Ergodicity in Nonautonomous Linear Differential Equations

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Let $\mathbb{R}^d$ and $\mathbb{R}^{d\times d}$ denote the d-dimensional space of real column vectors and the space of $d \times d$ matrices with real entries, respectively.

Let $\|\cdot\|$ denote any of the standard l_p-norms, $1 \le p \le \infty$, on $\mathbb{R}^d$ and the associated induced matrix norm on $\mathbb{R}^{d\times d}$.

The nonnegative cone $\mathbb{R}^d_+$ is the set of those vectors in $\mathbb{R}^d$ which have nonnegative components. The cone $\mathbb{R}^d_+$ induces a partial order on $\mathbb{R}^d$ by $x \le y$ if $y - x \in \mathbb{R}^d_+$.

Thus, $x \le y$ if and only if $x_i \le y_i$ for all i. We write $x < y$ if $x \le y$ and $x_i < y_i$ for some i and we write $x \ll y$ if $x_i < y_i$ for all i.

A vector x is called *nonnegative*, *positive* and *strongly positive* if $0 \le x$, $0 < x$ and $0 \ll x$, respectively. A similar notation and terminology is used for matrices.

Consider

$$x' = A(t)x, \qquad t \geq t_0, \tag{1}$$

where $t_0 \in \mathbb{R}$ and $A : [t_0, \infty) \to \mathbb{R}^{d\times d}$ is a continuous matrix function. We will assume the following standing assumptions:

(A_1) A is bounded on $[t_0, \infty)$, i.e.

$$\alpha = \sup_{t\geq t_0} \|A(t)\| < \infty,$$

(A_2) there exists an essentially nonnegative irreducible matrix $M \in \mathbb{R}^{d\times d}$ with

$$M \leq A(t) \quad \text{for all } t \geq t_0.$$

Recall that a matrix $M = (m_{ij}) \in \mathbb{R}^{d\times d}$ is *essentially nonnegative* if the off-diagonal elements of M are nonnegative, i.e. $m_{ij} \geq 0$ whenever $i \neq j$. An essentially nonnegative matrix $M \in \mathbb{R}^{d\times d}$ is *irreducible* if $e^{Mt} \gg 0$ for all $t > 0$.

It is known that under assumption (A_2), for every $t \geq t_0$, the matrix $A(t)$ is essentially nonnegative and irreducible.

As a consequence, Eq. (1) is cooperative and irreducible and therefore the solutions of Eq. (1) are strongly order preserving, i.e. $x(t_0) > 0$ implies that $x(t) \gg 0$ for all $t > t_0$.

Cooperative systems of ordinary differential equations play an important role in applications, see

H. Smith, *Monotone Dynamical Systems. An Introduction to the Theory of Competitive and Cooperative Systems*. AMS, Providence, RI, 1996.

T. Malik and H. Smith, *Does dormancy increase fitness of bacterial populations in time-varying environments?*, Bull. Math. Biol. 70, 1140–1162 (2008).

We will study the weak and strong ergodic properties of the positive solutions to (1) in the sense of the following definition.

Definition

The positive solutions of (1) are called

- *weakly ergodic* if for any two solutions x and y of (1) with initial values $x(t_0) > 0$ and $y(t_0) > 0$,

$$\frac{x(t)}{\|x(t)\|} - \frac{y(t)}{\|y(t)\|} \longrightarrow 0, \qquad t \to \infty,$$

- *strongly ergodic* if there exists a strongly positive vector $v_e \in \mathbb{R}^d$ such that for every solution x of (1) with initial value $x(t_0) > 0$,

$$\frac{x(t)}{\|x(t)\|} \longrightarrow v_e, \qquad t \to \infty. \tag{2}$$

The notions of weak and strong ergodicity originate from discrete age structured models in demography, where (using the l_1-norm) $\|x(t)\|$ denotes the total number of people at time t and the components of the age-structure vector $x(t)/\|x(t)\|$ describe the percentage of people in the given age classes at time t.

