# Two Limit Cycles in a Two-Species Reaction

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# Contents

1 Introduction
- 1.1 Existence of periodic trajectories
- 1.2 Exclusion of periodic trajectories
- 1.3 Limit cycles
- 1.4 The model with a large and a small limit cycle

2 Methods to be used
- 2.1 Poincaré compactification
- 2.2 Homogeneous directional blow-up

3 The model with two limit cycles
- 3.1 Preparation for the analysis of the model
- 3.2 Phase portrait on the Poincaré disk
- 3.3 The big and the small limit cycle
- 3.4 Plotting the limit cycles

4 References

# 1 Introduction

## 1.1 Existence of periodic trajectories

- Exotic behavior of chemical reactions: oscillation, multistability, multistationarity or chaos

- Oscillatory chemical reactions may also form the basis of periodic behavior in biological systems

- The second part of the 16th problem of David Hilbert (1900): find the number of limit cycles of two-dimensional autonomous polynomial differential systems.

- The problem seems to be very difficult even in the case of two-dimensional kinetic differential equations.

- Schlomiuk and Vulpe (2012): in the class of quadratic differential equations the number of different phase portraits is estimated to be more than 2000

- **Frank-Kamenetsky (1947)**: modelling the oscillation of cold flames

The Lotka-Volterra equation

$$x' = k_1 x - k_2 x y, \quad y' = k_2 x y - k_3 y$$

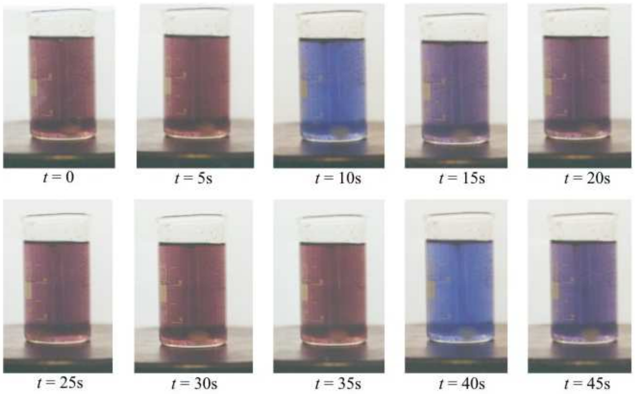
was reinterpreted as the induced kinetic differential equation of the reaction

$$X \xrightarrow{k_1} 2X, \quad X + Y \xrightarrow{k_2} 2Y, \quad Y \xrightarrow{k_3} 0$$

Nonlinear first integral $\Longrightarrow$ conservative oscillations, i.e. closed trajectories in the first quadrant

- **The Belousov-Zhabotinsky reaction** (Belousov: 1958, Zhabotinsky: 1964) is an oscillating chemical reaction.

https://en.wikipedia.org/wiki/Belousov-Zhabotinsky_reaction




- **Hsü (1976)**: the Oregonator model of the BZ reaction has periodic solutions by a theorem on Andronov-Hopf bifurcation

- **Field, Kőrös and Noyes (1972)**: Oscillations in chemical systems

## 1.2 Exclusion of periodic trajectories

Application of the theorem by Bendixson or by Bendixson and Dulac:

- **Wegscheider reaction**: The induced kinetic differential system

$$x' = -k_1 x + k_{-1} y - 2 k_2 x^2 + 2 k_{-2} y^2 =: f(x, y)$$
$$y' = k_1 x - k_{-1} y + 2 k_2 x^2 - 2 k_{-2} y^2 =: -f(x, y)$$

of the reversible reaction ($X \underset{k_{-1}}{\overset{k_1}{\rightleftarrows}} Y$, $2X \underset{k_{-2}}{\overset{k_2}{\rightleftarrows}} 2Y$) has no periodic trajectory.

- **Bautin (1954)**: within the class of equations

$$x' = x(a x + b y + c)$$
$$y' = y(d x + e y + f)$$

only those can have a periodic solution which are of the Lotka-Volterra form (Bendixson-Dulac theorem)

**- Póta-Hanusse-Tyson-Light theorem** (proofs: Póta, with a Dulac function, 1983; Schuman and Tóth, 2003): Among two-species second-order reactions the only oscillatory reaction is the Lotka-Volterra model.

Suppose that the coefficients of the equation $x' = a x^2 + b x y + c y^2 + d x + e y + f$
$y' = A x^2 + B x y + C y^2 + D x + E y + F$
obey the inequalities

(1) $0 \le c, e, f, A, D, F$ (the equation is kinetic)
(2) $0 \ge a, C$ (no steps like 2 X⟶3 X or 2 Y⟶3 Y occur)
(3) at most one of $b$ and $B$ is positive (no steps like X+Y ⟶2 X+Y or X+Y⟶X+2 Y occur)

Then, the only equation to have periodic solutions is of the Lotka-Volterra form, specifically, limit cycles cannot arise.

Interesting question: is condition (1) enough to exclude the emergence of limit cycles?

**-Escher (1980, 1981)**: chemical examples with two species and second-order reactions with even more than one limit cycles, but long product complexes are allowed

Both conservative oscillations and limit cycles: $2Y \rightleftarrows 2X \longrightarrow 3X, \quad X \rightleftarrows 0 \longleftarrow Y, \quad X + Y \longrightarrow 0$

## 1.3 Limit cycles

**Prigogine and Lefever (1968):** the Brusselator model with a limit cycle (a Hopf bifurcation emerges)

$$0 \underset{1}{\overset{1}{\rightleftarrows}} X \overset{b}{\longrightarrow} Y, \quad 2X + Y \overset{a}{\longrightarrow} 3X$$
$$x' = 1 - (b+1)x + a x^2 y, \qquad y' = b x - a x^2 y$$

**Gray and Scott (1986):** the Autocatalator model

$$0 \overset{k_0}{\longrightarrow} Y \overset{k_3}{\longrightarrow} X \overset{k_2}{\longrightarrow} 0, \quad 2X + Y \overset{k_1}{\longrightarrow} 3X$$
$$x' = k_3 y - k_2 x + k_1 x^2 y, \qquad y' = k_0 - k_3 y - k_1 x^2 y$$

**Erle (1998)**: If $0 \le m' < m;\ 0 < \beta < n';\ \alpha > 0$ then for the reaction $m X + n Y \underset{k_1'}{\overset{k_1}{\rightleftarrows}} m' X + n' Y, \quad \alpha X \underset{k_2'}{\overset{k_2}{\rightleftarrows}} 0 \underset{k_3'}{\overset{k_3}{\rightleftarrows}} \beta Y$
there exist reaction rate coefficients for which the reaction has an asymptotically orbitally stable closed orbit.

**Erle (2000)**: Nonoscillation in closed reversible chemical systems

**Schnakenberg (1979)**: For exhibiting limit cycle behavior a two-species reaction has to consist of at least three
reaction steps among which one must be autocatalytic of the type $2X + Y \longrightarrow 3X$. The possible candidates are those whose stationary state is an unstable focus.

**Császár, Jicsinszky and Turányi (1982)**: They used necessary conditions to construct candidate reactions with limit cycles. They have shown numerically that some of the reactions seem to have limit cycles.

**Schlosser and Feinberg (1994)**: a graph theoretical necessary condition of periodicity and multistationarity

### 1.4 The model with a large and a small limit cycle

The dynamical system investigated in [1] comes from a chemical model published by Császár et al. (1982):

$$0 \xrightarrow{K_1} V, \quad U \xrightarrow{K_2} 0, \quad V \xrightarrow{K_3} U, \quad 2\,U \xrightarrow{K_4} V, \quad 2\,U + V \xrightarrow{K_5} 3\,U$$

where $K_i > 0,\ i = 1,\ \ldots,\ 5$ are the reaction rate coefficients. The induced kinetic differential equations are

$$u' = -K_2\,u - 2\,K_4\,u^2 + K_3\,v + K_5\,u^2\,v$$
$$v' = K_1 + K_4\,u^2 - K_3\,v - K_5\,u^2\,v$$

**B. Ács, G. Szederkényi, Zs. Tuza, and Z. A. Tuza (2016)**: precisely 17160 reaction graphs with different structure can produce exactly the same dynamical behavior: they induce the same mass action type kinetic differential equations. However, the computed structures could not be used to show important dynamical properties of the model (e.g. the existence of positive equilibria or the boundedness of solutions).

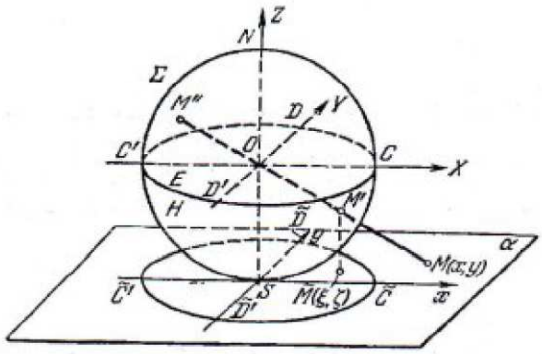
# 2 Methods to be used

## 2.1 Poincaré compactification

The Poincaré compactification is one of he tools to study the behavior of the trajectories of a planar differential system near infinity. Each polynomial vector $X = P\,\partial/\partial x + Q\,\partial/\partial y$ of degree $d$ of the system

$$x' = P(x,\ y),\ \ y' = Q(x,\ y)$$

can be analytically extended to the Poincaré sphere $\mathbb{S}^2 = \{(x,\ y,\ z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$. This can be done by central projection of the points $M(x,\ y,\ -1) \in \mathbb{R}^3$ onto $\mathbb{S}^2$.



