Analysis of fractional diffusion problems

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- Δ^α is a nonlocal operator, it is hard to define in nonhomogeneous cases.
- More equivalent formulations in L^p, C_0, C_{bu} spaces.

Equivalent formulations in $\mathbb{R}^d$

Let $\mathscr{L}$ be $-(-\Delta)^{\alpha/2}$ in $\mathbb{R}^d$.

- Fourier definition: $\mathscr{F}(\mathscr{L}f)(\xi) = -|\xi|^\alpha \mathscr{L}f(\xi)$
- distributional definition: $\int_{\mathbb{R}^d} \mathscr{L}f(y)\varphi(y)dy = \int_{\mathbb{R}^d} f(y)\mathscr{L}\varphi(y)dy$
- Bochner's definition: $\mathscr{L}f = \frac{1}{\gamma(-\alpha/2)} \int_0^\infty (e^{t\Delta}f - f)t^{-1-\alpha/2}dt$
- Balakrishnan's definition: $\mathscr{L}f = \frac{\sin(\alpha\pi/2)}{\pi} \int_0^\infty \Delta(sI - \Delta)^{-1} f s^{\alpha/2-1} ds$
- singular integral definition:
  $\mathscr{L}f = \lim_{r\to 0+} \frac{2^\alpha \gamma((d+\alpha)/2)}{\pi^{d/2}|\gamma(-\alpha/2)|} \int_{\mathbb{R}^d\setminus B(x,r)} \frac{f(\cdot+z)-f(\cdot)}{|z|^{d+\alpha}} dz$

Spectral definition in bounded domain

On an arbitrary Lipschitz domain $(-\Delta)^{-1}$ with homogeneous boundary conditions can be recognized as a $L_2(\Omega) \to H_0^1(\Omega)$ positive, self-adjoint, compact operator, and its eigenfunctions $\{\omega_j\}_{j\in\mathbb{N}}$ make a complete system with eigenvalues $\{\lambda_j\}_{j\in\mathbb{N}}$. We can define

$$((-\Delta)^{-1})^\alpha : L_2(\Omega) \to L_2(\Omega)$$

for $0 \le \alpha \le 1$ for $u = \sum_{j\in\mathbb{N}} u_j\omega_j$ with

$$((-\Delta)^{-1})^\alpha u = \sum_{j\in\mathbb{N}} \lambda_j^\alpha u_j\omega_j.$$

This also shows that the equation is well posed in this case.

Matrix transformation method for finite elements

Instead of constructing the appropriate bilinear form, we define directly the stiffness matrix corresponding to $(-\Delta)^\alpha$ by taking the α power of A_h, the stiffness matrix corresponding to $-\Delta$, for arbitrary finite elements.

- Elliptic equation: $(-\Delta)^\alpha u = f \;\; u|_{\partial\Omega} = 0$
- $u_{h,\alpha} = A_h^{-\alpha}\Pi_{0,h}f$, where $[A_h]_{i,j} = (\nabla\varphi_i|\nabla\varphi_j)_{L_2}$ and $\Pi_{0,h} : L_2(\Omega) \to V_h$ L_2-orthogonal projection
- It was proved for L_2 orthogonal finite element basis.
- Numerical experiments work in any basis.
- If $\|u - u_h\|_0 \le h^s\|f\|_0$, then $\|u_\alpha - u_{h,\alpha}\|_0 \le h^{s\alpha}\|f\|_0$

Matrix transformation method for finite elements

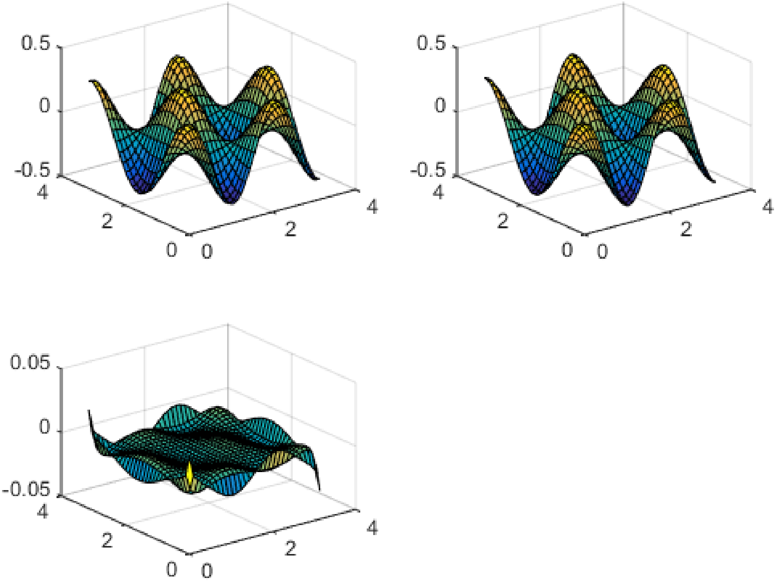
Space-fractional diffusion:

- $\partial_t u(t,x) = -(-\Delta)^\alpha u(t,x) \qquad u(0,x) = u_0(x)$
- Homogeneous Dirichlet or Neumann boundary conditions
- Method of lines: First finite element discretization in the spatial derivatives.
- Use some method to solve the ODE system.
- $$\frac{E_h^\alpha u_{n+1} - E_h^\alpha u_n}{\delta} = A_h^\alpha u_{n+1}$$
  $u_0 = \Pi_{h,1} u_0(x)$ where $[E_h]_{i,j} = (\varphi_i|\varphi_j)_{L_2}$
- Presumption: if $\max_j \|u^j - u_h^j\| = O(k + h^s)$, then $\max_j \|u^j - u_{h,\alpha}^j\| = O(k + h^{\alpha s})$

Numerical results

Homogeneous Neumann

- $$\partial_t u = -(1/13)^{0.7}(-\Delta)^{0.7} u$$
  $$\text{on } (0,1)x(0,\pi)^2$$
- $$u_0(x,y) = \cos(3x)\cdot\cos(2y)$$
- R4 elements
- spatial partition: N=15,30,60
- time partition: M=60
- L_2 norm of errors: $0.0263, 0.0133, 0.0077$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How to solve the equations for nonomogeneous boundary coditions?

A possible way: for the

$$(-\Delta)^\alpha u = f$$

$$u|_{\partial\Omega} = g$$

problem we should find an extension $\widehat{f}$ of f to $\mathbb{R}^d$, then solve the $(-\Delta)^\alpha \widehat{u} = \widehat{f}$, then if we restrict $\widehat{u}$ to ω, and $\widehat{u}|_{\partial\Omega} = u|_{\partial\Omega}$.

- Is there such an extension ?
- In one dimension the answer is yes.

Thank you for Your patience !