Roughly speaking, in this case ergodicity means that the age structure, in the long run, becomes independent of the initial population size.

Necessary and sufficient conditions for weak and strong ergodicity for discrete time equations can be found in the paper by

M. Golubitsky, E.B. Keeler and M. Rothschild, *Convergence of the age structure: Application of the projective metric*, Theoret. Population Biol. 7, 84–93 (1975).

and in the monographs by

- E. Seneta, *Non-negative Matrices and Markov Chains*, Series in Statistics, Springer, 1973
- J. Cushing, *An Introduction to Structured Population Dynamics*, SIAM, Philadelphia, 1998.
- D.J. Hartfiel, *Nonhomogeneous Matrix Products*, World Scientific, New Jersey, 2002.
- U. Krause, *Positive Dynamical Systems in Discrete Time*, Studies in Mathematics 62, de Gruyter, Berlin, 2015.

For further related results on weak ergodicity in various continuous time equations, we refer to

S.D. Tuljapurkar, *Population dynamics in variable environments. IV. Weak ergodicity in the Lotka equation*, J. Math. Biol. 14, 221–230 (1982).

H.R. Thieme, *Asymptotic proportionality (weak ergodicity) and conditional asymptotic equality of solutions to time-heterogeneous sublinear difference and differential equations*, J. Differential Equations 73, 237–268 (1987).

H. Inaba, *Weak ergodicity of population evolution processes*, Math. Biosci. 96, 195–219 (1989).

Weak Ergodicity

First we summarize some well-known consequences of the Perron-Frobenius theory for nonnegative matrices.

Denote by $\mathcal{M}$ the set of essentially nonnegative irreducible matrices in $\mathbb{R}^{d \times d}$.

Every $M \in \mathcal{M}$ has a unique strongly positive normalized eigenvector which will be denoted $v(M)$. We shall call it the *Perron vector* of M.

The *spectral abscissa* of M, defined by

$$s(M) = \max\{\Re\lambda : \lambda \in \sigma(M))\},$$

is an algebraically simple eigenvalue of M and every other eigenvalue of M has real part less than $s(M)$.

As usual, $\sigma(M) \subseteq \mathbb{C}$ denotes the spectrum of M.

The Perron vector corresponds to the spectral abscissa so that

$$Mv(M) = s(M)v(M), \qquad v(M) \gg 0, \qquad \|v(M)\| = 1.$$

Furthermore, every nonnegative eigenvector of M irrespectively of the eigenvalues is a positive multiple of $v(M)$.

If, in addition, M is nonnegative, then its spectral abscissa $s(M)$ coincides with the spectral radius $\rho(M)$.

Evidently, every strongly positive matrix M is irreducible.

It is known that the eigenvalues and the eigenvectors corresponding to algebraically simple eigenvalues depend continuously on the matrix elements, therefore the mappings $s : \mathcal{M} \to \mathbb{R}$ and $v : \mathcal{M} \to \mathbb{R}^d_+$ are continuous.

Recall that the *transition matrix* of (1), denoted by $X(t,s)$ for t, $s \in [t_0, \infty)$, is defined by

$$X(t,s) = \Phi(t)\Phi^{-1}(s) \quad \text{for all } t,\, s \in [t_0, \infty),$$

where Φ is a fundamental matrix solution of (1).

The transition matrix is the unique matrix solution of the initial value problem

$$D_1 X(t,s) = A(t)X(t,s), \qquad X(s,s) = I,$$

where t, $s \in [t_0, \infty)$ and I is the identity matrix. It has the *cocycle property*

$$X(t,s) = X(t,r)X(r,s) \quad \text{for all } t, r, s \in [t_0, \infty),$$

and every solution x of (1) can be represented as

$$x(t) = X(t,s)x(s) \quad \text{for all } t, s \in [t_0, \infty).$$

Lemma (Exponental Estimates for the Transition Matrix)

The transition matrix $X(t,s)$ of (1) satisfies

$$e^{M(t-s)} \leq X(t,s) \leq e^{N(t-s)} \qquad \text{whenever } t \geq s \geq t_0, \tag{3}$$

where $N = \sup_{t \geq t_0} A(t)$, the supremum being taken element-wise. In particular, $X(t,s) \gg 0$ for all $t > s \geq t_0$.