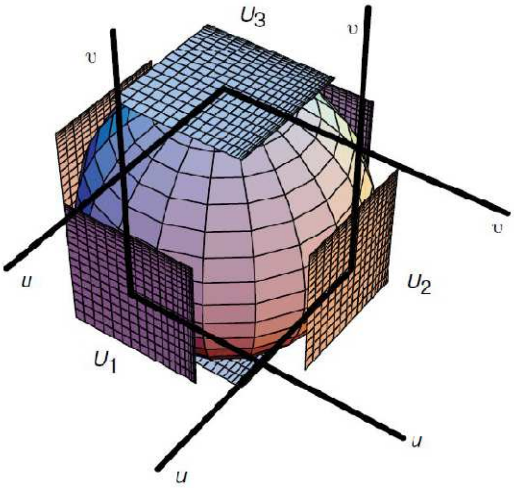

Northern hemisphere: $H_+ = \{(x,\ y,\ z) \in \mathbb{S}^2 : z > 0\}$; $\quad M''\left(\frac{x}{\sqrt{x^2+y^2+1}},\ \frac{y}{\sqrt{x^2+y^2+1}},\ \frac{1}{\sqrt{x^2+y^2+1}}\right) \in H_+$

Southern hemisphere: $H_- = \{(x,\ y,\ z) \in \mathbb{S}^2 : z < 0\}$; $\quad M'\left(\frac{-x}{\sqrt{x^2+y^2+1}},\ \frac{-y}{\sqrt{x^2+y^2+1}},\ \frac{-1}{\sqrt{x^2+y^2+1}}\right) \in H_-$

Equator: $\mathbb{S}^1 = \{(x,\ y,\ z) \in \mathbb{S}^2 : z = 0\}$

The finite points of the plane are projected to the northern hemisphere and the southern hemisphere and the infinite points to the equator.

For the investigation of the points on the sphere, six local charts will be used:
$U_i = \{s \in \mathbb{S}^2 : s_i > 0\}, \ i = 1, 2, 3$
$V_i = \{s \in \mathbb{S}^2 : s_i < 0\}, \ i = 1, 2, 3$




The vector fields on $\mathbb{S}^2$ in the charts are

$$U_1 : u' = v^d\, Q\left(\frac{1}{v}, \frac{u}{v}\right) - u\, v^d\, P\left(\frac{1}{v}, \frac{u}{v}\right), \quad v' = -v^{d+1}\, P\left(\frac{1}{v}, \frac{u}{v}\right)$$
$$U_2 : u' = v^d\, P\left(\frac{u}{v}, \frac{1}{v}\right) - u\, v^d\, Q\left(\frac{u}{v}, \frac{1}{v}\right), \quad v' = -v^{d+1}\, Q\left(\frac{u}{v}, \frac{1}{v}\right)$$
$$U_3 : u' = P(u, v), \quad v' = Q(u, v)$$

$$V_1 : u' = (-1)^{d-1}\left(v^d\, Q\left(\frac{1}{v}, \frac{u}{v}\right) - u\, v^d\, P\left(\frac{1}{v}, \frac{u}{v}\right)\right), \quad v' = (-1)^d\, v^{d+1}\, P\left(\frac{1}{v}, \frac{u}{v}\right)$$
$$V_2 : u' = (-1)^{d-1}\left(v^d\, P\left(\frac{u}{v}, \frac{1}{v}\right) - u\, v^d\, Q\left(\frac{u}{v}, \frac{1}{v}\right)\right), \quad v' = (-1)^d\, v^{d+1}\, Q\left(\frac{u}{v}, \frac{1}{v}\right)$$
$$V_3 : u' = (-1)^{d-1}\, P(u, v), \quad v' = (-1)^{d-1}\, Q(u, v)$$

where $d$ denotes the degree of the system. $V_i$ distinguishes from $U_i$ by the factor $(-1)^{d-1}$.

The finite points correspond to the charts $U_3$, $V_3$, respectively. The infinite points correspond to the charts $U_1$, $U_2$, $V_1$, $V_2$, where $v = 0$. It is enough to consider the points on $U_1 \mid_{v=0}$ and $U_2 \mid_{(0,0)}$ to understand the behavior of the infinite points.

Poincaré disk: the projejction of the points of the northern hemisphere of $\mathbb{S}^2$ to the equator.

**Example 1** $x' = x, \quad y' = -y$

Singular point: (0, 0), it is a saddle. Degree: $d = 1$.

Projection to $U_1$: $x = \frac{1}{v},\ y = \frac{u}{v} \implies u = \frac{y}{x},\ v = \frac{1}{x}$

The transformed system: $u' = -2\,u, \quad v' = -v$

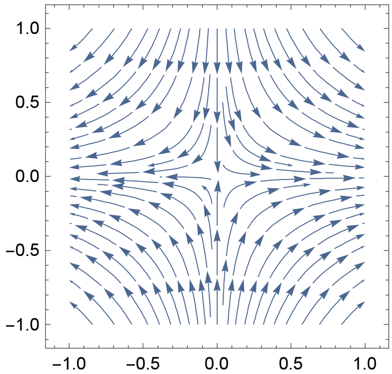
Singular point at $U_1$: (0, 0), it is a stable node at infinity. Since $d$ is odd, then the origin of $V_1$ is also a stable node.

Projection to $U_2$: $x = \frac{u}{v},\ y = \frac{1}{v} \implies u = \frac{x}{y},\ v = \frac{1}{y}$

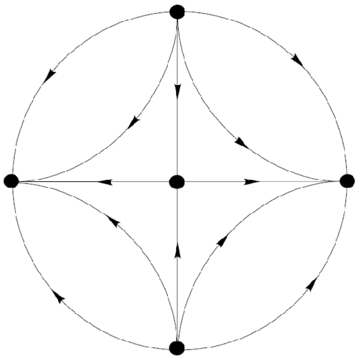
The transformed system: $u' = 2u, \quad v' = v$

Singular point at $U_2$: (0, 0), it is an unstable node at infinity. The origin of $V_1$ is also an unstable node.

The phase portrait on $\mathbb{R}^2$:



The phase portrait on the Poincaré disk:



**Example 2** $x' = -x - y^2, \quad y' = y + x^2$

Singular points: (0, 0): saddle, (-1,-1,): center (it can be seen with the first integral $H(x, y) = \frac{1}{3}x^3 + \frac{1}{3}y^3 + x\,y$). Degree: $d = 2$.

Projection to $U_1$: $x = \frac{1}{v},\ y = \frac{u}{v} \implies u = \frac{y}{x},\ v = \frac{1}{x}$

The transformed system: $u' = 1 + u^3 + 2\,u\,v, \quad v' = u^2\,v + v^2$

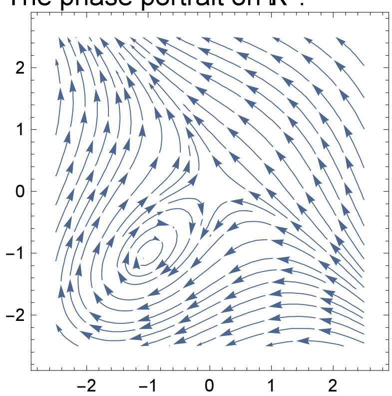
Singular point at $U_1$: $(-1, 0)$ (we need only the case $v = 0$), it is an unstable node at infinity. Since $d$ is even, the diametrically opposite point is a stable node in $V_1$.

Projection to $U_2$: $x = \frac{u}{v},\ y = \frac{1}{v} \implies u = \frac{x}{y},\ v = \frac{1}{y}$

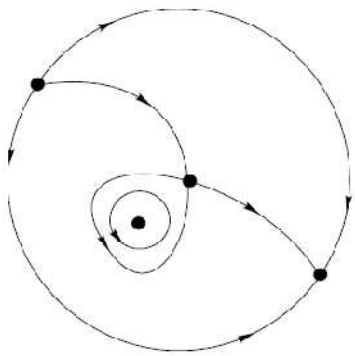
The transformed system: $u' = -1 - u^3 - 2\,u\,v, \quad v' = -u^2\,v - v^2$

Singular point at $U_2$: the only possibility is $u = 0,\ v = 0$, but (0, 0) is not a singular point, so there are no additional infinite singular points.

The phase portrait on $\mathbb{R}^2$:



The phase portrait on the Poincaré disk:



## 2.2 Homogeneous directional blow-up

Blow-up: transformation of the variables so that the behavior near a degenerate singular point is possible to determine.

Two types of blow-ups: "polar blow-up": singular point → circle
"directional blow-up": singular point → straight line

$x' = P(x, y) = P_m(x, y) + \ldots$
$y' = Q(x, y) = Q_m(x, y) + \ldots$

$P_m$, $Q_m$: homogeneous polynomials of degree $m \in \mathbb{N}^+$ and the dots mean higher order terms
We assume that the origin is a singular point, since $m > 0$.

The characteristic function to determine the direction of the blow-up:

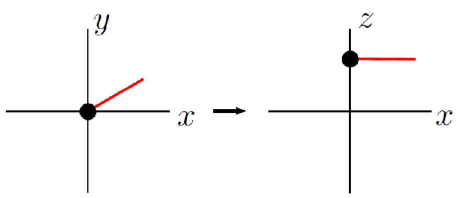
$$F(x, y) = x\, Q_m(x, y) - y\, P_m(x, y)$$

If $F \equiv 0$, then $P_m = x\, W_{m-1}$ and $Q_m = y\, W_{m-1}$, where $W_{m-1} \not\equiv 0$. The angle $\varphi \in [0, 2\pi)$ is the singular direction, if the factor of $W_{m-1}$ is of the form $y - v\, x$, where $v = \tan\varphi$.

