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Operator Method in the Theory of Differential Equations

April 6-th 2017

MUEGYETEM 1782

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Outline

1. Main concepts of theory and computer realization
2. Solving of Mathieu's equation
3. Solving of real problem: dynamical systems
4. Expressions of solutions of partial PDE using algebraic operators and algebraic convolution
5. Critic to Exp-function method and criterion to find exact ODE or PDE solutions
6. Special solutions of Huxley and Liouville's DE
7. New results of the research group based on operator method

Solving of differential equations using operator method (theory and realisation)

The basic principle

In this method convergence of power series in not analysed. In connection with this, algebraic operations (addition, multiplication etc.) on series involve only a finite number of terms. In this case power series are called algebraic series, and algorithms for finding solutions of these series – algebraic algorithms. This doesn’t put any obstacles in developing the theory of formal linear operators (in the linear space of algebraic series). It is worth noting that algebraic (formal) series were used by L.Euler.

Basic linear operators

a) **Down** $\underline{A}$ $\boxed{\underline{A}1 := 0,\ \underline{A}x^n := \lambda_n x^{n-1},\ \lambda_n \in \mathbf{C}\setminus\{0\},\ n \in \mathbf{N}}$

Example $Dx^n := nx^{n-1},\ \underline{X}x^n := x^{n-1}$

b) **Homothetic** H $\boxed{Hx^n := v_n x^n,\ v_n \in \mathbf{C},\ n \in \mathbf{Z_0}}$

Example $\mathit{1}x^n := x^n,\ \emptyset x^n := 0$

c) **Up** $\overline{A}$ $\boxed{\overline{A}x^n := \mu_n x^{n+1},\ \mu_n \in \mathbf{C}\setminus\{0\},\ n \in \mathbf{Z_0}}$

Example

$$\overline{X}x^n := x^{n+1},\ Lx^n := \frac{x^{n+1}}{n+1}$$

Special operators

1. **Pseudoinverse operator $\widetilde{A}^{-1}$ for operator A:**

$$\boxed{\boldsymbol{A\widetilde{A}^{-1}A = A}} \qquad \boxed{\boldsymbol{\widetilde{A}^{-1}A\widetilde{A}^{-1} = \widetilde{A}^{-1}}}$$

Example $\widetilde{\underline{\boldsymbol{X}}}^{-1} = \overline{\boldsymbol{X}}, \quad \widetilde{D}^{-1} = L$

2. **Perfect operators:**

$$e^{A} := \sum_{k=0}^{+\infty} \frac{1}{k!} A^{k}, \quad g(A) := \sum_{k=0}^{+\infty} A^{k}.$$

$$\boxed{\left(e^{A}\right)^{-1} = e^{-A}, \quad \left(g(A)\right)^{-1} = 1 - A}$$

Auxiliary results

$$\boldsymbol{D}_{s_k} s_k^n = n\boldsymbol{D}_{s_k} s_k^{n\text{-}1}, \quad \boldsymbol{D}_{s_k}\boldsymbol{D}_{s_l} s_k^n s_l^m = nm\, s_k^{n\text{-}1} s_l^{m\text{-}1}, \quad \boldsymbol{D}_{s_k}{}^0 s_k^n = s_k^n, \;\; \boldsymbol{L}x^n = \frac{x^{n+1}}{n+1},$$

generalized differential operator $\boxed{\boldsymbol{D} = P_1\boldsymbol{D}_{s_1} + \cdots + P_n\boldsymbol{D}_{s_n}}$ and $\boxed{\boldsymbol{G} = \sum_{k=0}^{+\infty}(\boldsymbol{L}_x\boldsymbol{D})^k}$

Theorem: Linear operator G is multiplicative

$$\boldsymbol{G}\;\; f(s_1,\ldots,s_n) = f(\boldsymbol{G}s_1,\ldots,\boldsymbol{G}s_n)$$

Example

$$\sum_{k=0}^{+\infty}\left(\boldsymbol{L}_x(t\boldsymbol{D}_s - s\boldsymbol{D}_t)\right)^k s^m t^n = (s\cos x + t\sin x)^m (t\cos x - s\sin x)^n$$

Solutions of n -th order ordinary differential equation

Let a differential equation $\boxed{y_x^{(n)} = P\left(x, y, y'_x, y''_x, \ldots, y_x^{(n-1)}\right)}$

$y(v; s_1, s_2, \ldots, s_n) = s_1$ $\quad (y(x; s_1, s_2, \ldots, s_n))'_x\big|_{x=v} = s_2$ $\quad (y(x; s_1, s_2, \ldots, s_n))''_x\big|_{x=v} = s_3$

$(y(x; s_1, s_2, \ldots, s_n))_x^{(n-1)}\big|_{x=v} = s_{n-1}$ be given. Then its solution $y(x; s_1, s_2, \ldots, s_{n-1})$

$$\boxed{y(x; s_1, s_2, \ldots, s_{n-1}) = \sum_{k=0}^{+\infty} p_k(s_1, s_2, \ldots, s_{n-1}, v)\frac{(x-v)^k}{k!}}$$

$$p_k(s_1, s_2, \ldots, s_{n-1}, v) = \left(\boldsymbol{D}_v + s_2\boldsymbol{D}_{s_1} + s_3\boldsymbol{D}_{s_2} + \ldots + s_{n-1}\boldsymbol{D}_{s_{n-2}} + P(v, s_1, s_2, \ldots, s_{n-1})\boldsymbol{D}_{s_{n-1}}\right)^k s_1$$

all coefficients $|p_k| < M^k$

obtained series converge

$x \in \boldsymbol{R}$

$P\left(x, y, y'_x, y''_x, \ldots, y_x^{(n-1)}\right)$ - polynomial or function

Example (Direct symbolic analysis)

Let a differential equation $\boxed{y' = y^2}$, $y(v) = s$ be given. We get that

$$\boxed{y = y(x;s) = \sum_{k=0}^{\infty} p_k \frac{(x-v)^k}{k!}}$$, when $p_k = p_k(s,v) = \left(\mathrm{D}_v + s^2 \mathrm{D}_s\right)^k s$

Then $p_0 = s,\quad p_1 = 1 \cdot s^2,\quad p_2 = 1 \cdot 2 s^3, \ldots, p_n = n!\, s^{n+1}$, i.e.

$$\boxed{y = y(x;s) = s \sum_{k=0}^{\infty} s^k (x-v)^k}$$, or $y = \dfrac{s}{1 - s(x-v)}$, when $|s(x-v)| < 1$

Solutions of a second order differential equation

$$\boxed{y''_{xx} = P(x, y, y'_x)} \qquad y(x;s,t)\big|_{x=v} = s \quad \left(y(x;s,t)\right)'_x\big|_{x=v} = t$$

expression of the solution

$$\boxed{y = y(x;s,t) = \sum_{k=0}^{+\infty} p_k(s,t,v)\frac{(x-v)^k}{k!}} \quad v \in R$$

$$p_k(s,t,v) = \left(\mathrm{D}_v + t\mathrm{D}_s + P(v,s,t)\mathrm{D}_t\right)^k s$$

using equalities

$$y(x,s_l,t_l) = y(x,s_{l+1},t_{l+1}) \longrightarrow y(x;s_1,t_1) = \sum_{k=0}^{+\infty} p_k(s_l,t_l,v_l)\frac{(x-v_l)^k}{k!} = y_l(x;s_l,t_l), \quad l = 1,2,\ldots$$

$$\boxed{s_{l+1} = \sum_{k=0}^{+\infty} p_k(s_l,t_l,v_l)\frac{(v_{l+1}-v_l)^k}{k!}} \qquad \boxed{t_{l+1} = \sum_{k=0}^{+\infty} p_{k+1}(s_l,t_l,v_l)\frac{(v_{l+1}-v_l)^k}{k!}}$$

sequence $v_2, v_3, \ldots$ are any variables, v_1, s_1, t_1 are given.

Computer realization

The solution of a differential equation is a series (in x) with coefficients p_k ,which, in their turn, are functions of both the initial conditions and the center v. Computer realization of the problem gives only a finite number of coefficients , and, every time, we get one or another approximation of the solution.

Having functions $p_k(s,t,v)$, k=0,...,N, we construct a polynomial

$$\hat{y}(x;s;t,v)=\sum_{k=0}^{N}p_k(s,t,v)\frac{(x-v)^k}{k!}.$$

Substituting concrete numerical values for the variables s, t and v we get an approximation of the solution – the polynomial $\hat{y}_1(x)$ in the neighborhood of the point v.