The proofs of the lemma is based on a comparison principle for Kamke type differential inequalities.

The next theorem shows that under assumptions (A_1) and (A_2) the positive solutions of (1) are weakly ergodic. In fact, we prove a stronger result which describes the asymptotic behavior of the normalized positive solutions of (1) as $t \to \infty$.

Theorem (Weak Ergodicity)

For every solution x of (1) with $x(t_0) > 0$,

$$\frac{x(t)}{\|x(t)\|} - \xi(t) \longrightarrow 0, \qquad t \to \infty,$$

where, for each $t > t_0$, $\xi(t) = v(X(t, t_0))$ is the Perron vector of the strongly positive transition matrix $X(t, t_0)$. In particular, the positive solutions of (1) are weakly ergodic.

The proof of the theorem is based on some properties of Hilbert's projective metric.

Let $\mathbb{R}^d_{++}$ denote the set of strongly positive vectors in $\mathbb{R}^d$.

For x, $y \in \mathbb{R}^d_{++}$, we define *Hilbert's projective metric* by

$$p(x,y) = \ln \frac{\max_{1\le i\le n} \frac{x_i}{y_i}}{\min_{1\le i\le n} \frac{x_i}{y_i}} = \max_{1\le i,j\le n} \ln \frac{x_i y_j}{x_j y_i}.$$

The projective metric p has the following properties:
For all x, y and $z \in \mathbb{R}^d_{++}$, we have

(i) $p(x,y) \ge 0$,

(ii) $p(x,y) = 0$ if and only if $y = \beta x$ for some positive constant β,

(iii) $p(x,y) = p(y,x)$,

(iv) $p(x,y) \le p(x,z) + p(z,y)$,

(v) $p(\beta x, \gamma y) = p(x,y)$ for any positive constants β and γ.

Recall that a matrix $S \in \mathbb{R}_+^{d\times d}$ is *row allowable*, if it has a positive entry in each of its rows.

An important property of Hilbert's projective metric is that strongly positive matrices act as contractions in this metric.

Lemma (Contractivity of Strongly Positive Matrices)

Let $S = (s_{ij}) \in \mathbb{R}^{d\times d}$ be a nonnegative row allowable matrix. Then for any x and $y \in \mathbb{R}_{++}^d$, we have

$$p(Sx, Sy) \le \tau_B(S)p(x, y),$$

where $\tau_B(S)$ is Birkhoff's contractivity coefficient defined by

$$\tau_B(S) = \frac{1-\sqrt{\phi(S)}}{1+\sqrt{\phi(S)}}, \qquad \phi(S) = \min_{1\le i,j,k,l\le n} \frac{s_{ik}s_{jl}}{s_{jk}s_{il}},$$

if $S \gg 0$ and $\tau_B(S) = 1$ if S has at least one 0 entry.

Now we can give a proof of the theorem.

By the lemma, the transition matrix $X(t, t_0)$ of (1) is strongly positive for $t > t_0$ and hence $x(t) = X(t, t_0)x(t_0) \gg 0$ for all $t > t_0$.

It is known that

$$\|x - y\| \leq 3(1 - e^{-p(x,y)}), \qquad x,\, y \in \mathbb{R}^d_{++},\ \|x\| = \|y\| = 1.$$

This, together with property (v) of p, implies that it is enough to show that

$$\lim_{t\to\infty} p(x(t), \xi(t)) = 0.$$

By the lemma, the transition matrix of (1) satisfies

$$0 \ll e^M \leq X(t+1, t) \leq e^N \qquad \text{for all } t \geq t_0.$$

Since $\mathcal{K} = \{S \in \mathbb{R}^{d\times d} \mid e^M \leq S \leq e^N\}$ is a compact set of strongly positive matrices on which τ_B is continuous, τ_B achieves its maximum $\theta < 1$ on $\mathcal{K}$.

