partial I}{\partial t} = S(t,x,y)J(t,x,y) - bI(t,x,y), \\ \dfrac{\partial R}{\partial t} = bI(t,x,y) \end{cases} \tag{4}$$

The required properties for this new equation

Theorem

If the condition

$$0 \leq J(t,x,y) = \theta I(t,x,y) + \phi_1 \frac{\partial^2 I(t,x,y)}{\partial x^2} + \phi_2 \frac{\partial^2 I(t,x,y)}{\partial y^2} \tag{5}$$

is satisfied, then the properties C_1, C_3 and C_4 are true for the solutions of (4). In this case $I(t,x,y)$ also tends to zero as t tends to infinity. C_2 is true without any restrictions.

Time discretisation using the forward Euler method

Our (continuous) equation was:

$$\begin{cases} \dfrac{\partial S}{\partial t} = -S(t,x,y)J(t,x,y), \\ \dfrac{\partial I}{\partial t} = S(t,x,y)J(t,x,y) - bI(t,x,y), \\ \dfrac{\partial R}{\partial t} = bI(t,x,y) \end{cases}$$

Applying the forward Euler method, we get:

$$\begin{cases} \dfrac{S_{k,l}^{n+1} - S_{k,l}^{n}}{\tau} = -aS_{k,l}^{n}J_{k,l}^{n}, \\ \dfrac{I_{k,l}^{n+1} - I_{k,l}^{n}}{\tau} = aS_{k,l}^{n}J_{k,l}^{n} - bI_{k,l}^{n}, \\ \dfrac{R_{k,l}^{n+1} - R_{k,l}^{n}}{\tau} = bI_{k,l}^{n}. \end{cases} \tag{6}$$

Time discretisation using the forward Euler method

Applying the forward Euler method, we get:

$$\begin{cases} \dfrac{S_{k,l}^{n+1} - S_{k,l}^{n}}{\tau} = -aS_{k,l}^{n} J_{k,l}^{n}, \\ \dfrac{I_{k,l}^{n+1} - I_{k,l}^{n}}{\tau} = aS_{k,l}^{n} J_{k,l}^{n} - bI_{k,l}^{n}, \\ \dfrac{R_{k,l}^{n+1} - R_{k,l}^{n}}{\tau} = bI_{k,l}^{n}. \end{cases}$$

in which we used the notation

$$J_{k,l}^{n} := \left(\theta I_{k,l}^{n} + \phi_1 \frac{I_{k-1,l}^{n} - 2I_{k,l}^{n} + I_{k+1,l}^{n}}{h_x^2} + \phi_2 \frac{I_{k,l-1}^{n} - 2I_{k,l}^{n} + I_{k,l+1}^{n}}{h_y^2} \right)$$

A sufficient condition for the required properties

Theorem

Property D_2 holds without restrictions, and if the step size satisfies

$$\tau \leq \min\left\{\frac{1}{b+2M\left(\frac{\phi_1}{h_x^2}+\frac{\phi_2}{h_y^2}\right)}, \frac{1}{M\left(\theta+2\left(\frac{\phi_1}{h_x^2}+\frac{\phi_2}{h_y^2}\right)\right)}\right\}$$

in which $M := \max_{(x,y)\in\Omega}\{S(0,x,y)+I(0,x,y)+R(0,x,y)\}$, *then properties D_1, D_3 and D_4 also hold.*

Time discretisation using an IMEX method

Our (continuous) equation was:

$$\begin{cases} \dfrac{\partial S}{\partial t} = -S(t,x,y)J(t,x,y), \\ \dfrac{\partial I}{\partial t} = S(t,x,y)J(t,x,y) - bI(t,x,y), \\ \dfrac{\partial R}{\partial t} = bI(t,x,y) \end{cases}$$

Applying an IMEX method, we get:

$$\begin{cases} \dfrac{S_{k,l}^{n+1} - S_{k,l}^{n}}{\tau} = -aS_{k,l}^{n} J_{k,l}^{n}, \\ \dfrac{I_{k,l}^{n+1} - I_{k,l}^{n}}{\tau} = aS_{k,l}^{n} J_{k,l}^{n} - bI_{k,l}^{n+1}, \\ \dfrac{R_{k,l}^{n+1} - R_{k,l}^{n}}{\tau} = bI_{k,l}^{n+1}. \end{cases}$$