Construction of solution

Summing up to N, we get a family of approximations - polynomials $\hat{y}_l(x)$ with centers v_l. The latter approximations "go away" from the exact solution, as the variable x "moves away" from the center. The approximation $y^*(x)$ (Fig. 1) of the solution is formed, using the family of approximations $\hat{y}_l(x)$ (Fig. 2), as follows:

$$y^*(x) = \hat{y}_l(x), \quad \text{for} \quad v_l \le x < v_{l+1} \ ,$$
$$l = 1,2,\ldots,n$$

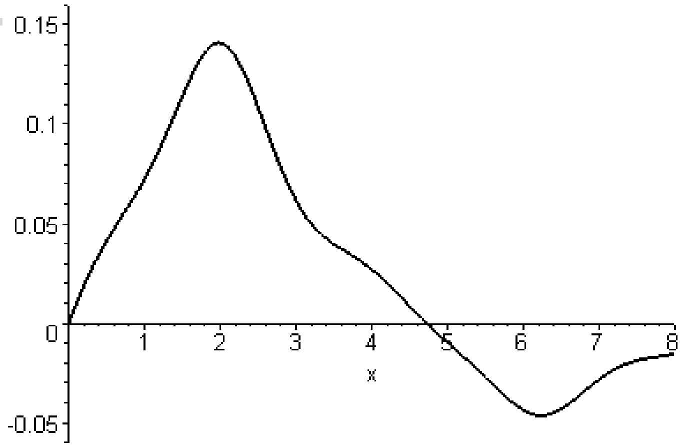


Fig. 1. Approximation of the solution

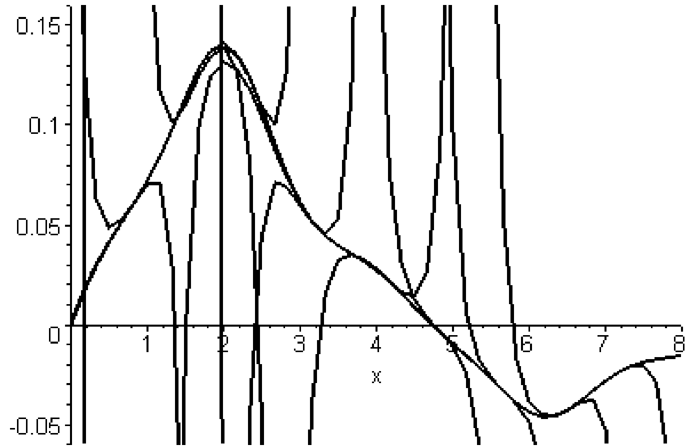
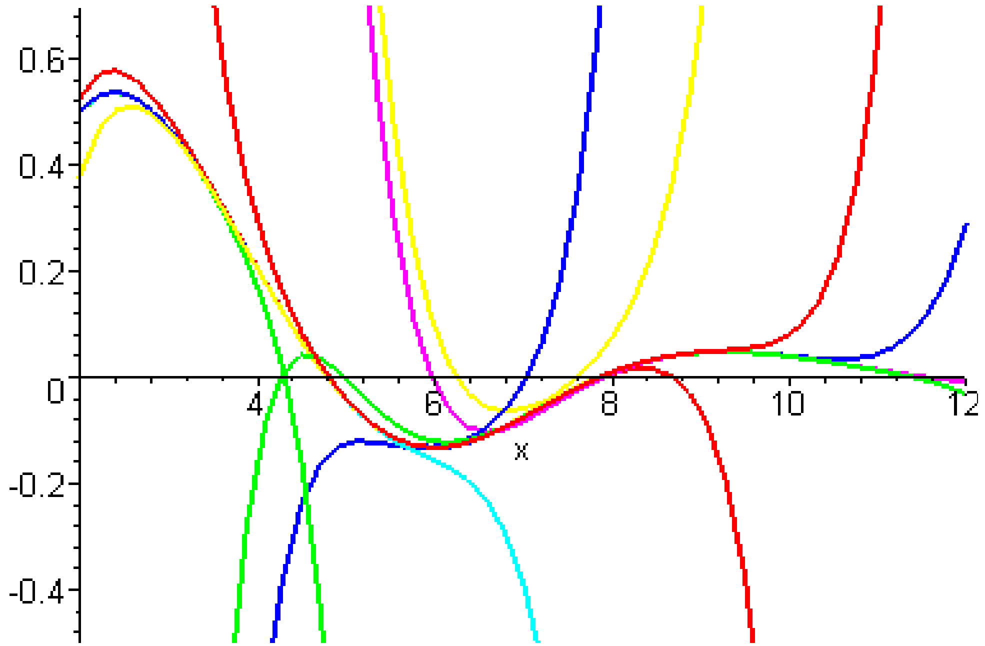
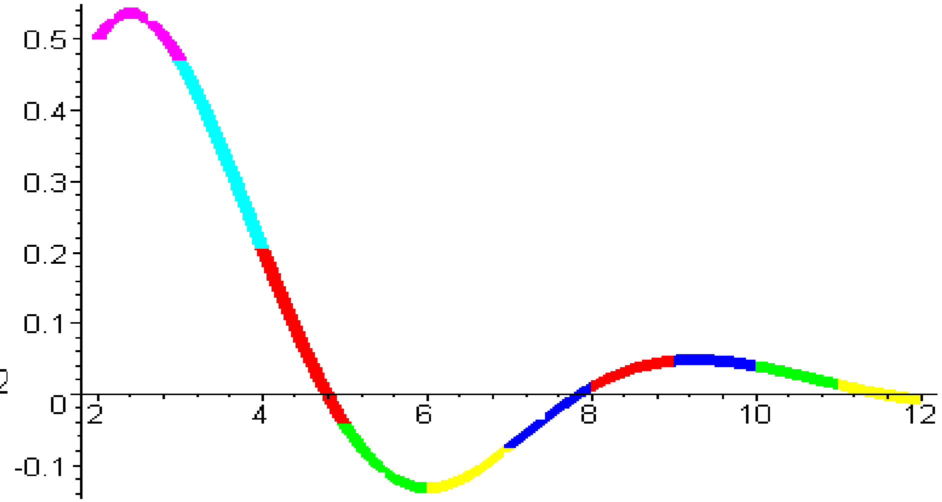


Fig. 2. The family of polynomials

Example of constructions



The family of polynomials



Bikulčienė, Liepa; Marcinkevičius, Romas; Navickas, Zenonas. Computer realization of the operator method for solving of differential equations // Numerical Analysis and its Applications : 3rd international conference NAA 2004, June 29 - July 3, 2004, Rousse, Bulgaria. Berlin: Springer, 2005. (Lecture Notes in Computer, Vol. 3401, p. 179-186.]

Example of approximation

$$
\begin{cases}
.5000 + .2000\,x - .1975\,x^2 - .01842\,x^3 + .02322\,x^4 + .002726\,x^5 - .002619\,x^6 & x < 1. \\
.5008 + .1953\,x + .0004179\,x^6 - .009449\,x^5 - .04089\,x^3 + .04514\,x^4 - .1839\,x^2 & x < 2. \\
.4889 + .2429\,x + .001063\,x^6 - .01205\,x^5 - .01623\,x^3 + .04385\,x^4 - .2384\,x^2 & x < 3. \\
.2421 + .8221\,x - .00009290\,x^6 + .004373\,x^5 + .3011\,x^3 - .05447\,x^4 - .8209\,x^2 & x < 4. \\
-1.170 + 3.093\,x - .0006531\,x^6 + .01662\,x^5 + .8516\,x^3 - .1666\,x^4 - 2.348\,x^2 & x < 5. \\
2.557 - 1.052\,x - .0006210\,x^6 + .01398\,x^5 + .4283\,x^3 - .1167\,x^4 - .4898\,x^2 & x < 6. \\
71.02 - 75.86\,x + .001956\,x^6 - .07004\,x^5 - 7.893\,x^3 + 1.027\,x^4 + 33.64\,x^2 & x < 7. \\
-165.6 + 137.9\,x - .0008542\,x^6 + .04125\,x^5 + 8.325\,x^3 - .8114\,x^4 - 46.94\,x^2 & x < 8. \\
-91.80 + 86.90\,x - .0007731\,x^6 + .03510\,x^5 + 6.239\,x^3 - .6495\,x^4 - 32.58\,x^2 & x < 9. \\
-67.44\,x + 119.5 - .00008656\,x^6 + .001249\,x^5 - 1.380\,x^3 + .04593\,x^4 + 14.38\,x^2 & 9. \le x
\end{cases}
$$

Example (first order differential equation)

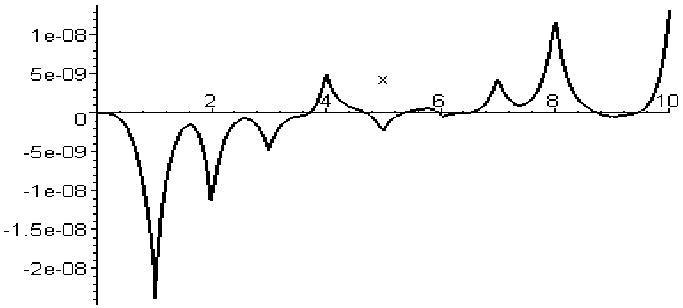
$$\boxed{y'_x = -5(y - \cos x)} \qquad y(0) = 1 + \frac{25}{26}$$

$$y = e^{-5x} + \frac{25}{26}\cos x + \frac{5}{26}\sin x \qquad \boxed{y(x,s,v) = \sum_{k=0}^{N} p_k(s,v)\frac{(x-v)^k}{k!}}$$

$$p_0(s,v) = s, \quad p_{k+1}(s,v) = \left(D_v - 5(s - \cos v)D_s\right)p_k(s,v)$$

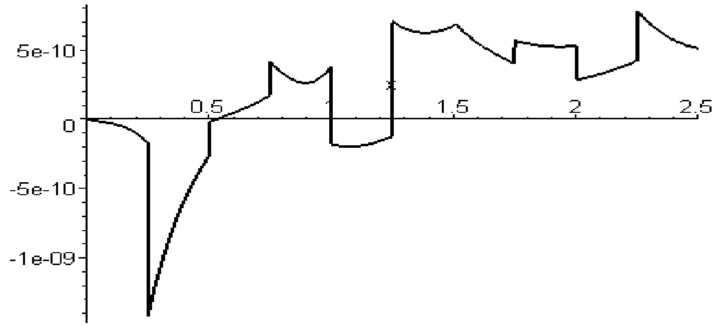
$$p_1(s,v) = -5s + 5\cos v, \quad p_2(s,v) = -5\sin v + 25s - 25\cos v, \quad p_3(s,v) = 120\cos v - 25\sin v - 125s, \ldots$$

$$\Delta = y'_x + 5\left(e^{-5x} + \frac{25}{26}\cos x + \frac{5}{6}\sin x - \cos x\right)$$



$N = 50, h = 0.25 \rightarrow$

$\leftarrow N = 50, h = 1$



Expresion of solution by standard functions (H-rank)