Let $t > t_0$ and $n = [t - t_0]$, where $[\cdot]$ denotes the greatest integer part. Using property (v) of p and the cocycle property of the transition matrix $X(t, s)$, we obtain

$$\begin{aligned} p(x(t), \xi(t)) &= p(X(t, t_0)x(t_0), X(t, t_0)\xi(t)) \\ &= p(X(t, t_0 + n)X(t_0 + n, t_0)x(t_0), X(t, t_0 + n)X(t_0 + n, t_0)\xi(t)). \end{aligned}$$

From this, by the application of lemma, we find that

$$p(x(t), \xi(t)) \leq p(X(t_0 + n, t_0)x(t_0), X(t_0 + n, t_0)\xi(t)).$$

Since

$$X(t_0+n, t_0) = X(t_0+n, t_0+n-1)X(t_0+n-1, t_0+n-2)\ldots X(t_0+1, t_0)$$

and each factor of the last product belongs to $\mathcal{K}$, we can repeatedly use the θ-contraction property of p, which yields

$$p(x(t), \xi(t)) \leq \theta^{n-1} p(X(t_0 + 1, t_0)x(t_0), X(t_0 + 1, t_0)\xi(t)). \quad (4)$$

If $U_+ = \{x \in \mathbb{R}^d_+ \mid \|x\| = 1\}$, then due to $S = X(t_0+1, t_0) \gg 0$, the image set $S(U_+)$ consists of strongly positive vectors.

Since $S : U_+ \to \mathbb{R}^d_{++}$ is continuous and U_+ compact, $S(U_+)$ is a compact subset of $\mathbb{R}^d_{++}$.

Let $v_0 = X(t_0+1, t_0)x(t_0) \gg 0$. Because $p(v_0, \cdot) : \mathbb{R}^d_{++} \to [0, \infty)$ is continuous and $S(U_+) \subset \mathbb{R}^d_{++}$ is compact, we have that $K = \sup_{x \in U_+} p(v_0, Sx) < \infty$.

Hence

$$p(X(t_0{+}1, t_0)x(t_0), X(t_0{+}1, t_0)\xi(t)) = p(v_0, S\xi(t)) \leq K, \qquad t > t_0.$$

This, together with (4), implies $p(x(t), \xi(t)) \to 0$ exponentially as $t \to \infty$.

The theorem implies that for the strong ergodicity of the positive solutions of (1) it is necessary and sufficient that the Perron vectors $\xi(t)$ of the transition matrix $X(t, t_0)$ converge to a strongly positive vector as $t \to \infty$.

However, in most cases we do not have an explicit formula for the transition matrix $X(t, t_0)$ and its Perron vector $\xi(t)$.

In the next theorem, under an additional assumption, we give a necessary and sufficient condition for the strong ergodicity of the positive solutions of (1) in terms of the Perron vectors of the coefficient matrix function A.

Theorem (Strong Ergodicity)

If additionally A is uniformly continuous on $[t_0, \infty)$, then for the strong ergodicity of the positive solutions of (1) with limiting vector $v_e \gg 0$ it is necessary and sufficient that

$$\nu(t) \longrightarrow v_e, \qquad t \to \infty, \tag{5}$$

where $\nu(t) = v(A(t))$ is the Perron vector of $A(t)$ for $t \geq t_0$.

Proof of Sufficiency. Suppose that $\nu(t) \longrightarrow v_e$ as $t \to \infty$. holds.

For $t \geq t_0$, define $\sigma(t) = s(A(t))$, the spectral abscissa of $A(t)$.

For each $t \geq t_0$, $A(t)$ belongs to the compact set of essentially nonnegative irreducible matrices $\mathcal{K} = \{S \in \mathbb{R}^{d \times d} \mid M \leq S \leq N\}$ with N as in the lemma about exponential estimates for the transition matrix.

As noted before, $s : \mathcal{M} \to \mathbb{R}$ is continuous and therefore it is uniformly continuous on the compact subset $\mathcal{K}$ of $\mathcal{M}$.