**The blow-up in the *x* direction** is the transformation $x \to x$, $y \to x\, z$, and the transformed system is

$$x' = P(x, x\, z),\quad z' = \frac{Q(x, x\, z) - z\, P(x, x\, z)}{x}$$

The singularity is transformed into the line $x = 0$ called the exceptional divisor. The appearing common factor, $x^{m-1}$, sometimes $x^m$ needs to be cancelled. The 2nd and 3rd quadrants are swapped.




$x > 0,\ y > 0 \implies z > 0$
$x < 0,\ y > 0 \implies z < 0$
$x < 0,\ y < 0 \implies z > 0$
$x > 0,\ y < 0 \implies z < 0$

**The blow-up in the *y* direction** is the transformation $x \to y\, z$, $y \to y$, and the transformed system is

$$z' = \frac{P(y\, z, y) - z\, Q(y\, z, y)}{y},\quad y' = Q(y\, z, y)$$

The singularity is transformed into the line $y = 0$. The 3rd and 4th quadrants are swapped.

**Example 1** $\quad x' = x^2 (1 + x^2) + y^2, \quad y' = x^3\, y \qquad (1)$

The characteristic function:
$x' = x^2 (1 + x^2) + y^2 = (x^2 + y^2) + x^4 \implies P_2(x, y) = x^2 + y^2$
$y' = x^3\, y \implies Q_2(x, y) = 0$
$F = x\, Q_2(x, y) - y\, P_2(x, y) = x \cdot 0 - y (x^2 + y^2) = -y (x^2 + y^2)$
$F = 0$ if $y = 0 \implies x$ directional blow-up

Blow-up in the $x$ direction: $\quad x = x,\ z = \dfrac{y}{x}$

The transformed system: $x' = x^2 (1 + x^2 + z^2)$ (2) Cancelling $x$: $x' = x (1 + x^2 + z^2)$ (3)

$z' = -x z (1 + z^2)$ $\qquad$ $z' = -z (1 + z^2)$

The origin is transformed into the line $x = 0$. On this line the only stationary point is (0, 0) with eigenvalues 1, −1 $\Longrightarrow$ (0, 0) is a saddle.

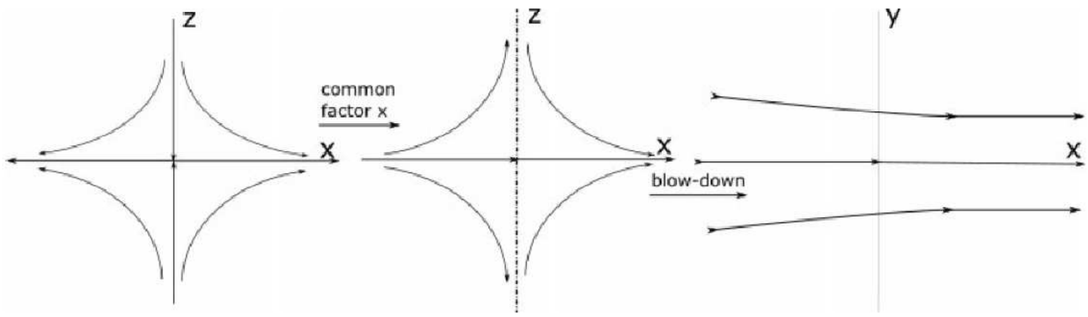
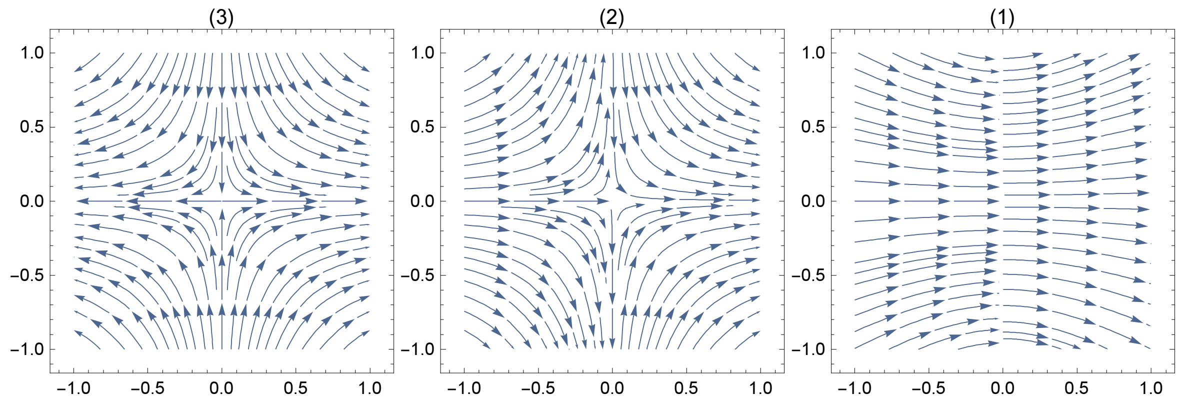



Figure 2.3: Blow down after $x$-directional blow-up




**Example 2** $\qquad$ $x' = -2 y + 3 y^2 - y^3, \; y' = -x^3$ (1)

The characteristic function: $F = x\, Q_1(x, y) - y\, P_1(x, y) = x \cdot 0 - y(-2 y) = 2 y^2$

$F = 0$ if $y = 0 \Longrightarrow$ blow – up in the $x$ direction : $x = x, \; z = \dfrac{y}{x}$

The transformed system: $x' = -2 x z + 3 x^2 z^2 - x^3 z^3$ (2)

$z' = -x^2 + 2 z^2 - 3 x z^3 + x^2 z^4$

The origin is still degenerate so an additional blow-up needs to be done.

The characteristic function: $F = x\, Q_2(x, z) - z\, P_2(x, z) = x(-x^2 + 2 z^2) - z(-2 x z) = x(-x^2 + 4 z^2)$

$F = 0$ if $x = 0 \Longrightarrow$ blow – up in the $z$ direction : $t = \dfrac{x}{z}, \; z = z$

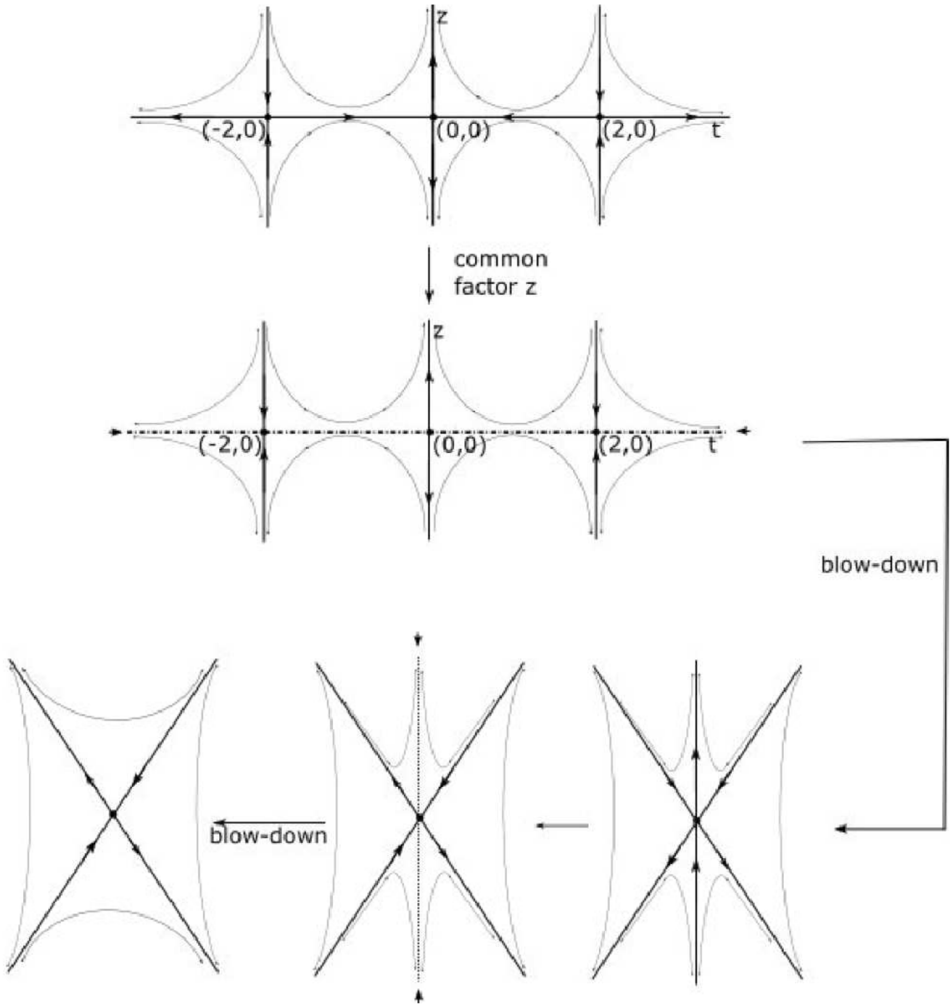
The transformed system: $t' = -t z (4 - t^2 - 6 t z^2 + 2 t^2 z^4)$ (3)

$z' = z^2 (2 - t^2 - 3 t z^2 + t^2 z^4)$

Cancelling $z$: $t' = -t\, (4 - t^2 - 6 t z^2 + 2 t^2 z^4)$ (4)

$z' = z (2 - t^2 - 3 t z^2 + t^2 z^4)$

The origin is transformed into the $t$ axis. Here (on the line $z = 0$) the stationary points are (0, 0), (2, 0), (−2, 0). The origin is a saddle with eigenvalues −4, 2.