Time discretisation using an IMEX method

Applying an IMEX method, we get:

$$\begin{cases} \dfrac{S_{k,l}^{n+1} - S_{k,l}^{n}}{\tau} = -aS_{k,l}^{n} J_{k,l}^{n}, \\ \dfrac{I_{k,l}^{n+1} - I_{k,l}^{n}}{\tau} = aS_{k,l}^{n} J_{k,l}^{n} - bI_{k,l}^{n+1}, \\ \dfrac{R_{k,l}^{n+1} - R_{k,l}^{n}}{\tau} = bI_{k,l}^{n+1}. \end{cases}$$

in which we used the notation

$$J_{k,l}^{n} := \left(\theta I_{k,l}^{n} + \phi_1 \frac{I_{k-1,l}^{n} - 2I_{k,l}^{n} + I_{k+1,l}^{n}}{h_x^2} + \phi_2 \frac{I_{k,l-1}^{n} - 2I_{k,l}^{n} + I_{k,l+1}^{n}}{h_y^2} \right)$$

A sufficient condition for the required properties

Theorem

Property D_2 holds without restrictions, and if the step size satisfies

$$\tau \leq \min \left\{ \frac{1}{b + 2M\left(\frac{\phi_1}{h_x^2} + \frac{\phi_2}{h_y^2}\right)}, \frac{1}{M\left(\theta + 2\left(\frac{\phi_1}{h_x^2} + \frac{\phi_2}{h_y^2}\right)\right)} \right\}$$

in which
$M := \max_{(x,y)\in\Omega}\{S(0,x,y) + I(0,x,y) + R(0,x,y)\}$, *then properties D_1, D_3 and D_4 also hold.*

Using numerical integration

Let us consider the rectangle $[-\delta_1, \delta_1] \times [-\delta_2, \delta_2]$, and examine an equidistant split of it:

$$x_i = -\delta_1 + ih, \quad i = 0, 1, \ldots, m, \quad h = \frac{2\delta_1}{m}, \tag{7}$$

$$y_j = -\delta_2 + jk, \quad j = 0, 1, \ldots, n, \quad k = \frac{2\delta_2}{n}, \tag{8}$$

We approximate the integrals of our initial equation:

$$T(t,h,k) \approx -\int_{-\delta_1}^{\delta_1}\int_{-\delta_2}^{\delta_2} f_1(|u_1|) f_2(|u_2|) I(t, x+u_1, y+u_2)\, du_1\, du_2$$

Using the two dimensional trapezoidal rule

$$T(t,h,k) \approx -\int_{-\delta_1}^{\delta_1}\int_{-\delta_2}^{\delta_2} f_1(|u_1|)f_2(|u_2|)I(t,x+u_1,y+u_2)du_1\,du_2$$

Let us use the notation

$$F_I(t,u_1,u_2) := f_1(|u_1|)f_2(|u_2|)I(t,x+u_1,y+u_2).$$

Applying the trapezoidal rule for the integral:

$$T(t,h,k) = \frac{1}{4}hk\left(\sum_{corners} F_I(t,u_1^i,u_2^j))+\right.$$

$$\left.+2\sum_{edges} F_I(t,u_1^i,u_2^j)) + 4\sum_{inner} F_I(t,u_1^i,u_2^j))\right)$$

The new equation

This way, our new equation takes the form:

$$\begin{cases} \dfrac{dS(t,x,y)}{dt} = -aS(t,x,y)T(t,h,k), \\ \dfrac{dI(t,x,y)}{dt} = aS(t,x,y)T(t,h,k) - bI(t,x,y), \\ \dfrac{dR(t,x,y)}{dt} = bI(t,x,y), \end{cases}$$

Theorem

Properties C_1, C_2, C_3 and C_4 hold without any restrictions.

Using forward Euler method

$$\begin{cases} \dfrac{dS(t,x,y)}{dt} = -aS(t,x,y)T(t,h,k), \\ \dfrac{dI(t,x,y)}{dt} = aS(t,x,y)T(t,h,k) - bI(t,x,y), \\ \dfrac{dR(t,x,y)}{dt} = bI(t,x,y), \end{cases}$$

Using forward Euler method, we get

$$\begin{cases} S^{n+1} = S^n - a\tau S^n T^n, \\ I^{n+1} = I^n + a\tau S^n T^n - b\tau I^n, \\ R^{n+1} = R^n + b\tau I^n, \end{cases} \tag{9}$$

A sufficient condition for the required properties

Theorem

Property D_2 holds without restrictions, and if the step size satisfies

$$\tau \leq \min\left\{\frac{1}{\frac{1}{4}hkN\tilde{M}}, \frac{1}{b}\right\}$$

for every n, where

$$\tilde{M} = \max_{(x,y)\in\Omega}\left(\max_{u_1,u_2\in B_{max(\delta_1,\delta_2)}(x,y)} f(|u_1|)f(|u_2|)I(0,x+u_1,y+u_2)\right)$$

and N is the number of the interpolation points in the numerical integral, then properties D_1, D_3 and D_4 also hold.

