Dynamical Analysis of a Discretized System Modelling Somitogenesis

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$$\Omega_1 \subset \mathbb{R}, \quad \Omega_2 \subset \mathbb{R}^{d+1}, \quad \mathbf{x} : \Omega_1 \to \mathbb{R}^d, \quad \mathbf{F} : \Omega_2 \to \mathbb{R}^d, \quad \mathbf{x}, \mathbf{F} \in \mathfrak{C}^1$$

$$\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t), p)$$

discretizing with a Mickens' type non-standard finite difference scheme

$$\rightsquigarrow$$

$$\mathbf{x}_{n+1} = \mathbf{x}_n + \varphi(h)\mathbf{F}(\mathbf{x}_n, p) \qquad (n \in \mathbb{N})$$

Previously on My Doctoral Research

Proposition:

Supppose that $d \in \mathbb{N}$, $A \in \mathbb{R}^{d \times d}$, $B := I_d + hA$, furthermore conditions

$$\lambda \in \sigma(A) \qquad \text{and} \qquad \varphi(h) < \frac{-\max^2(\Im(\lambda)) - 2s(A)}{s^2(A)}$$

hold. Then $s(A) < 0$ implies $\rho(B) < 1$.

Proposition:

Suppose that $d \in \mathbb{N}$, $A \in \mathbb{R}^{d \times d}$,

$$B := I_d + \varphi(h)A.$$

Then $s(A) > 0$ implies $\rho(B) > 1$ independent of $\varphi(h)$.

$$\alpha(p) \pm \beta(p)i \quad \in \sigma\big(\text{Jacobian of the RHS at any EP in the CS}\big)$$

$$\Rightarrow$$

$$1 + \varphi(h)\alpha(p) \pm \varphi(h)\beta(p)i \ \in \sigma\big(\text{Jacobian of the RHS at the same EP in the DS}\big)$$

$$\Rightarrow$$

Proposition:

If Hopf bifurcation occurs from an EP with p_H critical value, then Neimark-Sacker bifurcation cannot occur from the same EP with the same critical value.

NS Bifurcation after discretization

$$(NS_1) \quad \sqrt{\det(J_f(p_{NS}))\varphi(h)^2 + \mathrm{Tr}(J_f(p_{NS}))\varphi(h) + 1} = 1$$

$$(NS_2) \quad \frac{\partial \sqrt{\det(J_f(p))\varphi(h)^2 + \mathrm{Tr}(J_f(p))\varphi(h) + 1}}{\partial p}\Big|_{p=p_{NS}} \neq 0$$

$$(NS_3) \quad \sqrt{\det(J_f(p_{NS}))\varphi(h)^2 + \mathrm{Tr}(J_f(p_{NS}))\varphi(h) + 1}^{\,k} \neq 1 \quad (k \in \{1, 2, 3, 4\})$$

Finding the critical value of the the Neimark-Sacker bifurcation parameter p_{NS} after distretization

$$D(p) := \det(\text{Jacobian of the RHS at any EP in the CS})$$

$$T(p) := \text{Tr}(\text{Jacobian of the RHS at any EP in the CS})$$

$$\Rightarrow$$

$$D(p)\varphi(h) + T(p) = 0$$

Background of the model describing somitogenesis

Based on the chemical scheme:

$$A \xrightarrow{k_1} R, \qquad 2A + B \xrightarrow{k_2} 3A, \qquad B \underset{k'_{-3}}{\overset{k_3}{\rightleftharpoons}} R$$

Annie Lemarchand and Bogdan Nowakowski proposed the following reaction-diffusion system

$$\left.\begin{array}{rcl} \partial_t A & = & d_A \Delta_{\mathbf{r}} A + f_A(A, B), \\ \partial_t B & = & d_B \Delta_{\mathbf{r}} B + f_B(A, B) \end{array}\right\} \qquad (1)$$

where

- $d_A, d_B > 0$ represent the diffusion coefficients,
- $A(\mathbf{r}, t)$ and $B(\mathbf{r}, t)$ are the concentrations of the species;
- $\alpha > 0$ and $\delta > 0$ are annihilation rates of the species A and B respectively, $\gamma > 0$ represents the input of the species B and $\beta > 0$ is the conversion rate.