# 3 The model with two limit cycles

## 3.1 Preparation for the analysis of the model

The number of parameters in the following system can be decreased by two:

$$u' = -K_2\, u - 2\, K_4\, u^2 + K_3\, v + K_5\, u^2\, v$$
$$v' = K_1 + K_4\, u^2 - K_3\, v - K_5\, u^2\, v$$

Let $u(t) = a\, x(t\, \tau)$, $v(t) = b\, y(t\, \tau)$, where $a$, $b$, $\tau$ are constants $\Longrightarrow$ $u'(t) = a\tau\, x(t\, \tau)$, $v'(t) = b\tau\, y(t\, \tau)$.
The new system is:

$$x' = -\frac{K_2\, x}{\tau} - \frac{2\, a\, K_4\, x^2}{\tau} + \frac{b\, K_3\, y}{a\, \tau} + \frac{a\, b\, K_5\, x^2\, y}{\tau}$$
$$y' = \frac{K_1}{b\, \tau} + \frac{a^2\, K_4\, x^2}{b\, \tau} - \frac{K_3\, y}{\tau} - \frac{a^2\, K_5\, x^2\, y}{\tau}$$

Let the coefficient of $x^2\, y$ and $x^2$ be equal to 1 $\Longrightarrow$ $a = b$, $\tau = a^2\, K_5$ and $a = \frac{K_4}{K_5}$.

If $k_1 = \frac{K_1\, K_5^2}{K_4^3}$, $k_2 = \frac{K_2\, K_5}{K_4^2}$, $k_3 = \frac{K_3\, K_5}{K_4^2}$ then the new system is:

$$x' = -k_2\, x - 2\, x^2 + k_3\, y + x^2\, y \qquad (1)$$
$$y' = k_1 + x^2 - k_3\, y - x^2\, y$$

which can be obtained from the original one if $K_4 = K_5 = 1$ and the notation is changed appropriately.

## 3.2 Phase portrait on the Poincaré disk

**Theorem 1.** For any positive values of parameters $k_i$ the corresponding system (1) has a unique singular point in the first quadrant and all trajectories of this quadrant tend to this point or to a limit cycle surrounding it when $t \to +\infty$.

**Proof.**

**1. The singular points of system (1)**

$x' = -k_2\, x - 2\, x^2 + k_3\, y + x^2\, y = 0 \implies$

$y' = k_1 + x^2 - k_3\, y - x^2\, y = 0$

$$A\left(\frac{-k_2 + \sqrt{4\, k_1 + k_2^2}}{2},\ \left(2\, k_1 \sqrt{4\, k_1 + k_2^2}\right) \Big/ \left(k_2 (k_3 - k_1) + (k_3 + k_1) \sqrt{4\, k_1 + k_2^2}\right)\right)$$

$$B\left(\frac{-k_2 - \sqrt{4\, k_1 + k_2^2}}{2},\ \left(2\, k_1 \sqrt{4\, k_1 + k_2^2}\right) \Big/ \left(k_2 (k_1 - k_3) + (k_3 + k_1) \sqrt{4\, k_1 + k_2^2}\right)\right)$$

$A$ is located in the first quadrant and $B$ is in the second one, so only the point $A$ is of interest for us.

The vector field on the coordinate axes bounding the first quadrant is directed inside the quadrant.

If $x = 0$: $x' = k_3\, y > 0$, $\qquad y' = k_1 - k_3\, y$ $(> 0,\ = 0,\ < 0)$

If $y = 0$: $x' = -k_2\, x - 2\, x^2 < 0$, $\qquad y' = k_1 + x^2 > 0$

Thus, to understand the behavior of the trajectories in the quadrant we have to study the singular points of the system at infinity.

**2. Behavior of the trajectories at the ends of the $O\,x$ axis**

**2. a)** Projection to the Poincaré sphere, substitution and time rescaling: $u = \frac{y}{x}$, $z = \frac{1}{x}$, $d\tau \to \frac{1}{z^2} dt$

$$u' = -u - u^2 + z + 2\, u\, z + k_2\, u\, z^2 - k_3\, u\, z^2 - k_3\, u^2\, z^2 + k_1\, z^3 = U(u, z) \qquad (2)$$
$$z' = -u\, z + 2\, z^2 + k_2\, z^3 - k_3\, u\, z^3 = Z(u, z)$$

Singular points at the equator $z = 0$: $C(0, 0)$ and $D(-1, 0)$, but only $C$ corresponds to the first quadrant.

$C$ is degenerate $\implies$ blow-up

First degree approximations of $U$ and $Z$: $U_1 = -u + z$, $\quad Z_1 \equiv 0$

Characteristic function: $F = u\, Z_1 - z\, U_1 = z(u - z)$

$F = 0$ when $z = 0$ or $u = z$ $\implies$ trajectories of (2) tend to $C$ tangentially to the lines $z = 0$ and $u = z$

**2. b)** Blow-up of the singular point (0, 0) in (2), substitution: $X = u$, $Y = \frac{z}{u}$

$$X' = -X\left(1 + X - Y - 2\, X\, Y - k_2\, X^2\, Y^2 + k_3\, X^2\, Y^2 + k_3\, X^3\, Y^2 - k_1\, X^2\, Y^3\right) \qquad (3)$$
$$Y' = -Y\left(-1 + Y - k_3\, X^2\, Y^2 + k_1\, X^2\, Y^3\right)$$

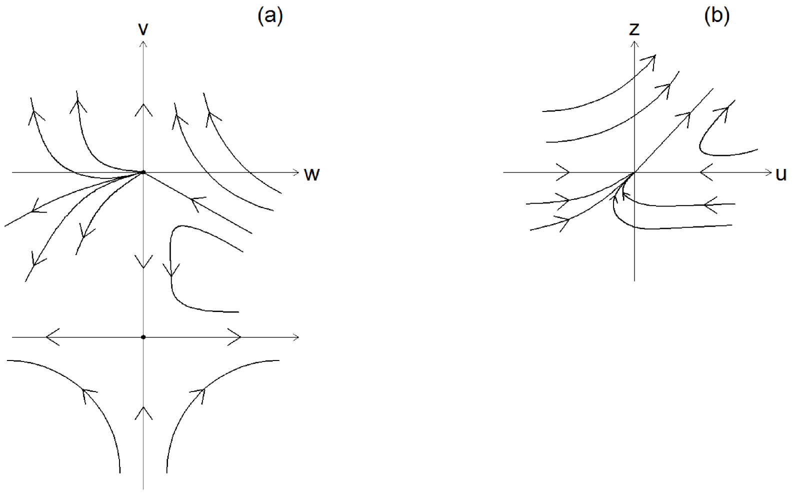
Singular points at the axis $X = 0$: (0, 0) and (0, 1), where (0, 0) is a saddle. We investigate (0, 1) further.

**2. c)** Moving the singular point (0, 1) to (0, 0) using the substitution $w = X$, $v = Y - 1$ and time rescaling $dt \to -dt$

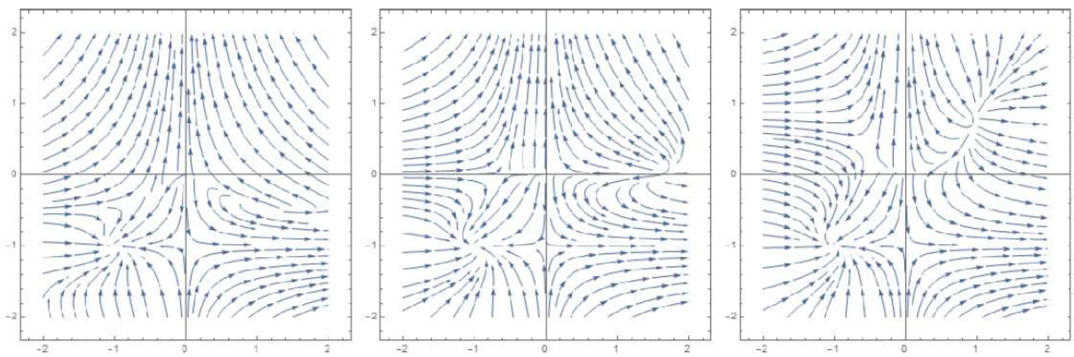
$$w' = -v\,w - w^2 - 2\,v\,w^2 - k_1\,w^3 - k_2\,w^3 + k_3\,w^3 - 3\,k_1\,v\,w^3 - 2\,k_2\,v\,w^3 + 2\,k_3\,v\,w^3 - 3\,k_1\,v^2\,w^3 - k_2\,v^2\,w^3 + k_3\,v^2\,w^3 - k_1\,v^3\,w^3 + k_3\,w^4 + 2\,k_3\,v\,w^4 + k_3\,v^2\,w^4$$
$$v' = v + v^2 + k_1\,w^2 - k_3\,w^2 + 4\,k_1\,v\,w^2 - 3\,k_3\,v\,w^2 + 6\,k_1\,v^2\,w^2 - 3\,k_3\,v^2\,w^2 + 4\,k_1\,v^3\,w^2 - k_3\,v^3\,w^2 + k_1\,v^4\,w^2$$
(4)

**Application of a theorem by Andronov at al. (1973)**:
Let (0, 0) be an isolated equilibrium state of the system
$w' = P_2(w, v) = P(w, v)$, $v' = v + Q_2(w, v) = Q(w, v)$.
Let $v = \phi(w)$ be a solution of the equation $v + Q_2(w, v) = 0$ in the neighborhood of (0, 0) and assume that $P_2(w, \phi(w)) = -w^2 + h.o.t.$ Then the origin is a saddle-node.

(a) Phase portrait of system (4)
(b) Phase portrait of system (2) after the blow-down (the direction of the trajectories changes since we divided by −1)




Phase portraits of system (4) for fixed values of $k_1$ and $k_3$ when $k_2 = 1$.