Using an IMEX method

$$\begin{cases} \dfrac{dS(t,x,y)}{dt} = -aS(t,x,y)T(t,h,k), \\ \dfrac{dI(t,x,y)}{dt} = aS(t,x,y)T(t,h,k) - bI(t,x,y), \\ \dfrac{dR(t,x,y)}{dt} = bI(t,x,y), \end{cases}$$

Using an IMEX method, we get

$$\begin{cases} S^{n+1} = S^n - a\tau S^n T^n, \\ I^{n+1} = I^n + a\tau S^n T^n - b\tau I^{n+1}, \\ R^{n+1} = R^n + b\tau I^{n+1}, \end{cases} \tag{10}$$

A sufficient condition for the required properties

Theorem

Property D_2 holds without restrictions, and if the step size satisfies

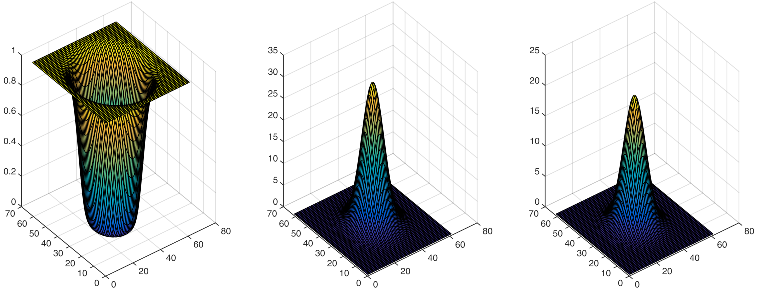
$$\tau \leq \frac{1}{\frac{1}{4} hkN\tilde{M}}$$

for every n, where

$$\tilde{M} = \max_{(x,y)\in\Omega} \left(\max_{u_1, u_2 \in B_{max(\delta_1, \delta_2)}(x,y)} f(|u_1|) f(|u_2|) I(0, x+u_1, y+u_2) \right)$$

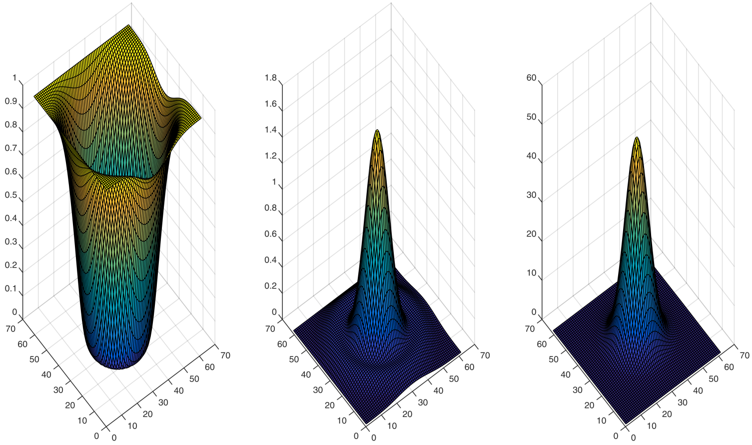
and N is the number of the interpolation points in the numerical integral, then properties D_1, D_3 and D_4 also hold.

A good step size, T=20



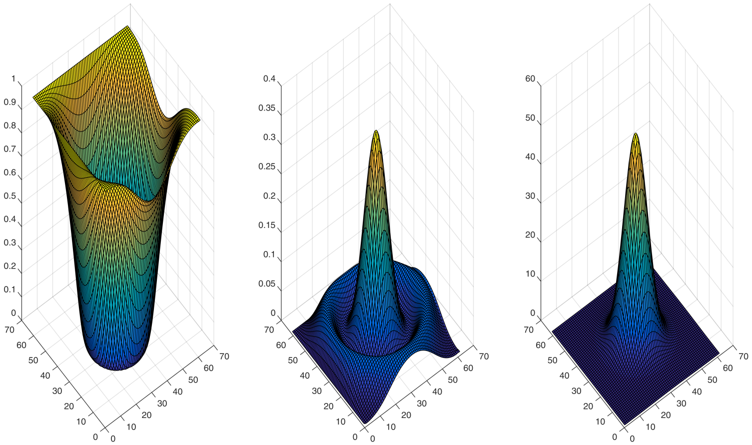
$$b = 0.1, h = k = 0.1, \tau = T/3000, \Omega = [0,3]^2, a = 1300$$