Condition of expression by exponential functions

$$\boxed{F(x) = \sum_{k=0}^{+\infty} p_k \frac{x^k}{k!},\ k = 0,1,2,\ldots} \Rightarrow \boxed{F(x) = \sum_{r=1}^{m_0} \lambda_r \exp(\mu_r x)}$$

$$\begin{vmatrix} p_0 & p_1 & p_2 & \cdots & p_{(n+1)} \\ p_1 & p_2 & p_3 & \cdots & p_{(n+2)} \\ p_2 & p_3 & p_4 & \cdots & p_{(n+3)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{(n+1)} & p_{(n+2)} & p_{(n+3)} & \cdots & p_{2(n+1)} \end{vmatrix} = 0,$$

exist

$m_0 \in \mathbf{N},\ n > m_0$

$$p_k = \sum_{r=1}^{m_0} \lambda_r \mu_r^k,\ k = 0,1,2,\ldots$$

Condition of expression by trigonometric functions

$$F(x) = \sum_{k=0}^{+\infty} p_{2k} \frac{x^{2k}}{(2k)!} + \sum_{k=0}^{+\infty} p_{2k+1} \frac{x^{2k+1}}{(2k+1)!}$$

$$F(x) = \sum_{r=0}^{m_1} \lambda_r \cos \mu_r x + \sum_{r=0}^{m_2} \hat{\lambda}_r \sin \hat{\mu}_r x$$

exist

$m_1 \in \mathbf{N},\ n > m_1$ $\qquad$ $m_2 \in \mathbf{N},\ n > m_2$

$$\begin{vmatrix} p_0 & p_2 & p_4 & \cdots & p_{2(n+1)} \\ p_2 & p_4 & p_6 & \cdots & p_{2(n+2)} \\ p_4 & p_6 & p_8 & \cdots & p_{2(n+3)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{2(n+1)} & p_{2(n+2)} & p_{2(n+3)} & \cdots & p_{4(n+1)} \end{vmatrix} = 0, \qquad \begin{vmatrix} p_1 & p_3 & p_5 & \cdots & p_{2n+3} \\ p_3 & p_5 & p_7 & \cdots & p_{2n+5} \\ p_5 & p_7 & p_9 & \cdots & p_{2n+7} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{2n+3} & p_{2n+5} & s_{2n+7} & \cdots & p_{4n+5} \end{vmatrix} = 0$$

Expression of series by sum of trigonometric functions

$$p_{2k}=\sum_{r=0}^{m_1}\lambda_r\left(-\mu_r^2\right)^k, k=0,1,2,\ldots,2\left(m_1+1\right)$$

even part

$$\rho_1=-\mu_0^2,\rho_2=-\mu_1^2,\quad \rho_3=-\mu_2^2,\ldots,\rho_{m_1+1}=-\mu_{m_1}^2,\quad \begin{vmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{m_1+1} \\ p_0 & p_2 & p_4 & \cdots & p_{2(m_1+1)} \\ p_2 & p_4 & p_6 & \cdots & p_{2(m_1+2)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{2m_1} & p_{2(m_1+1)} & p_{2(m_1+3)} & \cdots & p_{4m_1+2} \end{vmatrix}=0$$

.

$$p_{2k+1}=\sum_{r=0}^{m_2}\lambda_r(-1)^k\,\overline{\mu}_r^{2k+1}, k=0,1,2,\ldots,2\left(m_2+1\right)$$

odd part

$$\rho_1=-\overline{\mu}_0^2,\rho_2=-\overline{\mu}_1^2,\rho_3=-\overline{\mu}_2^2,\ldots,\rho_{m_2+1}=-\overline{\mu}_{m_2}^2$$

Example 4

Let $p_1 = p_3 = \ldots = p_{2k+1} = \ldots = 0$ and $p_{2k} = 2(-1)^k + (-2)^k$. Then

$$p_0 = 3,\ p_2 = -4,\ p_4 = 6,\ p_6 = -10,\ p_8 = 18,\ p_{10} = -34,\ p_{12} = 66, \ldots$$

$$|3| = 3, \quad \begin{vmatrix} 3 & -4 \\ -4 & 6 \end{vmatrix} = 34, \quad \begin{vmatrix} 3 & -4 & 6 \\ -4 & 6 & -10 \\ 6 & -10 & 18 \end{vmatrix} = 0, \quad \begin{vmatrix} 3 & -4 & 6 & -10 \\ -4 & 6 & -10 & 18 \\ 6 & -10 & 18 & -34 \\ -10 & 18 & -34 & 66 \end{vmatrix} = 0, \ldots$$

$$\rho_{1,2} = \pm i, \rho_{3,4} = \pm\sqrt{2}\, i.$$

$$F(x) = e^{ix} + e^{-ix} + \frac{e^{\sqrt{2}\, ix} + e^{-\sqrt{2}\, ix}}{2} = 2\cos x + \cos\sqrt{2}\, x$$

Navickas, Zenonas; Bikulčienė, Liepa. Expressions of solutions of ordinary differential equations by standard functions // Proceedings of the 10th International Conference Mathematical Modelling and Analysis 2005 and 2nd International Conference Computational Methods in Applied Mathematics, June 1-5, 2005, Trakai, Lithuania. Vilnius: Technika, 2005, ISBN 9986059240. p. 485-491.

Perturbation method

differential equation

$$y'' = P(y, y') \qquad y = y(0, s, t) = s \qquad y'(x, s, t)\big|_{x=0} = t$$

expression of the solution in operator form

$$y = g\left(L_x t D_s + L_x P(s, t) D_t\right) s$$

With parameter of perturbations

$$z'' = \varepsilon P(z, z') \qquad z(x, s, t, \varepsilon) = \sum_{k=0}^{+\infty} z_k(x, s, t)\varepsilon^k$$

$$y(x, s, t) = z(x, s, t, 1) \qquad y \approx z_0 + z_1 + \ldots + z_n$$

Perturbation method

$$z = g\left(L_x tD_s + \varepsilon L_x P(s,t) D_t\right)s = g\left(g\left(L_x tD_s\right)\varepsilon L_x P(s,t) D_t\right)g\left(L_x tD_s\right)s =$$

$$= \sum_{k=0}^{+\infty}\left(L_x^k\left(\left(g\left(L_x tD_s\right)P(s,t)D_t\right)^k (s+tx)\right)\right)\varepsilon^k .$$

$$q_0(x,s,t) = s + tx \qquad q_{k+1}(x,s,t) = \left(g\left(L_x tD_s\right)P(s,t)D_t\right)q_k(x,s,t)$$

$$z = \sum_{k=0}^{+\infty}\left(L_x^k q_k(x,s,t)\right)\varepsilon^k$$

Bikulčienė, Liepa; Navickas, Zenonas. Practice of operator relationships in symbolical calculus // Acta Universitatis Apulensis. Alba Iulia: 1 Decembrie 1918 University of Alba Iulia. ISSN 1582-5329. 2008, no. 15, p. 373-378.

Example

$$y'' = -\sin y \qquad y'(0,s,t) = t \qquad y = y(0,s,t) = s$$

$$y = \sum_{k=0}^{+\infty} \left(L_x t D_s - L_x \sin s D_t\right)^k s$$

$$\boxed{z'' = -\varepsilon \sin z}$$

$$z = \sum_{k=0}^{+\infty} \left(L_x \left(g\left(L_x t D_s\right)\left(-\sin s\right) D_t\right)^k \left(st + x\right)\right)\varepsilon^k$$

$$q_1 = -L_x \sin(s + tx),$$

$$q_2 = L_x^2\left(2\sin(2s + 2tx) - \frac{3}{2}\sin(2s + tx) - \frac{tx}{2}\cos(2s + tx)\right) - L_x\left(\frac{x}{2}\sin tx\right)$$

Solving of
Mathieu’s equation

Mathieu's equation

$$\boxed{y'' + Hy' + \beta^2 (1 + a \cos wx) \sin y = 0} \quad y(x)\big|_{x=v} = s \quad (y(x))'_x\big|_{x=v} = t$$

$$\boxed{y = y(x; s, t) = \sum_{k=0}^{+\infty} p_k(s, t, v) \frac{(x - v)^k}{k!}}$$

$$p_0(s, t, v) = s,$$

$$p_{k+1}(s,t,v) = \left(\mathrm{D}_v + t\mathrm{D}_s - \left(Ht + \beta^2(1 + a\cos wv)\sin s\right)\mathrm{D}_t\right) p_k(s,t,v),$$

Then

$$p_1(s,t,v) = t, \qquad p_2(s,t,v) = -\left(Ht + \beta^2(1 + a\cos wv)\sin s\right),$$