(a) $k_1 = 4, k_3 = 1$ (b) $k_1 = k_3 = 1$ (c) $k_1 = 1, k_3 = 2$

**3. Behavior of the trajectories at the ends of the $O\,y$ axis**

**3. a)** Projection to the Poincaré sphere, substitution and time rescaling: $u = \frac{x}{y}$, $z = \frac{1}{y}$, $d\tau \to \frac{1}{z^2}\,dt$

$$u' = u^2 + u^3 - 2u^2 z - u^3 z + k_3 z^2 - k_2 u z^2 + k_3 u z^2 - k_1 u z^3 = U(u, z) \qquad (5)$$
$$z' = u^2 z - u^2 z^2 + k_3 z^3 - k_1 z^4 = Z(u, z)$$

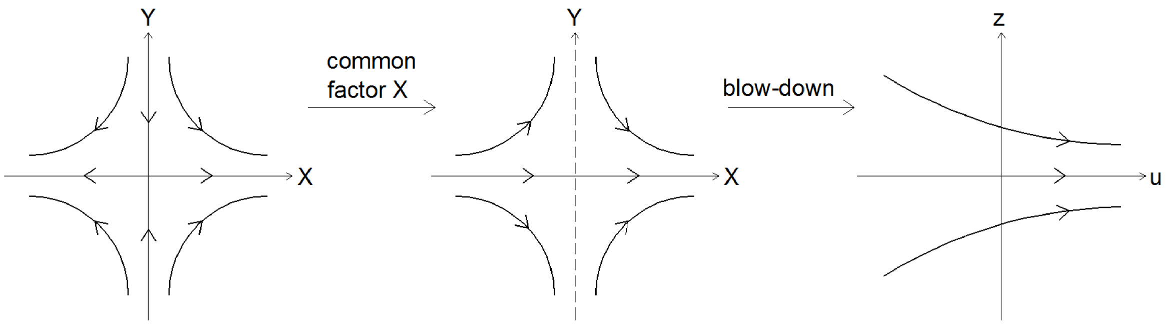
The origin is a degenerate singular point at the ends of $Oy$ axis $\Longrightarrow$ blow-up
Second degree approximations of $U$ and $Z$: $U_2 = u^2 + k_3 z^2$, $Z_2 \equiv 0$
Characteristic function: $F = u Z_2 - z U_2 = -z(u^2 + k_3 z^2)$

$F = 0$ when $z = 0$ $\Longrightarrow$ the characteristic direction is $z = 0$ and a $u$-directional blow-up needs to be done

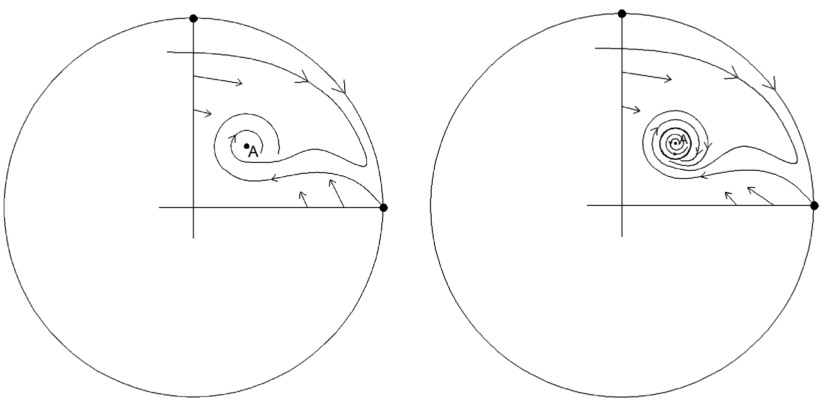
**3. b)** Blow-up in the $u$ direction, substitution and time rescaling: $u = X$, $z = X Y$, $dt \to X\, dt$, and dividing by the common factor $X$

$$X' = -X\left(-1 - X + 2XY + X^2 Y - k_3 Y^2 + k_2 X Y^2 - k_3 X Y^2 + k_1 X^2 Y^3\right) \qquad (6)$$
$$Y' = Y\left(-1 + 2XY - k_3 Y^2 + k_2 X Y^2\right)$$

Phase portrait of system (6) and phase portrait of system (5) after the blow-down:




**4. Phase portrait of system (1) on the Poincaré disk**




There is exactly one singular point, the point $A$, in the first quadrant. Thus, when the time increases, each trajectory of the first quadrant either reach the singular point $A$ or a limit cycle surrounding $A$.

## 3.3 The big and the small limit cycle

To simplify the further analysis, we assume that the singular point $A(x_0, y_0)$ of system (1) is on the straight line $x = 1$. Then

$$\begin{aligned} x' &= -k_2 - 2 + k_3 y + y = 0 \\ y' &= k_1 + 1 - k_3 y - y = 0 \end{aligned} \quad \Longrightarrow \quad k_1 = k_2 + 1,\ x_0 = 1,\ y_0 = \frac{2 + k_2}{1 + k_3}$$

**Theorem 2.** If In system (1) all parameters are positive, $k_3 < 1$, $k_1 = k_2 + 1$, $k_2 = -\frac{k_3^2 + 6 k_3 + 1}{k_3 - 1}$ and

$$g_1 = \left(-1 - 43 k_3 + 9 k_3^2 + 3 k_3^3\right) \Big/ \left((-1 + k_3)(3 + k_3)\sqrt{3 - 2 k_3 - k_3^2}\right) > 0$$

then system (1) has a stable limit cycle and an additional unstable limit cycle bifurcates from the singular point $A$ after small perturbations of the parameters.

**Proof.**

**a)** The singular point $A\left(1, \frac{2 + k_2}{1 + k_3}\right)$ of (1) is shifted into (0, 0):

$$x_1' = \frac{1}{1 + k_3} \left(k_2 x_1 - 4 k_3 x_1 - k_2 k_3 x_1 + k_2 x_1^2 - 2 k_3 x_1^2 + y_1 + 2 k_3 y_1 + k_3^2 y_1 + 2 x_1 y_1 + 2 k_3 x_1 y_1 + x_1^2 y_1 + k_3 x_1^2 y_1\right)$$

$$y_1' = -\frac{1}{1 + k_3} \left(2 x_1 + 2 k_2 x_1 - 2 k_3 x_1 + x_1^2 + k_2 x_1^2 - k_3 x_1^2 + y_1 + 2 k_3 y_1 + k_3^2 y_1 + 2 x_1 y_1 + 2 k_3 x_1 y_1 + x_1^2 y_1 + k_3 x_1^2 y_1\right)$$

(7)

System (7) has only one singular point in the first quadrant: (0, 0).

**b)** The Jacobian at the origin

The trace of the Jacobian matrix of (7) at the origin is $\text{tr} = -\frac{1 - k_2 + 6 k_3 + k_2 k_3 + k_3^2}{1 + k_3}$

$$\text{tr} = 0 \iff k_2 = -\frac{1 + 6 k_3 + k_3^2}{-1 + k_3}$$

The eigenvalues are pure imaginary, $\lambda_{1,2} = \pm i \beta$, where $\beta = \frac{(1 + k_3)\sqrt{3 + k_3}}{\sqrt{1 - k_3}}$, if $k_3 < 1$.

**c)** The matrix $S$ that transforms the Jacobian at the origin into Jordan canonical form:

$$S = \begin{pmatrix} 1 + k_3 & \frac{(1 + k_3)\sqrt{3 + k_3}}{\sqrt{1 - k_3}} \\ \frac{4 (1 + k_3)}{-1 + k_3} & 0 \end{pmatrix}$$

In (7) we introduce the change of coordinates

$$x_1 = \left((1 + k_3)\left(\sqrt{1 - k_3}\, u + \sqrt{3 + k_3}\, v\right)\right) \Big/ \left(\sqrt{1 - k_3}\right)$$

$$y_1 = \frac{4 (1 + k_3) u}{-1 + k_3}$$

and time rescaling: $dt \to -dt \frac{(1 + k_3)\sqrt{3 + k_3}}{\sqrt{1 - k_3}}$

**d)** The transformed system

(8):

$u' =$

$$\frac{1}{2\sqrt{3-2k_3-k_3^2}}\Big(3u^2-4k_3u^2+k_3^2u^2+2u^3-2k_3^2u^3-2\sqrt{3-2k_3-k_3^2}\,v+2\sqrt{3-2k_3-k_3^2}\,uv-$$
$$2k_3\sqrt{3-2k_3-k_3^2}\,uv+4\sqrt{3-2k_3-k_3^2}\,u^2v+4k_3\sqrt{3-2k_3-k_3^2}\,u^2v-$$
$$3v^2-4k_3v^2-k_3^2v^2+6uv^2+8k_3uv^2+2k_3^2uv^2\Big)$$

$$v' = \frac{1}{2(-1+k_3)(3+k_3)}\Big(-6u+4k_3u+2k_3^2u-11u^2+13k_3u^2-k_3^2u^2-k_3^3u^2-6u^3-2k_3u^3+6k_3^2u^3+$$
$$2k_3^3u^3-10\sqrt{3-2k_3-k_3^2}\,uv+8k_3\sqrt{3-2k_3-k_3^2}\,uv+2k_3^2\sqrt{3-2k_3-k_3^2}\,uv-$$
$$12\sqrt{3-2k_3-k_3^2}\,u^2v-16k_3\sqrt{3-2k_3-k_3^2}\,u^2v-4k_3^2\sqrt{3-2k_3-k_3^2}\,u^2v+$$
$$3v^2+19k_3v^2+9k_3^2v^2+k_3^3v^2-18uv^2-30k_3uv^2-14k_3^2uv^2-2k_3^3uv^2\Big)$$

The origin in (8) is either a center or a focus. To distinguish between the two cases we look for a Lyapunov function.