Whence, σ is a composition of two uniformly continuous functions, the restriction of s to $\mathcal{K}$ and A. This implies that σ is also uniformly continuous on $[t_0, \infty)$.

Then

$$y(t) = x(t) \exp\left(- \int_{t_0}^{t} \sigma(u)\, du \right), \qquad t \geq t_0$$

is a solution of the differential equation

$$y' = B(t)y, \qquad B(t) = A(t) - \sigma(t)I, \qquad t \geq t_0. \qquad (6)$$

Clearly, B is uniformly continuous on $[t_0, \infty)$. For $t \geq t_0$, we have

$$|\sigma(t)| = |s(A(t))| \leq \rho(A(t)) \leq \|A(t)\| \leq \alpha$$

with α as in (A_1). Hence,

$$\sup_{t \geq t_0} \|B(t)\| \leq 2\alpha$$

and

$$\tilde{M} \leq B(t) \leq \tilde{N} \quad \text{for all } t \geq t_0, \qquad (7)$$

where $\tilde{M} = M - \alpha I$ is essentially nonnegative and irreducible and $\tilde{N} = N + \alpha I$.

Since the Perron vector $\nu(t)$ of $A(t)$ corresponds to the the spectral abscissa $\sigma(t)$ of $A(t)$, we have

$$B(t)\nu(t) = 0 \quad \text{for all } t \geq t_0. \tag{8}$$

In view of the identity $\frac{x(t)}{\|x(t)\|} = \frac{y(t)}{\|y(t)\|}$ for $t \geq t_0$, it is enough to show that

$$\frac{y(t)}{\|y(t)\|} \longrightarrow v_e, \qquad t \to \infty. \tag{9}$$

Let w be an arbitrary accumulation point of $\frac{y(t)}{\|y(t)\|}$ as $t \to \infty$, i.e.

$$\frac{y(t_n)}{\|y(t_n)\|} \longrightarrow w, \qquad n \to \infty,$$

for some $t_n \to \infty$. We need to show that $w = v_e$.

Define

$$z_n(t) = \frac{y(t_n + t)}{\|y(t_n\|}$$
$$B_n(t) = B(t_n + t)$$

for each t satisfying $t_n + t \geq t_0$. From (6), we find that

$$z_n'(t) = B_n(t) z_n(t) \tag{10}$$

for every t for which $t_n + t \geq t_0$. By known estimates for the growth of the solutions of ODE's, we have

$$\|y(\tau)\| \exp\left(-\int_\tau^t (\mu \circ (-B))\right) \leq \|y(t)\| \leq \|y(\tau)\| \exp\left(\int_\tau^t (\mu \circ B)\right)$$

for all $t \geq \tau \geq t_0$, where $\mu : \mathbb{R}^{d \times d} \to \mathbb{R}$ is the logarithmic norm. Due to the estimate $|\mu(B(t))| \leq \|B(t)\|$ for $t \geq t_0$, we have

$$\|y(\tau)\| e^{-2\alpha(t-\tau)} \leq \|y(t)\| \leq \|y(\tau)\| e^{2\alpha(t-\tau)} \quad \text{for all } t \geq \tau \geq t_0.$$

From this and (10), we find that

$$e^{-2\alpha|t|} \leq \|z_n(t)\| \leq e^{2\alpha|t|}, \tag{11}$$

$$\|z_n'(t)\| \leq 2\alpha e^{2\alpha|t|} \tag{12}$$

for each t such that $t_n + t \geq t_0$. From (11) and (12) and from the boundedness and uniform continuity of B, it follows that the functions z_n and B_n are uniformly bounded and equicontinuous on every compact subinterval of $(-\infty, \infty)$. Referring to the Arzelà-Ascoli theorem, combined with Cantor's diagonalization argument, it follows that there exists a subsequence (t_{n_k}) of (t_n) such that for every $t \in \mathbb{R}$ the limits

$$z(t) = \lim_{k\to\infty} z_{n_k}(t) = \lim_{k\to\infty} \frac{y(t_{n_k} + t)}{\|y(t_{n_k}\|}$$

$$C(t) = \lim_{k\to\infty} B_{n_k}(t) = \lim_{k\to\infty} B(t_{n_k} + t)$$

exist and the convergence is uniform on every compact subinterval of $(-\infty, \infty)$.