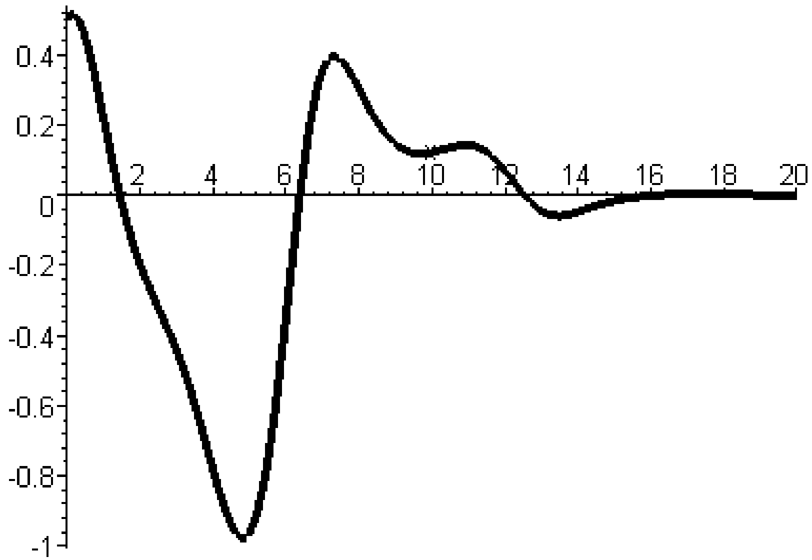
$$p_3(s,t,v) = \beta^2 aw \sin s \sin wv - \beta^2 t \cos s (1 + a\cos wv) +$$

$$+ H\left(Ht + \beta^2(1 + a\cos wv)\sin s\right)\sin s, \ldots$$

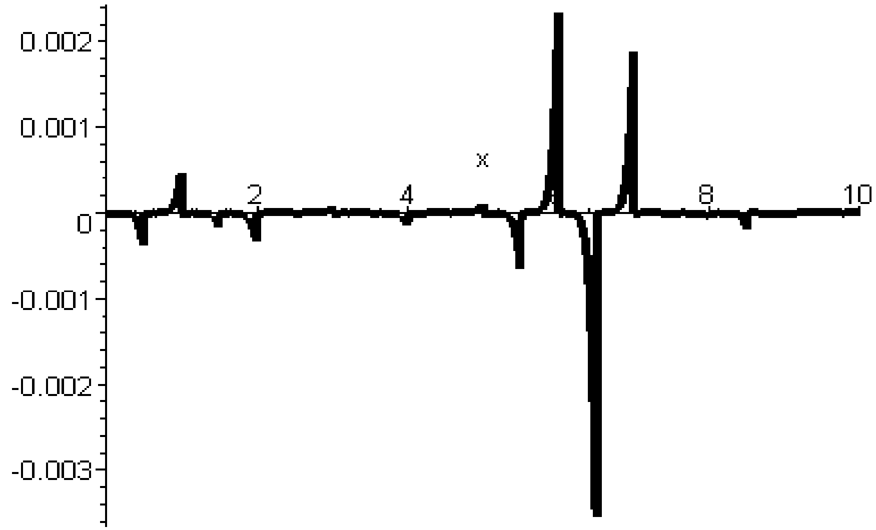
Plots of approximation and error estimates (Mathieu's equation)

$$H = 0.7,\ \beta = 0.9, a = 2,\ w = 1,\ h = 1$$

$$y(0) = 0.5 \quad y'(0) = 0.2,\ x = 0..20$$



Approximation of the solution



The error estimates of the solution.

The error estimates and the calculation time

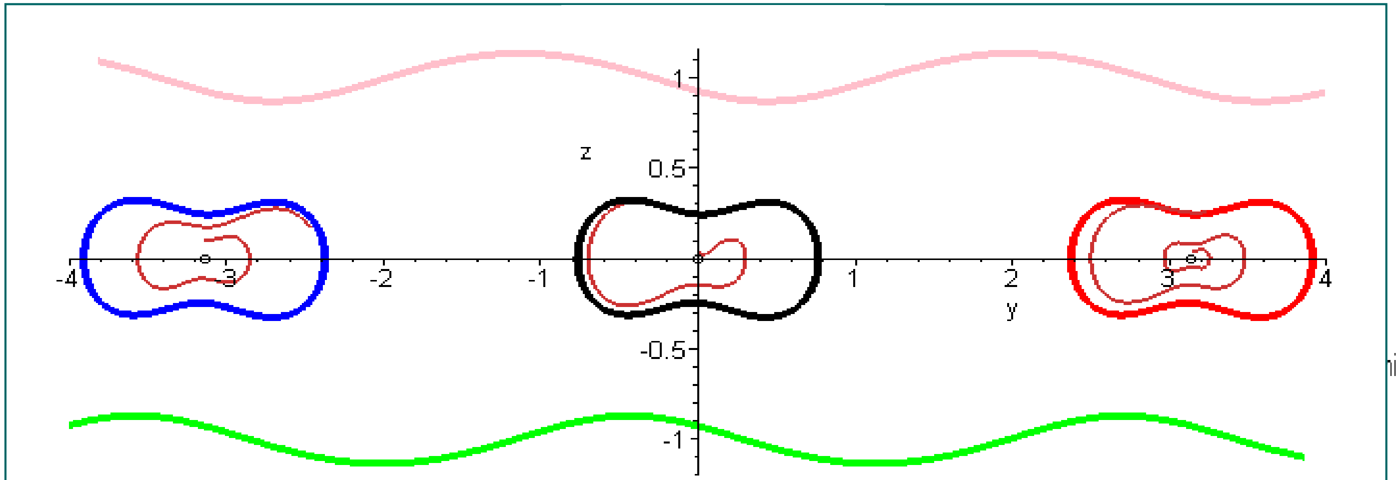
N	h	Δ	T,s	N	h	Δ	T,s
8	1	0,034	12	9	0,5	0,004	54
8	0,75	0,028	12	9	0,25	0,006	62
8	0,5	0,025	14	10	1	0,0011	124
8	0,25	0,032	15	10	0,75	0,0078	127
9	1	0,005	42	10	0,5	0,0084	145
9	0,75	0,003	45	10	0,25	0,0014	160

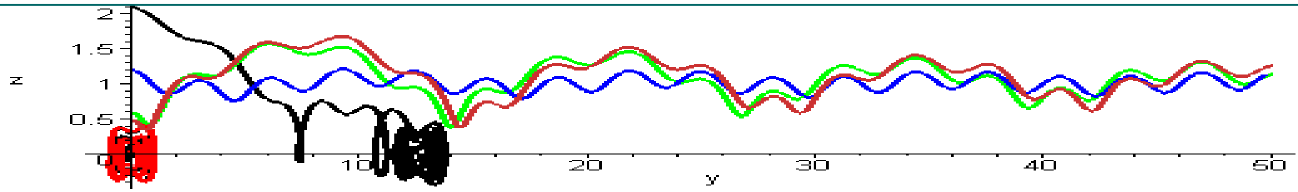
Investigations af steady regimes

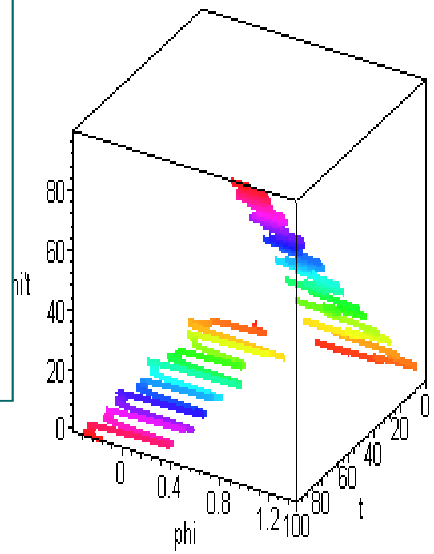
Solutions of the Mathieu's differential equation have various standing modes. Finding of attractor zones and their limits is important. Reduced equation

$$\boxed{y'' + Hy' + \beta \cos wx \sin y = 0} \qquad H = 0.05 \;\; \beta = 0.5 \qquad w = 1$$

have two standing modes:

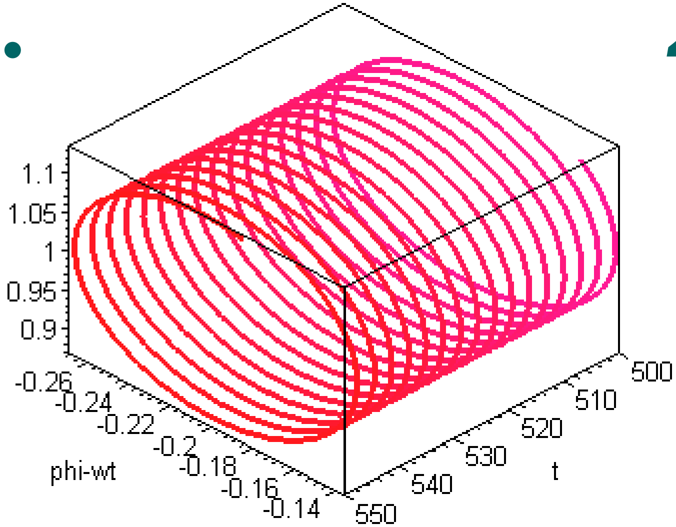


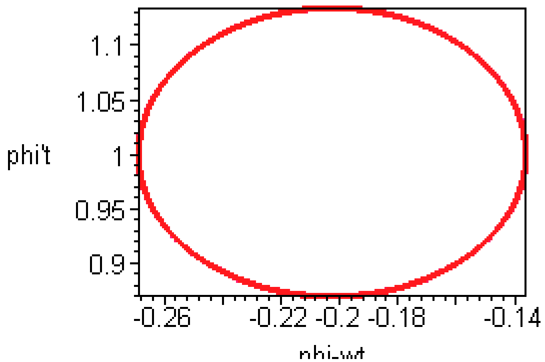




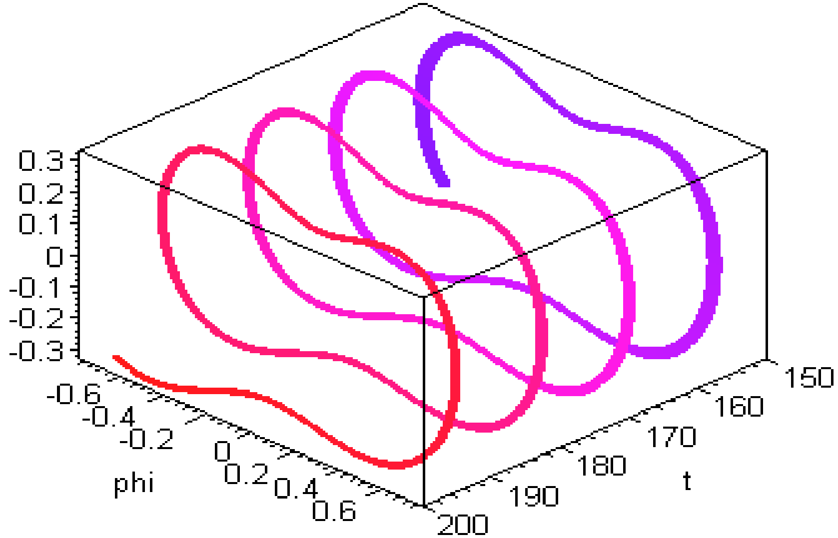
Two steady regimes

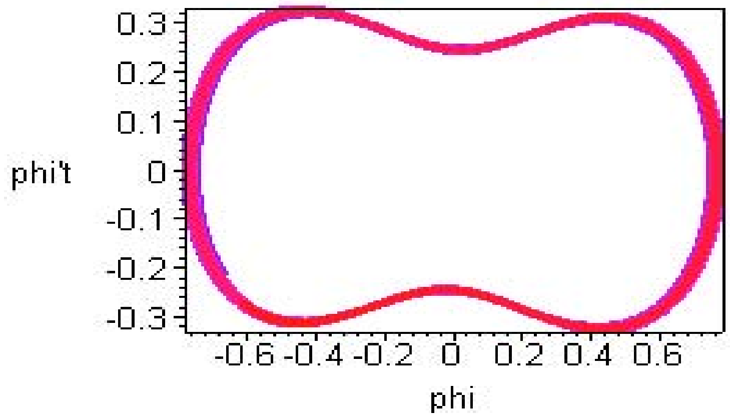
1.



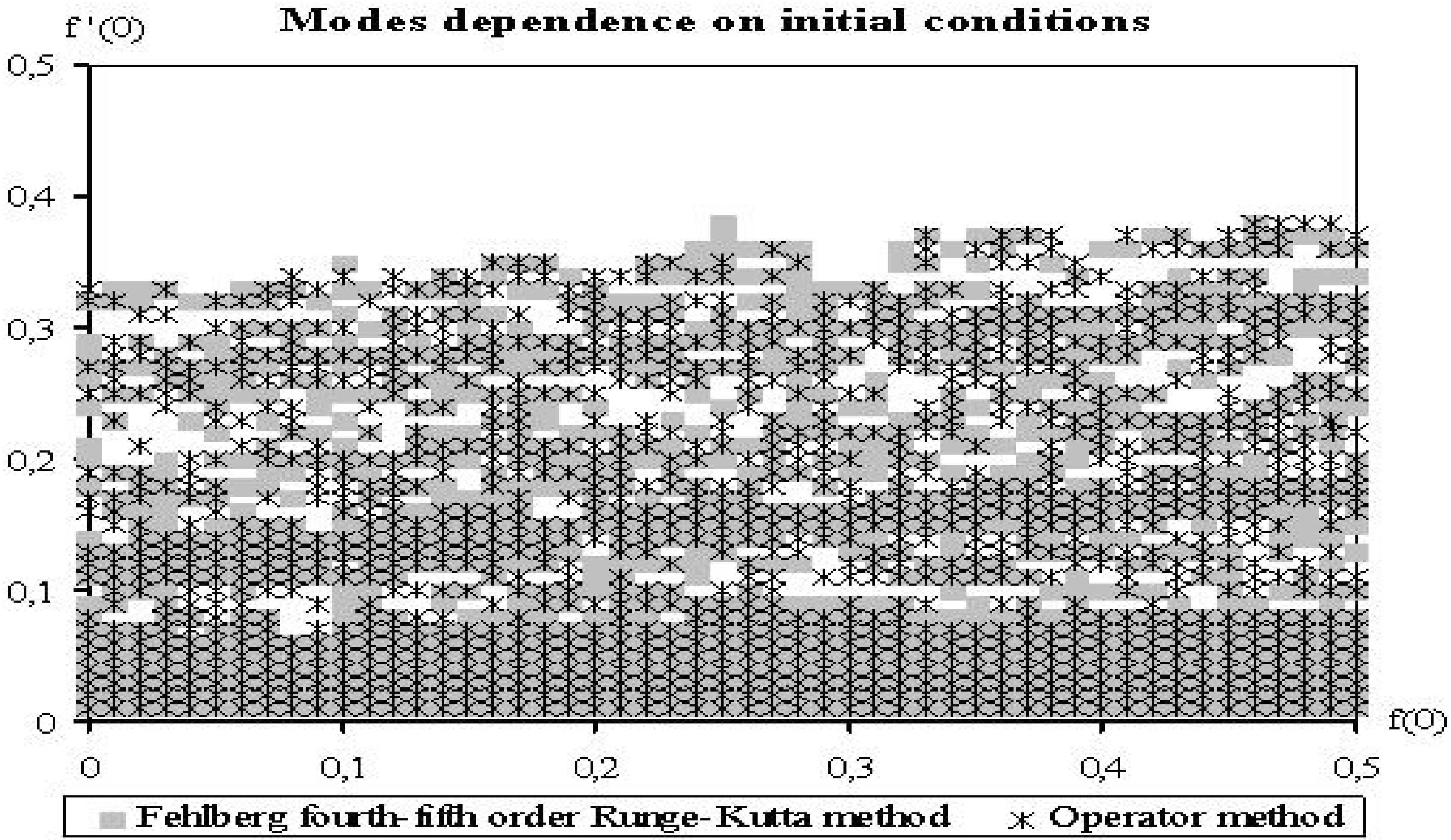


2.



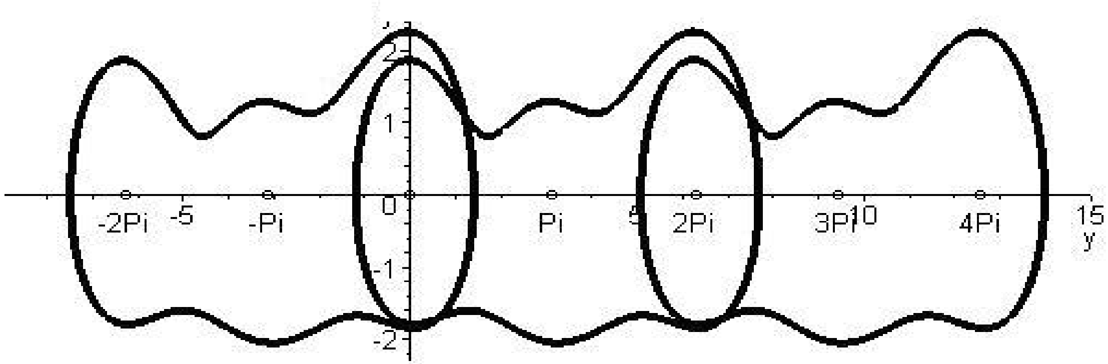


The results of two methods

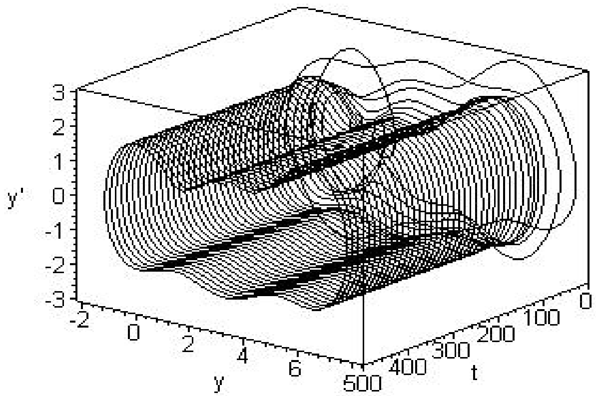


Results with other constants of Mathieu's equation

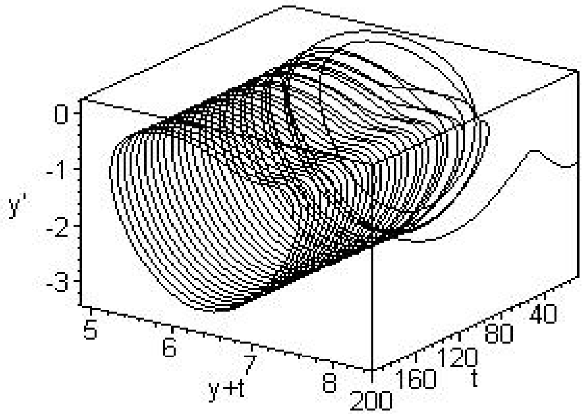
$$y'' + 0{,}05\,y' + 2\cos t \sin\, y + \sin\, y = 0$$