**e)** Lyapunov’s theorem. Let Φ be of the form

$$\Phi(u,\ v) = u^2+v^2+\sum_{k+m=3}^{\infty}\phi_{k\,m}\,u^k\,v^m$$

and quantities $g_i$ satisfying the identity

$$\frac{\partial\Phi}{\partial u}u'+\frac{\partial\Phi}{\partial v}v' = g_1\left(u^2+v^2\right)^2+g_2\left(u^2+v^2\right)^3+\ldots$$

By comparing the coefficients of the corresponding powers on both sides:

$$g_1 = \left(-1-43k_3+9k_3^2+3k_3^3\right)\Big/\left((-1+k_3)(3+k_3)\sqrt{3-2k_3-k_3^2}\right)$$

Remark: If $\frac{\partial\Phi}{\partial u}u'+\frac{\partial\Phi}{\partial v}v' = g_1u^4+g_2u^6+\ldots$ then $g_1$ is a constant multiple of the previous result.

When $g_1>0$ then $\Phi(u,\ v)$ is a positively defined Lyapunov function, whose derivative is also positively defined.

Lyapunov instability theorem $\Longrightarrow$ the point $A$ is an unstable focus

Theorem 1 $\Longrightarrow$ there is at least one stable limit cycle surrounding the point $A$

From the trace $\mathrm{tr} = -\frac{1-k_2+6k_3+k_2k_3+k_3^2}{1+k_3}$ we can see that we can slightly perturb the parameter $k_2$ in such a way that the stability of the singular point A is changed.

Therefore, an unstable limit cycle appears near $A$ as the result of the Andronov-Hopf bifurcation.

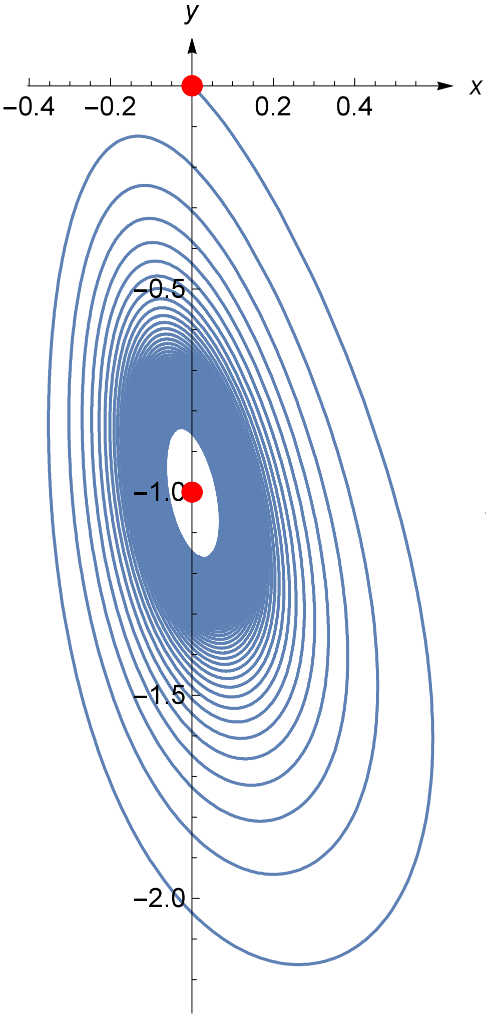
Since the stable limit cycle is preserved after small perturbations, the perturbed system has at least two limit cycles.

## 3.4 Plotting the limit cycles

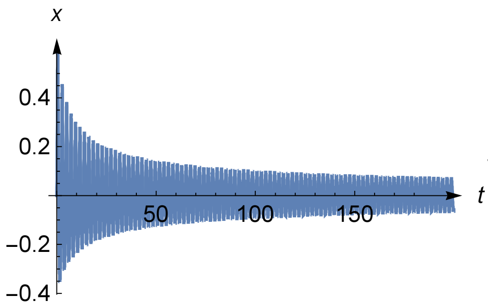
```
Quit

SetOptions[#, AxesStyle → Arrowheads[Automatic]] & /@
  {Plot, ParametricPlot, ListPlot, ListLinePlot};
SetDirectory[NotebookDirectory[]];
SetOptions[#, AxesStyle → Arrowheads[Automatic]] & /@ {Plot, ListPlot,
   ListLinePlot, ListLogLogPlot, ParametricPlot, DateListPlot, DiscretePlot};

ClearAll[k, p, q, x, y, g];
k3 = 3/10; (* 0 < k3 < 1 *)
k4 = 1; k5 = 1; k2 = -(k3^2 + 4 k3 k4 + 2 k3 k5 + k5^2)/(k3 - k5) - 1/10000; k1 = k2 + k4;
p[x_, y_] := -k2 x - 2 k4 x^2 + k3 y + k5 x^2 y;
q[x_, y_] := k1 + k4 x^2 - k3 y - k5 x^2 y;

ClearAll[nsol, ev, plotter];
nsol = First@NSolve[Join @@ Thread /@ {{p[x, y], q[x, y]} == 0, {x, y} > 0}, {x, y}, 20];
ev = Eigenvalues[D[{p[x, y], q[x, y]}, {{x, y}}] /. nsol];
plotter[τ_, shift_, ag_: Automatic, pg_: Automatic, pp_: 1000, ar_: Automatic, opts___] :=
 Module[{startingpoint, sys, solution},
  startingpoint = ({x, y} /. nsol) + shift;
  sys := NDSolveValue[Join[{u'[t] == p[u[t], v[t]], v'[t] == q[u[t], v[t]]},
      Thread[{u[0], v[0]} == startingpoint]],
     {u, v}, {t, τ}, AccuracyGoal → ag, PrecisionGoal → pg, opts];
  trafo[point_] := 1/shift[[2]] (point - startingpoint);
  solution[t_] := trafo[Through[sys[t]]];
  {ParametricPlot[Evaluate[solution[t]], {t, 0, τ},
    Epilog → {Red, PointSize[0.05], Point[{0, 0}], Point[trafo[{x, y} /. nsol]]},
    PlotRange → All, PlotPoints → pp, AspectRatio → ar,
    AxesLabel → {x, y}, LabelStyle → Directive[14], ImageSize → 250],
   Plot[Evaluate[solution[t][[1]]], {t, 0, τ}, PlotRange → All, PlotPoints → pp,
    AxesLabel → {t, x}, LabelStyle → Directive[12], ImageSize → 250],
   Plot[Evaluate[solution[t][[2]]], {t, 0, τ}, PlotRange → All, PlotPoints → pp,
    AxesLabel → {t, y}, LabelStyle → Directive[12], ImageSize → 250]}]
```

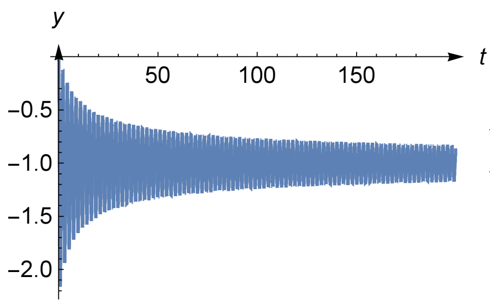
## Big cycle, trajectory going inward. Distance from the singular point: 1

```
Figure1 = plotter[200, {0, 1}]
```

{ 

,

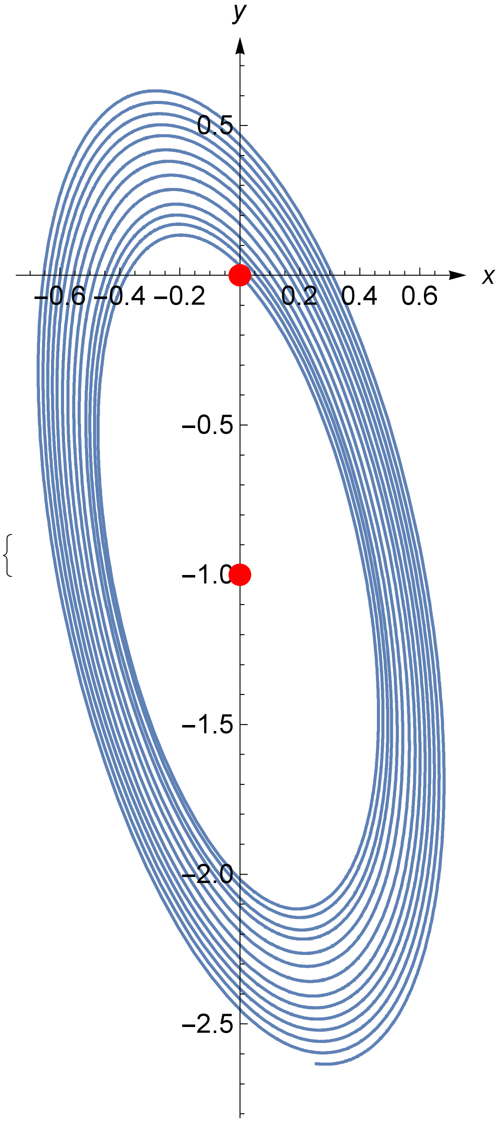


,


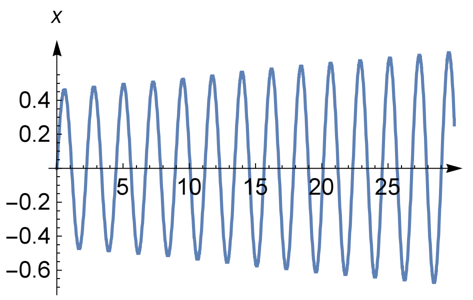
}

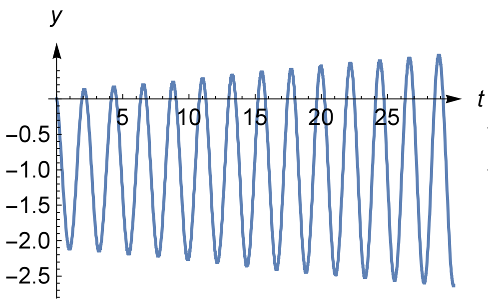
## Big cycle, trajectory going outward. Distance from the singular point: $10^{-5}$

```
Figure2 = plotter[30, {0, 0.00001}]
```