By passing to the limit in the integrated form of (10),

$$z_n(t) = z_n(0) + \int_0^t B_n(u) z_n(u)\, du,$$

we find that

$$z(t) = z(0) + \int_0^t C(u) z(u)\, du, \qquad t \in \mathbb{R}.$$

Therefore, $z : \mathbb{R} \to \mathbb{R}^d$ is an entire solution of the equation

$$z' = C(t) z, \qquad t \in \mathbb{R}. \tag{13}$$

Clearly, $z(t) \geq 0$ for all $t \in \mathbb{R}$ and $z(0) = w$. Since $\|z(0)\| = 1$, z is a nontrivial solution and hence $z(t) > 0$ for all $t \in \mathbb{R}$. The limiting function $C(t)$ inherits the estimates in (7) for all $t \in \mathbb{R}$.

Consequently, by the lemma, the transition matrix $Z(t,s)$ of (13) satisfies

$$0 \ll e^{\tilde{M}(t-s)} \leq Z(t,s) \leq e^{\tilde{N}(t-s)} \quad \text{for all } t > s. \tag{14}$$

Hence $z(t) = Z(t,s)z(s) \gg 0$ for all $t > s$. For all $t \in \mathbb{R}$, we have

$$z(t+1) = Z(t+1,t)z(t).$$

From (5) and (8), we find that

$$C(t)v_e = 0 \qquad \text{for all } t \in \mathbb{R},$$

which implies that $\tilde{z}(t) \equiv v_e$ is a constant solution of (13). Whence,

$$v_e = Z(t+1,t)v_e \quad \text{for all } t \in \mathbb{R}$$

and applying the projective metric, we find

$$p(z(t+1), v_e) = p(Z(t+1,t)z(t), Z(t+1,t)v_e) \quad \text{for all } t \in \mathbb{R}.$$

By virtue of (14), for each $t \in \mathbb{R}$, $Z(t+1, t)$ belongs to the compact set of strongly positive matrices

$$\tilde{\mathcal{K}} = \{S \in \mathbb{R}^{d \times d} \mid e^{\tilde{M}} \leq S \leq e^{\tilde{N}}\}$$

on which Birkhoff's contractivity function τ_B achieves its maximum $\tilde{\theta} < 1$. By the lemma on contractivity, we have

$$p(z(t+1), v_e) \leq \tilde{\theta} p(z(t), v_e) \quad \text{for all } t \in \mathbb{R}. \tag{15}$$

From (14), we obtain

$$e^{\tilde{M}} z(t) \leq Z(t+1, t) z(t) \leq e^{\tilde{N}} z(t)$$

and hence

$$e^{\tilde{M}} \frac{z(t)}{\|z(t)\|} \leq \frac{z(t+1)}{\|z(t)\|} \leq e^{\tilde{N}} \frac{z(t)}{\|z(t)\|} \quad \text{for all } t \in \mathbb{R}.$$

Let

$$\delta_i = \min_{x \in U_+} (e^{\tilde{M}} x)_i \qquad \text{and} \qquad \eta_i = \max_{x \in U_+} (e^{\tilde{N}} x)_i, \qquad 1 \le i \le d,$$

where $U_+ = \{x \in \mathbb{R}^d_+ \mid \|x\| = 1\}$. Taking into account that $e^{\tilde{M}} \gg 0$, we have

$$0 \ll \delta \le \frac{z(t+1)}{\|z(t)\|} \le \eta \quad \text{for all } t \in \mathbb{R},$$

with $\delta = (\delta_1, \dots, \delta_d)^T$ and $\eta = (\eta_1, \dots, \eta_d)^T$. The continuity of $p(\cdot, v_e)$ on the compact order interval $[\delta, \eta] \subset \mathbb{R}^d_{++}$, combined with property (v) of p, implies

$$\Delta = \sup_{t \in \mathbb{R}} p(z(t+1), v_e) = \sup_{t \in \mathbb{R}} p\left(\frac{z(t+1)}{\|z(t)\|}, v_e\right) < \infty.$$