The kinetic system

The kinetic part of the above system

$$\left.\begin{array}{rcccl} \dot{A} & = & f_A(A,B) & := & -\alpha A + \beta A^2 B, \\ \dot{B} & = & f_B(A,B) & := & \gamma - \delta B - \beta A^2 B \end{array}\right\} \qquad (2)$$

inspired from the Schnakenberg model

$$\dot{A} = A^2 B - A, \qquad \dot{B} = -A^2 B + k_{Sch} \qquad (3)$$

and the Gray-Schott model

$$\dot{A} = -AB^2 - k_{GS}^1 A + k_{GS}^2, \qquad \dot{B} = AB^2 - k_{GS}^3 B - k_{GS}^4, \qquad (4)$$

was examined earlier by

Sándor Kovács, Szilvia György and Noémi Gyúró-Magyar.

The discretized system

Using the nonstandard discretization method developed by Mickens we obtain the following discrete model

$$\left.\begin{aligned} A_{n+1} &= A_n + \varphi(h)\left(-\alpha A_n + \beta A_n^2 B_n\right), \\ B_{n+1} &= B_n + \varphi(h)\left(\gamma - \delta B_n - \beta A_n^2 B_n\right) \end{aligned}\right\} \qquad (5)$$

where $h > 0$ is the time step size and the nonnegative function satisfies

$$0 < \varphi(h) = h + \mathcal{O}(h^2) \quad (h \to 0).$$

Clearly, if φ is the identity function then we have the continuous system (2) discretized by the explicit Euler method:

$$\left.\begin{aligned} A_{n+1} &= A_n + h\left(-\alpha A_n + \beta A_n^2 B_n\right), \\ B_{n+1} &= B_n + h\left(\gamma - \delta B_n - \beta A_n^2 B_n\right). \end{aligned}\right\} \qquad (6)$$

Biological feasibility of the discretized system

Proposition:

If $A_0 > 0$, $B_0 > 0$ and

$$\varphi(h) < h^* := \min\{1/\alpha,\ 1/\delta\} \tag{7}$$

then the for solutions of (5) (and of (6), too) $A_n > 0$, $B_n > 0$ hold for any $n \in \mathbb{N}$.

Proposition:

If condition $\varphi(h) < 1/c$ holds then there is a suitable constant $k > 0$ s.t.

$$\left\{(A, B) \in \mathbb{R}_+^2 :\ A + B \leq \frac{k}{\mu} + \varepsilon, \text{ for any } \varepsilon > 0\right\} \tag{8}$$

is positively invariant where $0 < \mu < c := \min\{\alpha, \delta\}$.

Equilibria (fixed points) of (5), resp. (6)

The sign of

$$K := \beta\gamma^2 - 4\alpha^2\delta$$

decides on the number of interior equilibria. If

- $K < 0 \rightsquigarrow$ we have only

  $$\mathbf{E}_b = \left(0, \frac{\gamma}{\delta}\right)$$

  and no interior equilibrium.
- $K = 0 \rightsquigarrow$ there is a unique interior equilibrium:

  $$\overline{\mathbf{E}} := (\overline{A}, \overline{B}) := \left(\frac{\gamma}{2\alpha}, \frac{\gamma}{2\delta}\right);$$
- $K > 0 \rightsquigarrow$ there are two interior equilibria: $\mathbf{E}_\pm := (A_\mp, B_\pm)$ where

$$A_\pm := \frac{\beta\gamma \pm \sqrt{\beta K}}{2\alpha\beta} \quad \text{and} \quad B_\pm := \frac{\alpha}{\delta} \cdot A_\pm.$$

Stability of the boundary equilibrium $\mathbf{E}_b$

For system (2) $\mathbf{E}_b$ is asymptotically stable. but discretizing $\rightsquigarrow$ $\mathbf{E}_b$ is

- a sink, if
$$0 < \varphi(h) < 2/\alpha \quad \text{and} \quad 0 < \varphi(h) < 2/\delta$$
- a source, if
$$\varphi(h) > 2/\alpha \quad \text{and} \quad \varphi(h) > 2/\delta$$
- a saddle, if
$$0 < \varphi(h) < 2/\alpha, \varphi(h) > 2/\delta \quad \text{or} \quad \varphi(h) > 2/\alpha, 0 < \varphi(h) < 2/\delta$$
- nonhyperbolic, if
$$\varphi(h) = 2/\alpha \quad \text{or} \quad \varphi(h) = 2/\delta$$

.