A good step size, T=35



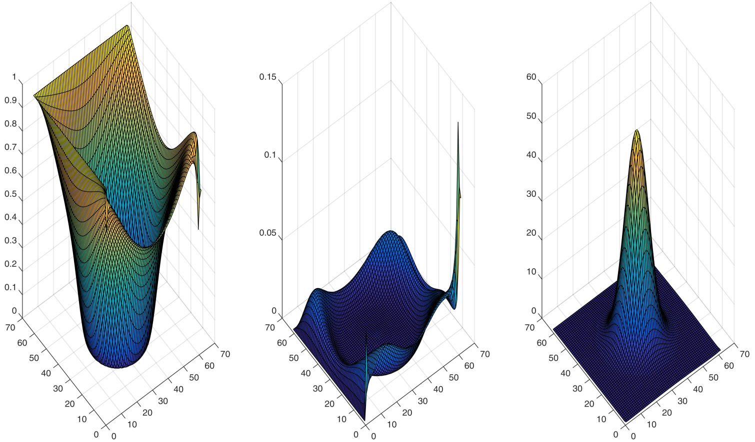
$$b = 0.1, h = k = 0.1, \tau = T/3000, \Omega = [0,3]^2, a = 1300$$

A good step size, T=50



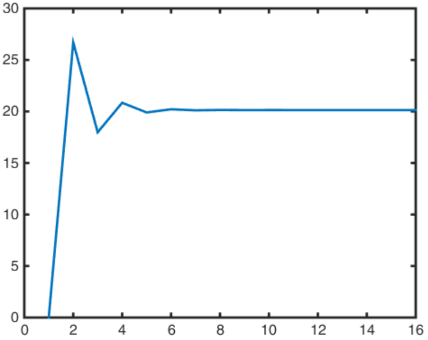
$$b = 0.1, h = k = 0.1, \tau = T/3000, \Omega = [0,3]^2, a = 1300$$

A good step size, T=100



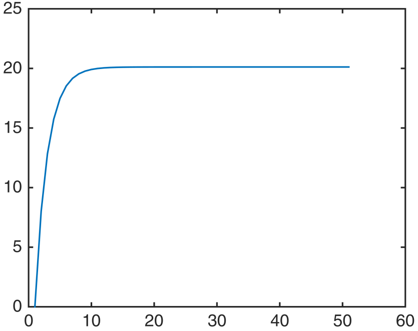
$$b = 0.1, h = k = 0.1, \tau = T/3000, \Omega = [0,3]^2, a = 1300$$

A bad step size (R at (40,32))



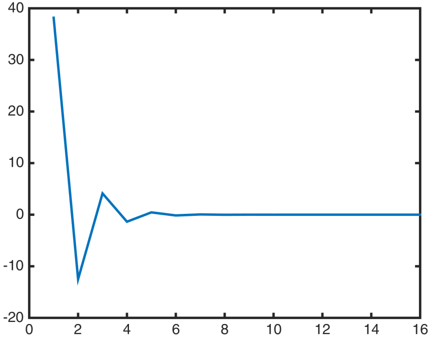
$$b = 0.1, h = k = 0.1, T = 200, \tau = T/13, \Omega = [0,3]^2, a = 10$$

How it should look like (R at (40,32))



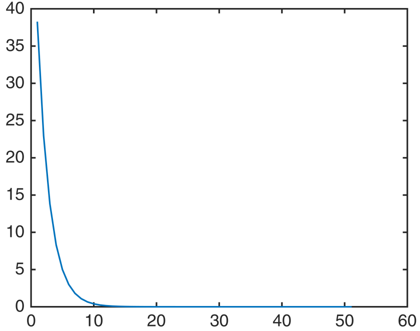
$$b = 0.1, h = k = 0.1, T = 200, \tau = T/1300, \Omega = [0,3]^2, a = 10$$

A bad step size (I at (37,32))



$$b = 0.1, h = k = 0.1, T = 200, \tau = T/13, \Omega = [0,3]^2, a = 10$$

How it should look like (I at (37,32))



$$b = 0.1, h = k = 0.1, T = 200, \tau = T/1300, \Omega = [0,3]^2, a = 10$$

Conclusions

- Two approaches were investigated.
- Necessary conditions of proper behavior were given.
- Numerical experiments were conducted.

Future work

- Investigate other behaviors (i.e. the stability of the wave)
- Adding a diffusion term to the equation
- Applying to an arbitrary domain

  $\Longrightarrow$ Shortley-Weller method or finite element methods (presented by M. Polner)
- Adding delay to the equation

  $\Longrightarrow$ system of delayed integro-differential equations

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Thank you for your attention!