From (15), we get $\Delta \le \tilde{\theta} \Delta$ with $\tilde{\theta} < 1$. Hence $\Delta = 0$ and thus $p(z(0), v_e) = 0$. Since $\|z(0)\| = \|v_e\| = 1$, this yields $w = z(0) = v_e$.

We give an example which illustrates the importance of the assumption of uniform continuity of A in the theorem about strong ergodicity.

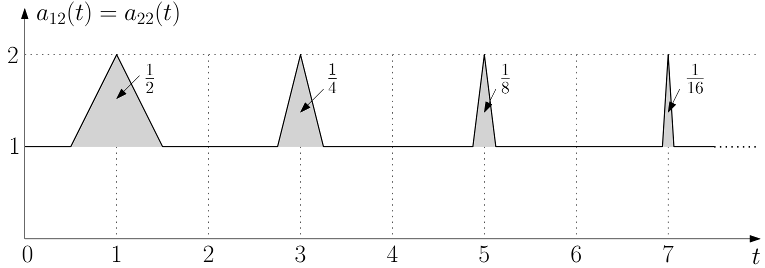
Example. Let

$$B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad \text{and} \qquad C = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$$

such that the corresponding spectral abscissae and Perron vectors with respect to the l_1-norm are

$$s(B) = 2, \quad v(B) = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \qquad \text{and} \qquad s(C) = 3, \quad v(C) = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}.$$

Define the entries of a continuous matrix function $A : [0, \infty) \to \mathbb{R}^{2\times 2}$ such that $a_{11}(t) = a_{12}(t) = 1$ for all $t \geq 0$ and the entries $a_{21}(t) = a_{22}(t)$ are defined as suggested in the figure



The entries a_{21} and a_{22} of the matrix function A. The areas of the shaded triangles with peaks at the odd numbers $2n-1$ forms a geometric sequence $\frac{1}{2^n}$, $n \in \mathbb{N}$.

In this manner, we can construct a continuous function A with the following properties:

$$B \leq A(t), \qquad t \geq 0,$$

$$A(2n) = B, \qquad v(A(2n)) = v(B), \quad n \in \mathbb{N}, \tag{16}$$

$$A(2n+1) = C, \qquad v(A(2n+1)) = v(C), \quad n \in \mathbb{N}, \tag{17}$$

$$\int_t^{t+1} \|A(s) - B\| \, ds \longrightarrow 0, \qquad t \to \infty,$$

$$\int_0^{\infty} \|A(s) - B\| \, ds < \infty.$$

Because of $v(B) \neq v(C)$, the relations (16) and (17) guarantee that the limit $\lim_{t\to\infty} \nu(t) = \lim_{t\to\infty} v(A(t))$ does not exist. Nevertheless, we can show that the positive solutions of (1) are strongly ergodic with limiting vector $v_e = v(B)$.

Indeed, by a Perron type theorem for positive solutions, for every solution x of (1) with $x(0) > 0$, we have

$$\lim_{t \to \infty} \frac{\ln \|x(t)\|}{t} = 2$$

and therefore, according to an asymptotic result due to Coppel, there exists a $\gamma > 0$ so that

$$x(t) = \gamma e^{2t} v(B) + o(e^{2t}), \qquad t \to \infty.$$

Hence $x(t)e^{-2t} \longrightarrow \gamma v(B)$ as $t \to \infty$, which readily implies

$$\frac{x(t)}{\|x(t)\|} = \frac{e^{-2t}x(t)}{\|e^{-2t}x(t)\|} \longrightarrow \frac{\gamma v(B)}{\|\gamma v(B)\|} = v(B), \qquad t \to \infty.$$

Thus, the positive solutions of (1) are strongly ergodic.