Stability of the equilibrium $\overline{\mathbf{E}}$, $\mathbf{E}_+$, $\mathbf{E}_-$

For system (2)

- $\overline{\mathbf{E}}$ may or may be not stable, but discretizing $\rightsquigarrow$ $\overline{\mathbf{E}}$ unstable
- $\mathbf{E}_+$ unstable, and its is unstable with respect (5), resp. (6), too
- the stability of $\mathbf{E}_-$ depends on the sign of $\beta - \frac{\alpha^4}{\gamma^2(\alpha-\delta)}$, but discretizing $\rightsquigarrow$ depends on $\beta - \frac{\alpha^2(\alpha-2\delta\varphi(h))^2}{\gamma^2(\alpha-\delta-\delta\varphi(h))}$

Bifurcation around $\mathbf{E}_b$

- If
$$\varphi(h) = \frac{2}{\alpha} \qquad \text{and} \qquad \alpha > \delta$$
then $\mathbf{E}_b$ undergoes a period-doubling (flip) bifurcation.

- saddle-node bifurcation **cannot** occur.

- Neimark-Sacker bifurcation **cannot** occur

Bifurcation around $\overline{\mathbf{E}}$

- If
$$\beta\gamma^2 = 4\alpha^2\delta$$
then a saddle-node bifurcation occurs independent of the step size/step function.

- flip bifurcation **cannot** occur

- Neimark-Sacker bifurcation **cannot** occur

Bifurcation around $\mathbf{E}_{-}$

- If $2\alpha^3 < \gamma\left(\sqrt{\beta K} + \beta\gamma\right)$ and

$$\varphi(h) = \frac{\frac{\gamma\left(\sqrt{\beta K}+\beta\gamma\right)-2\alpha^3}{2\alpha^2} + \sqrt{\left(\frac{\gamma\left(\sqrt{\beta K}+\beta\gamma\right)-2\alpha^3}{2\alpha^2}\right)^2 - \frac{2\left(K+\gamma\sqrt{\beta K}\right)}{\alpha}}}{\frac{K+\gamma\sqrt{\beta K}}{2\alpha}}$$

  flip bifurcation may occur.

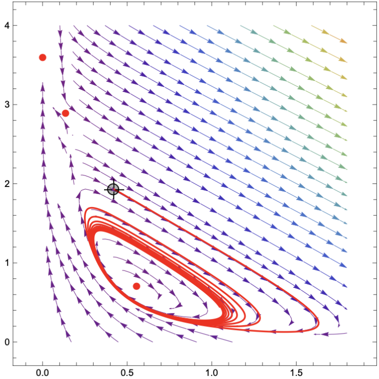
Proposition:
If

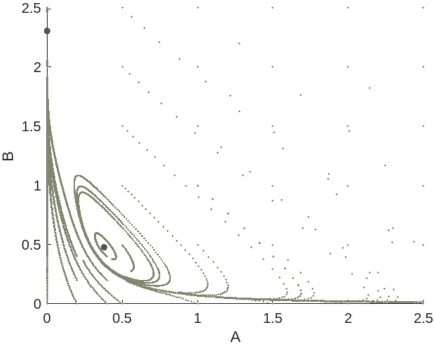
$$\beta = \frac{\alpha^2(\alpha - 2\delta\varphi(h)))^2}{\gamma^2(\alpha - \delta - \delta\varphi(h))}$$

Neimark-Sacker bifurcation occurs

- saddle-node bifurcation **cannot** occur

closed invariant curve remains after discretization





Andronov-Hopf bifurcation:

a = 2.29; b = 5.85; c = 1.58195; d = 0.44

Neimark-Sacker bifurcation: $h = \frac{1}{150}$

a = 3.29, b = 18.0339, c = 1.58195, d = 0.686499

Thank you for your attention!