{ 

,



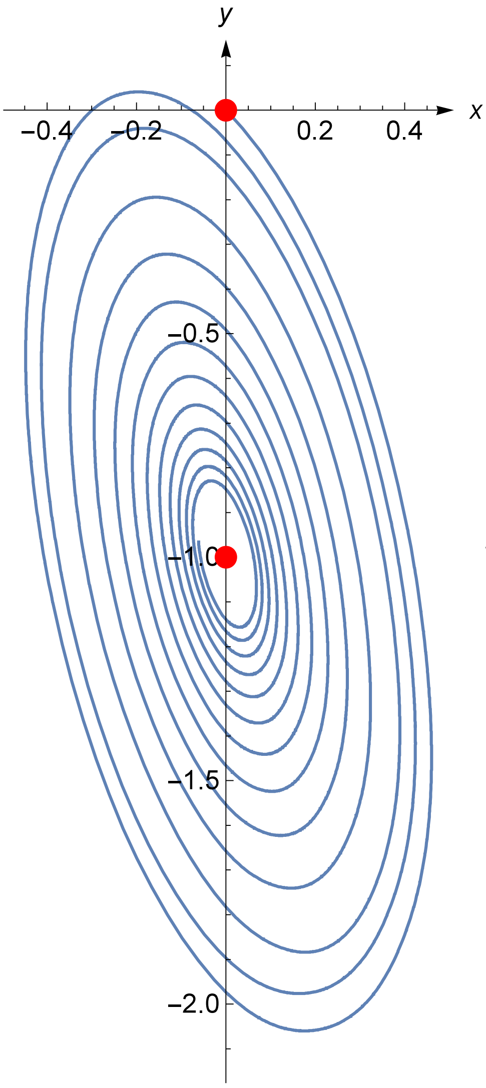
,


}

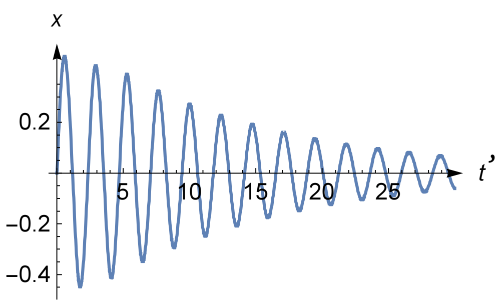
## Small cycle, trajectory going inward. Distance from the singular point: $6 \times 10^{-7}$

```
Figure3 = plotter[30, {0, 6. × 10^-7}]
```

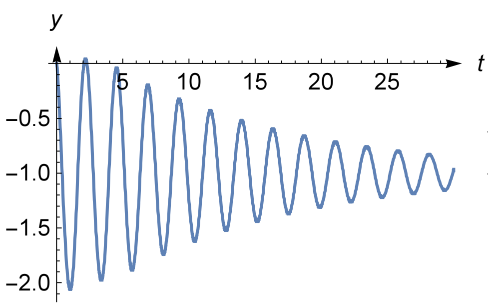
{




,



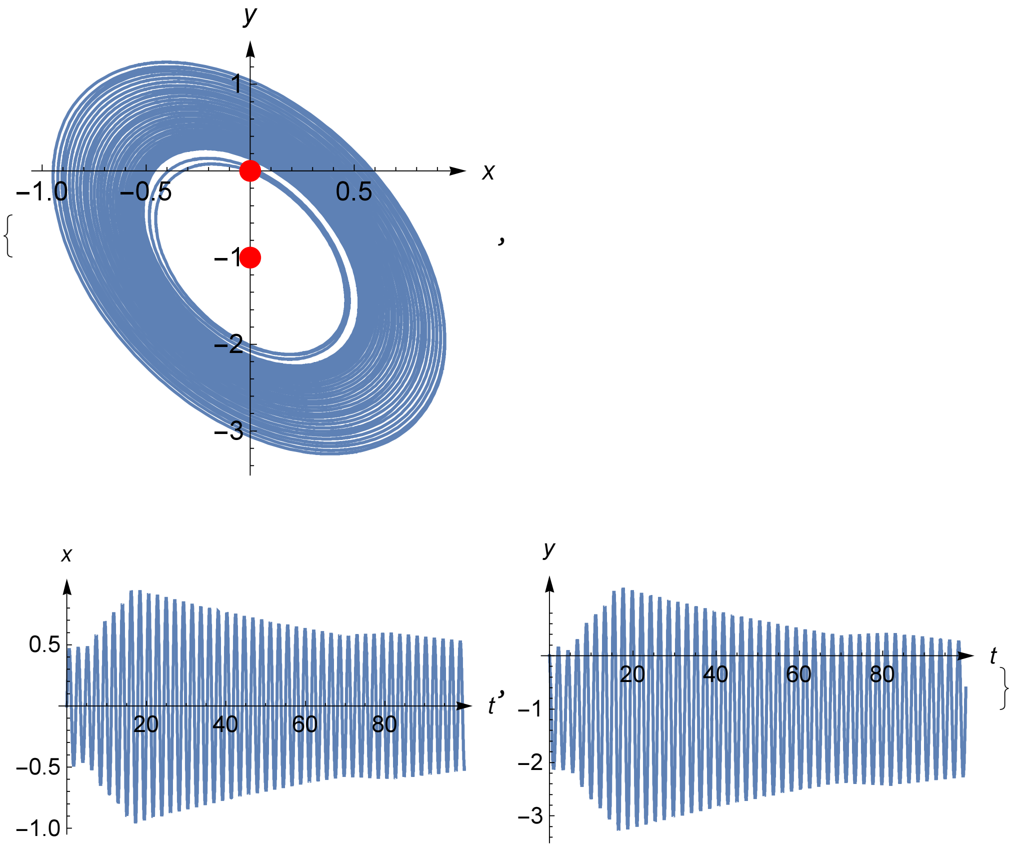

,



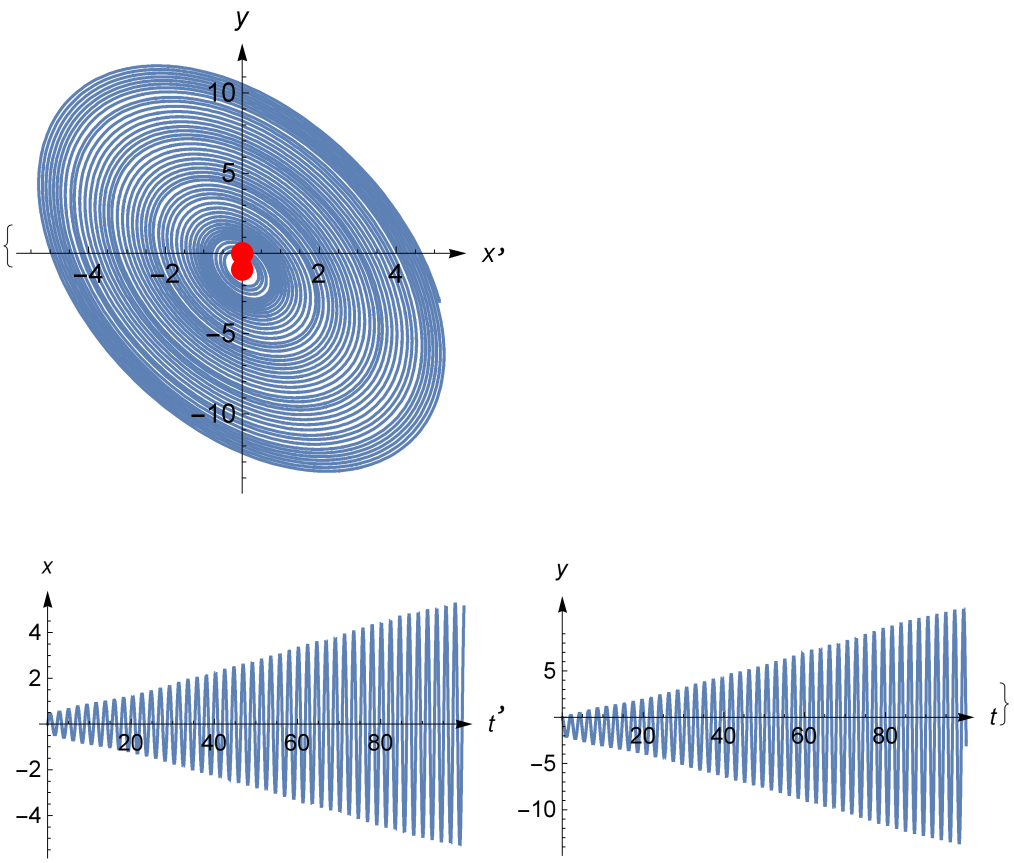

}

## Small cycle, trajectory going inward and outward

```
Figure41 = plotter[100, {0, 4.×10^-12}, 13, 100, 10000, 1]
```

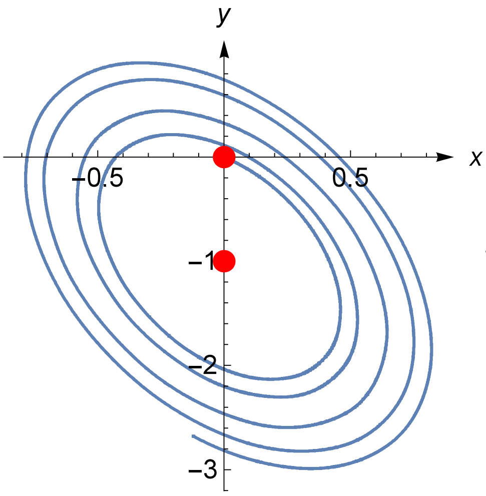



```
Figure42 = plotter[100, {0, 4.×10^-12}, 13, 100, 10000, 1, Method → "BDF"]
```



```
Figure43 = plotter[10, {0, 4. × 10^-12}, 13, 100, 10000, 1, Method → "BDF"]
```