Stable mode in the phase plane *(y/y')*

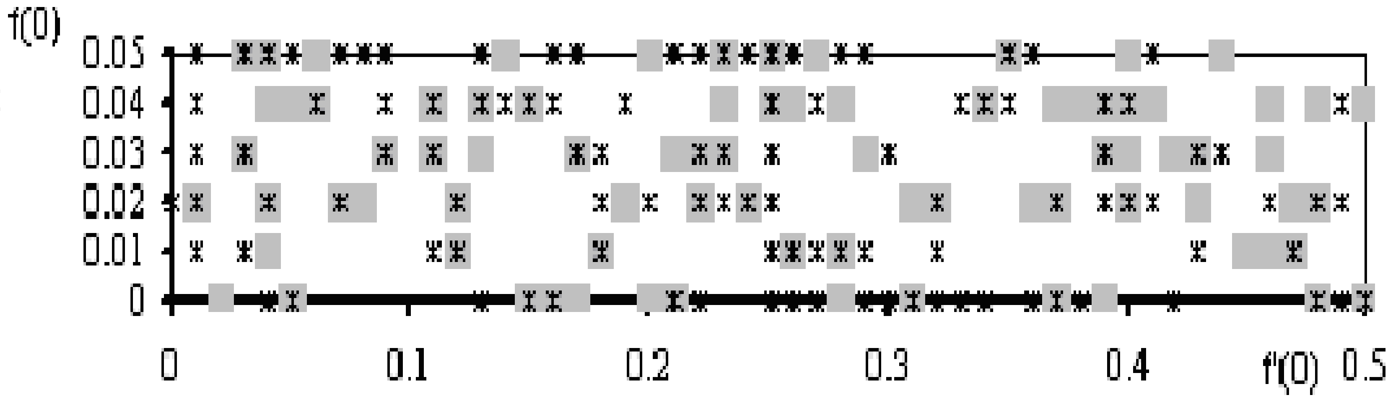


Stable mode in the phase space *(t/y/y')*



Other stable mode in the phase space *(t/y+t/y')*

Comparison of methods



Modes dependence on initial conditions

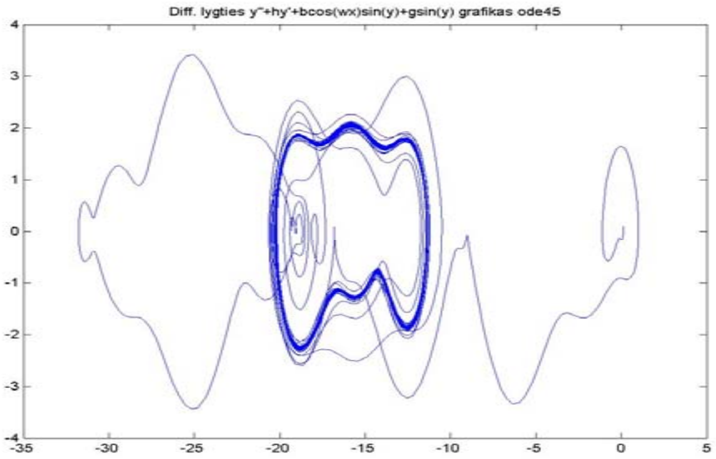
$$t = 0.5, \quad y(0) = 0.1, \quad y'(t)\Big|_{t=0} = 0.1$$

Fehlberg fourth-fifth order Runge-Kutta method	0.14240203322550688307
eventh-eighth order continuous Runge-Kutta method	0.14240203459506171859
Taylor series method	0.14240203459505515063
operator method, order of polynomial=8	0.14240202093159788066
operator method, order of polynomial=10	0.14240203481552959490
operator method, order of polynomial=12	0.14240203459505913253

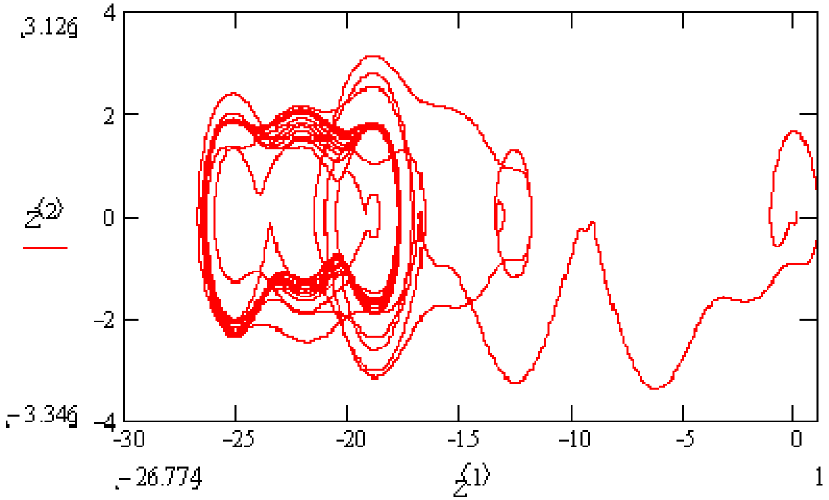
Values of solution obtained using various methods

Runge-Kutta metods

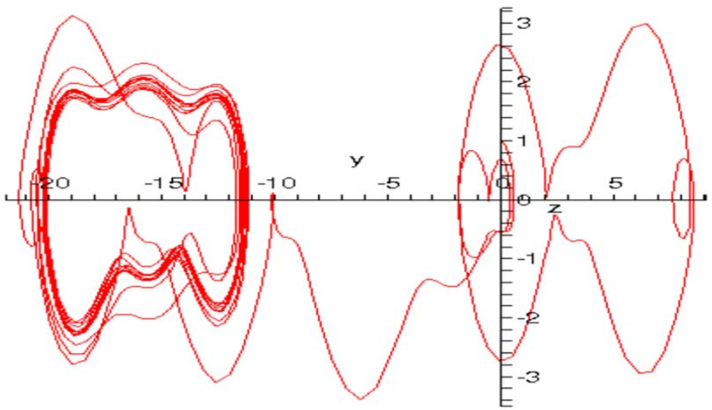
Matlab



MathCad



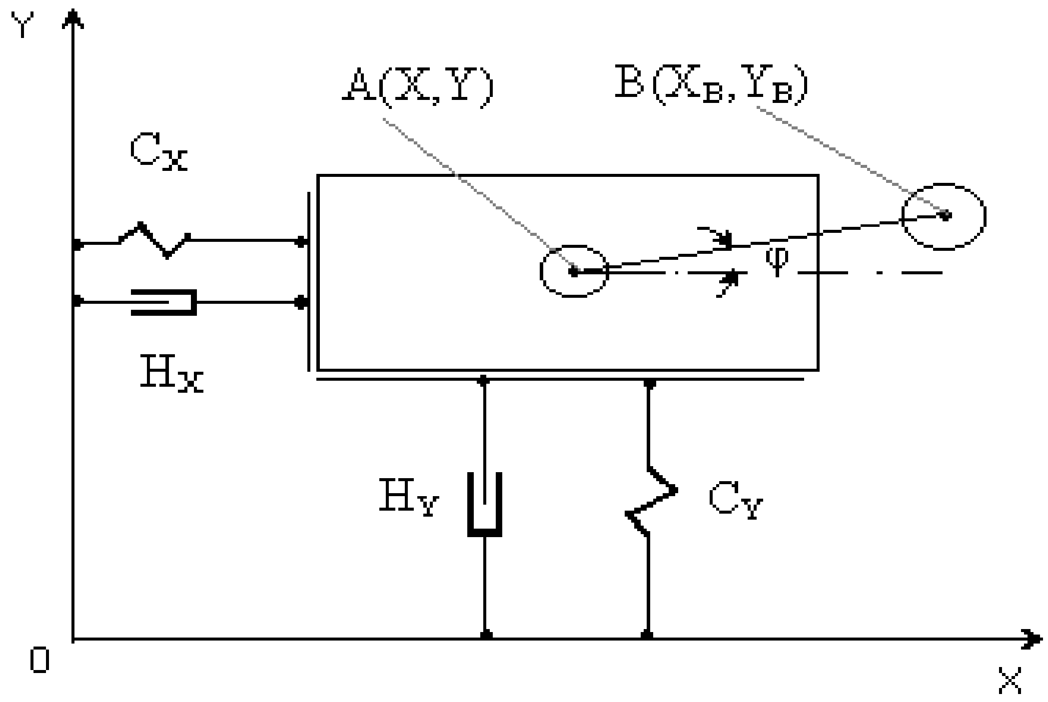
Maple



Bikulčienė, Liepa; Marcinkevičius, Romas; Navickas, Zenonas. Adapted operator method for solving of ordinary differential equations // ITI 2005 : proceedings of the 27th international conference on Information Technology Interfaces, June 20-23, 2005, Cavtat, Croatia. Zagreb: SRCE University Computing Centre, University of Zagreb, 2005, ISBN 953713802X. p. 611-616.

Solving of real problem: dynamical systems

Non autonomous system of motion transformation



The wave (or, flowing liquid) motion straight dynamically, without any kinematics forced relationships, stimulates rotational motion of the output term.

The specifying system of the model

$$\begin{cases} x'' + h_x x' + p_x^2 x = X, \\ y'' + h_y y' + p_y^2 y = Y \\ I\varphi'' + \Lambda = 0, \end{cases}$$

$$\Rightarrow$$

$$x'' = \frac{\Delta_x}{\Delta} = P,$$

$$y'' = \frac{\Delta_y}{\Delta} = Q,$$

$$\varphi'' = \frac{\Delta_\varphi}{\Delta} = R$$

$$X = f_x \cos \omega t + \mu r\left(\varphi'' \sin \varphi + \varphi'^2 \cos \varphi\right)$$

$$Y = \mu_{xy} f_y \sin \omega t - \mu_{xy} \mu r\left(\varphi'' \cos \varphi - \varphi'^2 \sin \varphi\right)$$

$$\Lambda = mr\left(-x'' \sin \varphi + y'' \cos \varphi\right) + H_\varphi \varphi' - M_\varphi$$