{

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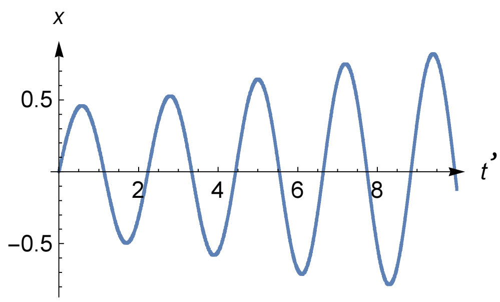


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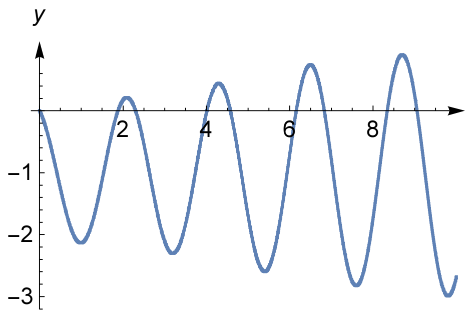
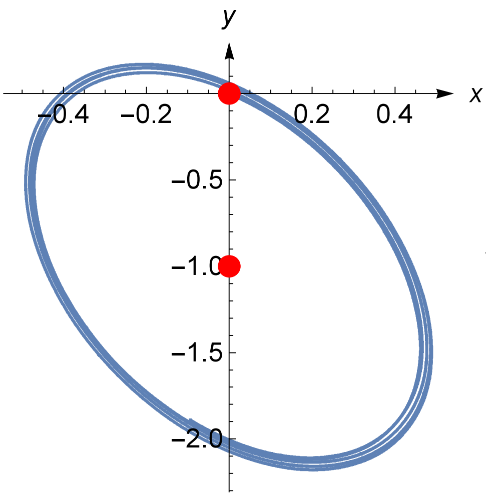



}

```
Figure51 = plotter[10, {0, 10.^-11}, 13, 100, 10000, 1, Method → "BDF"]
```

{

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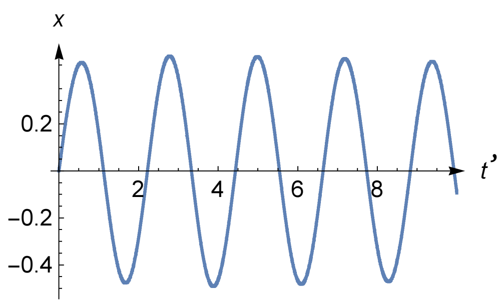


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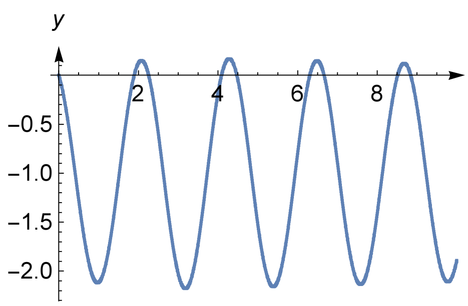
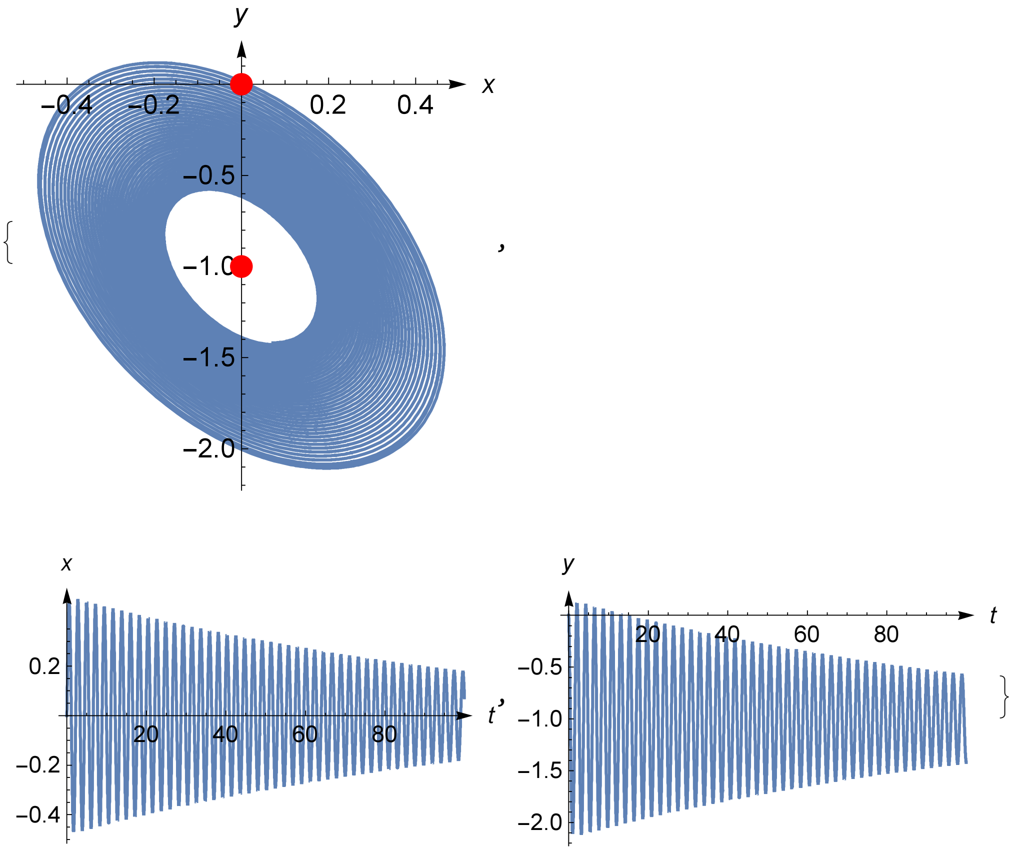
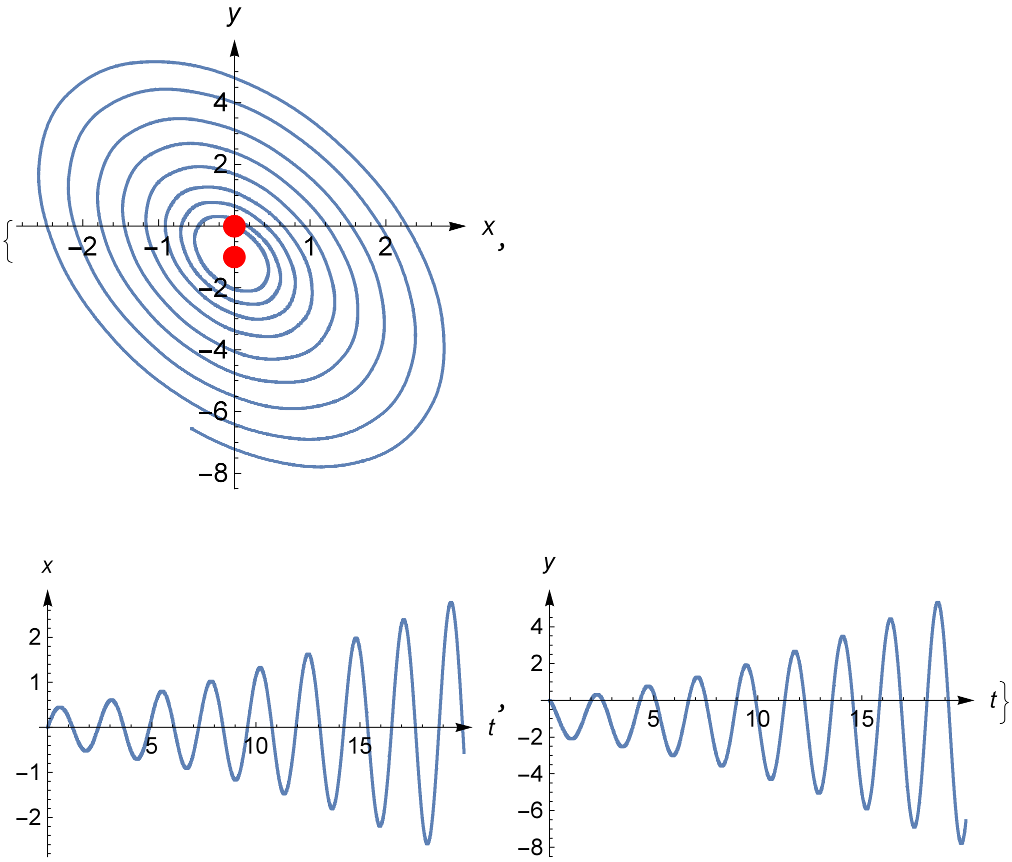



}

```
Figure52 = plotter[100, {0, 2. × 10^-11}, 13, 100, 10000, 1, Method → "BDF"]
```




```
Figure53 = plotter[20, {0, 10.^-12}, 13, 100, 10000, 1, Method → "BDF"]
```



# 4 References

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[2] B. Ferčec, I. Nagy, V. G. Romanovski, G. Szederkényi, and J. Tóth. Calculations for two limit cycles in a two-species reaction with wolfram mathematica,
(available 27 June 2018). http://math.bme.hu/~nagyi/Mathematica_notebooks/index.html

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