Notations

$$\Delta = m\mu r^2\left(\sin^2\varphi + \mu_{xy}\cos^2\varphi\right) - I$$

$$\Delta_x = \left(-I + \mu_{xy}\mu r^2\cos^2\varphi\right)\cdot F_x + \mu m r^2\cos\varphi\sin\varphi\cdot F_y + \mu r\sin\varphi\cdot F_\varphi\,,$$

$$\Delta_y = \left(-I + m\mu r^2\sin^2\varphi\right)\cdot F_y - \mu_{xy}m\mu r^2\cos\varphi\sin\varphi\cdot F_x - \mu_{xy}\mu r\cos\varphi\cdot F_\varphi\,,$$

$$\Delta_\varphi = F_\varphi - mr\sin\varphi F_x + mr\cos\varphi F_y$$

$$F_x = f_x\cos\omega t - \varphi'^2\mu r\cos\varphi - h_x x' - p_x^2 x$$

$$F_y = f_y\mu_{xy}\sin\omega t + \varphi'^2\mu_{xy}\mu r\sin\varphi - h_y y' - p_y^2 y$$

$$F_\varphi = H_\varphi\varphi' - M_\varphi$$

Solutions of system

$$\begin{cases} x_t'' = P(t, x, x_t', y, y_t', \varphi, \varphi_t') \\ y_t'' = Q(t, x, x_t', y, y_t', \varphi, \varphi_t') \\ \varphi_t'' = R(t, x, x_t', y, y_t', \varphi, \varphi_t') \end{cases}$$

$$x(v) = s_1 \qquad y(v) = s_2 \qquad \varphi(v) = s_3$$

$$x_t'(t)\big|_{t=v} = t_1 \qquad y_t'(t)\big|_{t=v} = t_2 \qquad \varphi_t'(t)\big|_{t=v} = t_3$$

$$\mathbf{D} = \mathbf{D}_v + t_1\mathbf{D}_{s1} + P\mathbf{D}_{t1} + t_2\mathbf{D}_{s2} + Q\mathbf{D}_{t2} + t_3\mathbf{D}_{s3} + R\mathbf{D}_{t3}$$

$$x = \sum_{k=0}^{+\infty} p_k \frac{(t-v)^k}{k!}, \qquad p_k = \mathbf{D}^k s_1$$

$$y = \sum_{k=0}^{+\infty} q_k \frac{(t-v)^k}{k!}, \qquad q_k = \mathbf{D}^k s_2$$

$$\varphi = \sum_{k=0}^{+\infty} r_k \frac{(t-v)^k}{k!}, \qquad r_k = \mathbf{D}^k s_3$$

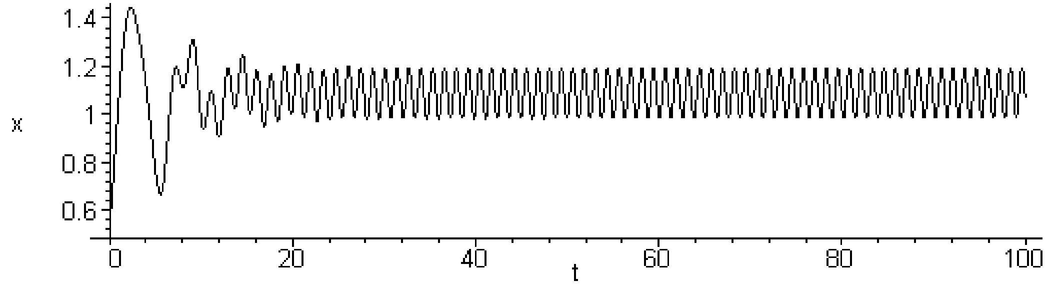
$$P = P(v, s_1, t_1, s_2, t_2, s_3, t_3)$$

$$Q = Q(v, s_1, t_1, s_2, t_2, s_3, t_3)$$

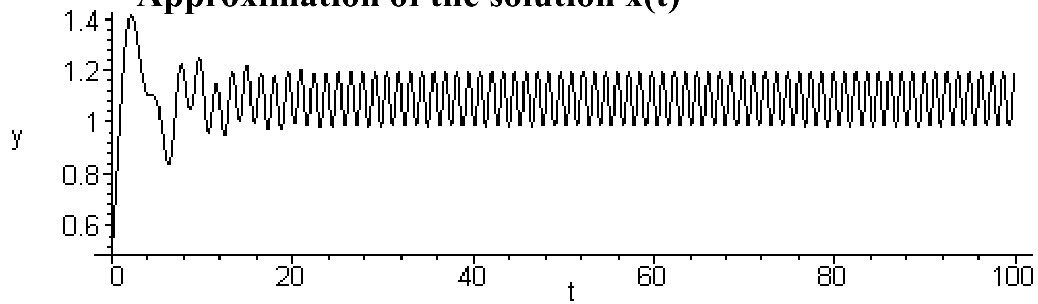
$$R = R(v, s_1, t_1, s_2, t_2, s_3, t_3)$$

- polynomials or functions

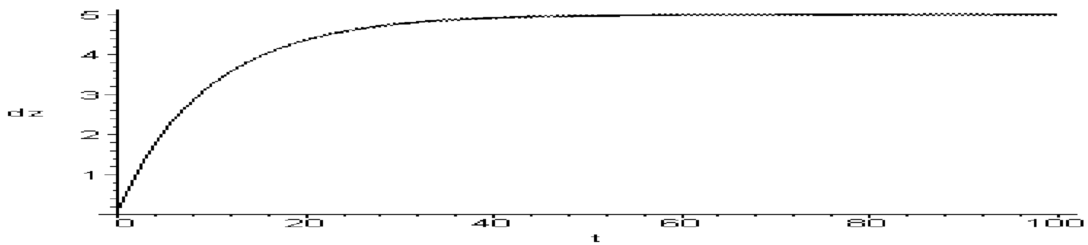
The steady mode (example)



Approximation of the solution x(t)



Approximation of the solution y(t)



Approximation of the solution φ'(t)

$$h_x = h_y = h_\varphi = 0.1$$

$$q_{1x} = q_{1y} = 0.3$$

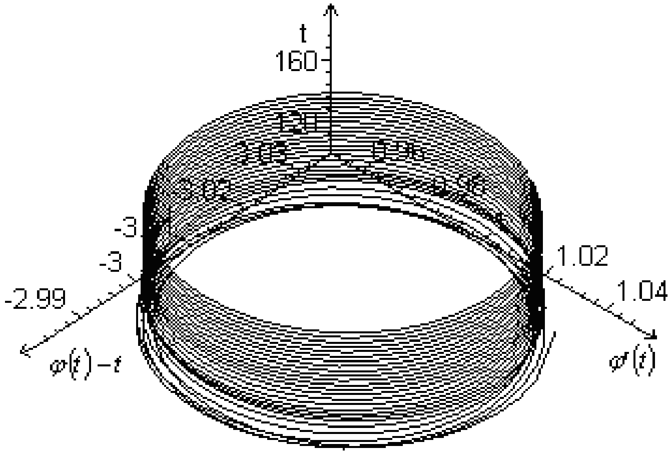
$$q_{3x} = q_{3y} = \mu = \mu^* = \mu_\varphi = 1$$

$$x(0) = 0.5 \quad x_t'(t)\big|_{t=0} = 0.5$$

$$y(0) = 0.5 \quad y_t'(t)\big|_{t=0} = 0.5$$

$$\varphi(0) = 0.1 \quad \varphi_t'(t)\big|_{t=0} = 0.1$$

Expression by trigonometric functions

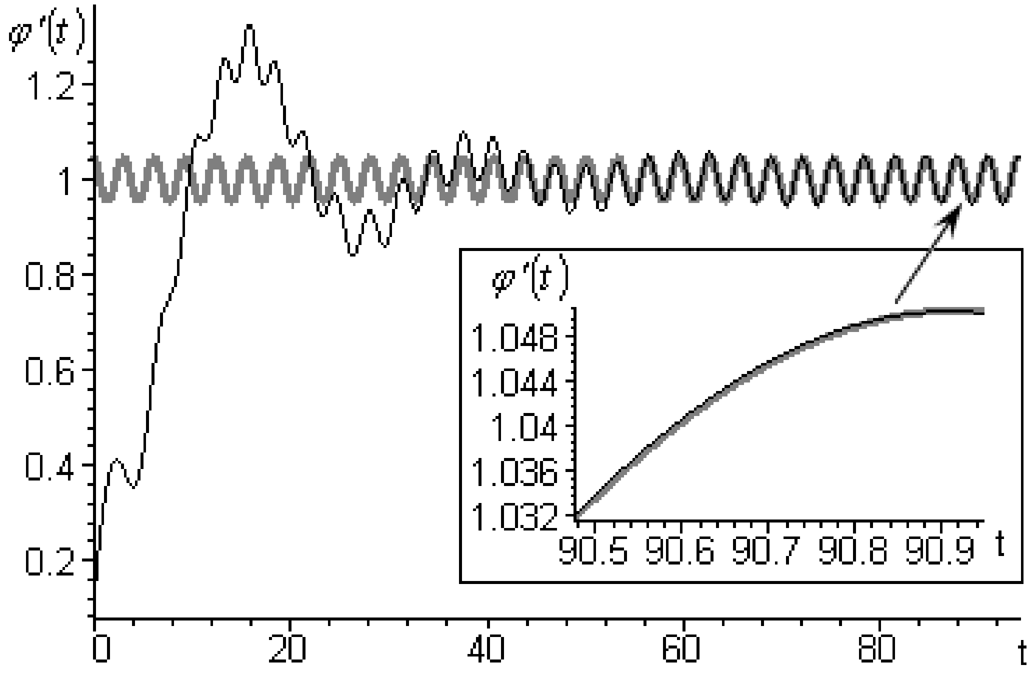


Steady mode in the phase space

$(\varphi'(\mathbf{t})/\varphi(\mathbf{t}) - \mathbf{t}/\mathbf{t})$

$$\varphi(t) = 1{,}047 - 0{,}036t - 0{,}094t^2 + 0{,}024t^3 + \\ + 0{,}0313\,t^4 - 0{,}0048\,t^5 - 0{,}0042\,t^6 + \\ + 0{,}0005\,t^7 + 0{,}0003t^8 + \ldots$$

$$\varphi'(t) = 1{,}0006\cos(0{,}0373t) + 0{,}0464\cos(2.0064t) - \\ - 0.018\sin(2t).$$



Dynamical characteristics

Average capacity of effective resistance powers

$$N_n = \frac{1}{T}\int_t^{t+T} M_\varphi \dot{\varphi}\, dt$$

Average capacity of dissipative powers

$$N_d = \frac{1}{T}\int_t^{t+T}\left(H_x \dot{x}^2 + H_y \dot{y}^2 + H_\varphi \dot{\varphi}^2\right)dt$$

Average capacity of driving powers

$$N_v = \frac{1}{T}\int_t^{t+T}\left(F_x \dot{x} + F_y \dot{y}\right)dt$$

Coefficient of effective action

$$\eta = \frac{N_n}{N_v} = \frac{N_n}{N_n + N_d}$$

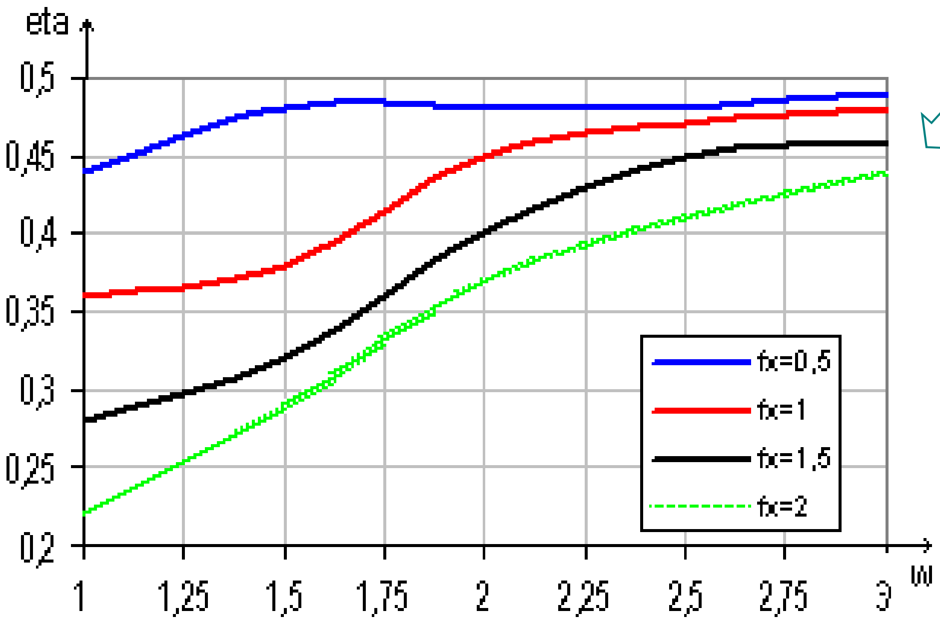
Average speed

$$\overline{\dot{\varphi}} = \frac{1}{T}\int_t^{t+T} \dot{\varphi}\, dt$$

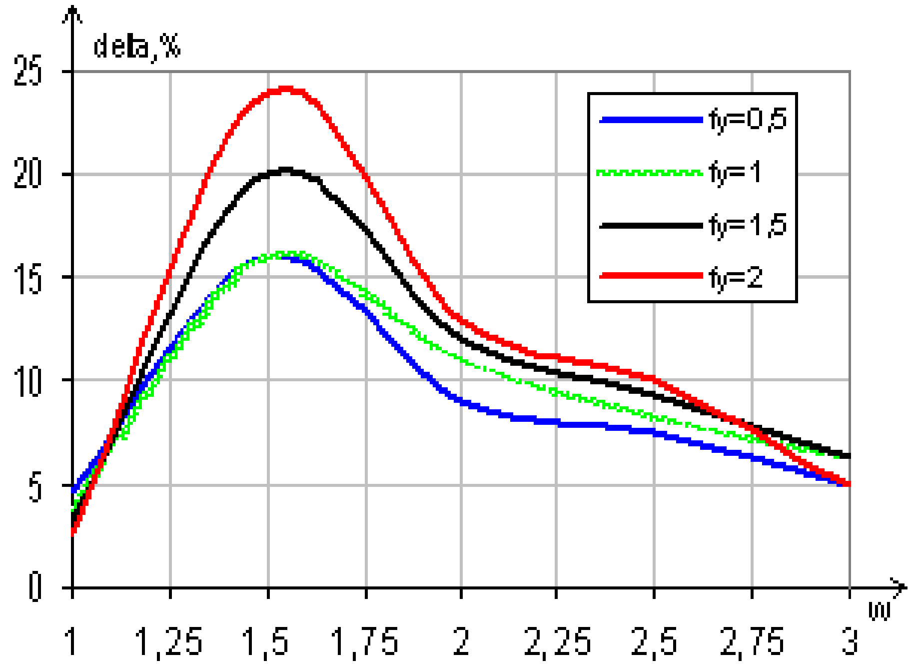
Coefficient of non-uniformity

$$\delta\dot{\varphi} = \frac{\dot{\varphi}_{\max} - \dot{\varphi}_{\min}}{2\overline{\dot{\varphi}}}$$

Coefficients of effective action and non-uniformity



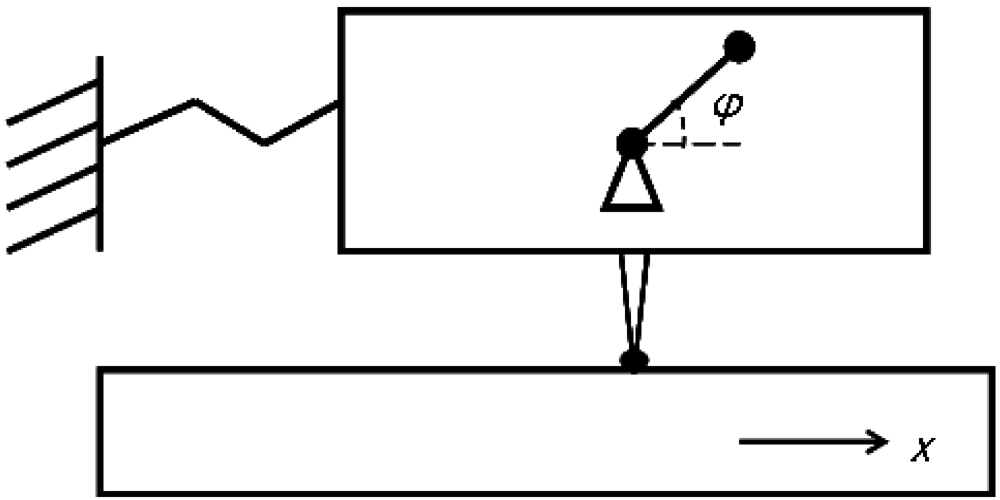
Coefficient of effective action ↑ when frequency of system and driving powers ↑



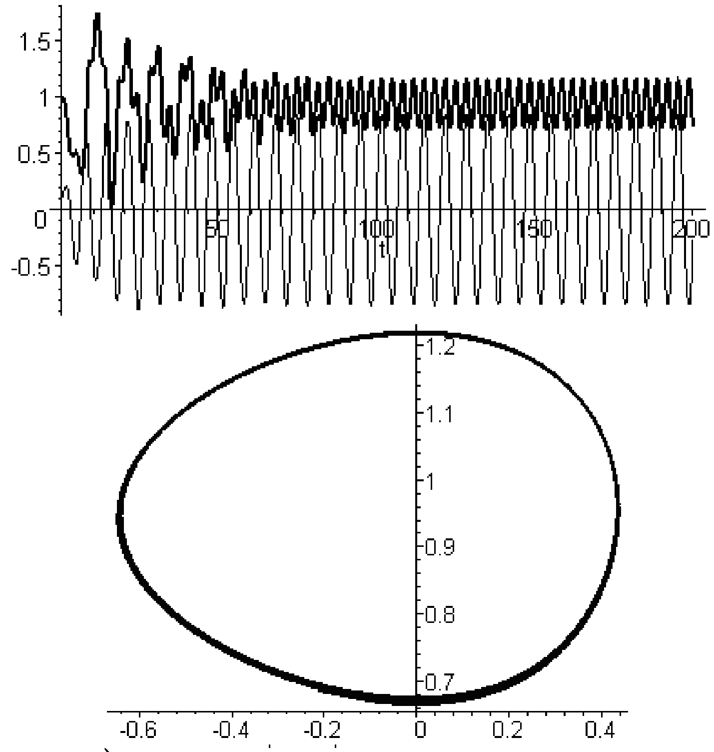
Coefficient of non-uniformity ↑ when driving powers ↑

Autonomous case

Model



Example of steady mode



$$\begin{cases} \varphi'' - x''\sin\varphi + H\varphi' = 0, \\ x'' + p^2 x - q_1 x' + q_3 x'^3 - \mu\left(\varphi''\sin\varphi + \varphi'^2\cos\varphi\right) = 0. \end{cases}$$

Solutions of a system

$$\begin{cases} x'' = \dfrac{\mu\varphi'^2\cos\varphi - p^2 x + q_1 x' - q_3 x'^3 - \mu H\varphi'\sin\varphi}{1-\mu\sin^2\varphi} = P \\ \varphi'' = \dfrac{\left(\mu\varphi'^2\cos\varphi - p^2 x + q_1 x' - q_3 x'^3\right)\sin\varphi - H\varphi'}{1-\mu\sin^2\varphi} = Q \end{cases}$$

$$x(v) = s_1,\ \ x'_t(v) = s_2$$

$$\varphi(v) = s_3,\ \varphi'_t(v) = s_4$$

$$x = \sum_{k=0}^{+\infty} p_k \frac{(t-v)^k}{k!},\ \ p_k = \boldsymbol{D}^k s_1$$

$$\varphi = \sum_{k=0}^{+\infty} q_k \frac{(t-v)^k}{k!},\ \ q_k = \boldsymbol{D}^k s_3$$

$$P = P(v, s_1, s_2, s_3, s_4)$$

$$Q = Q(v, s_1, s_2, s_3, s_4)$$

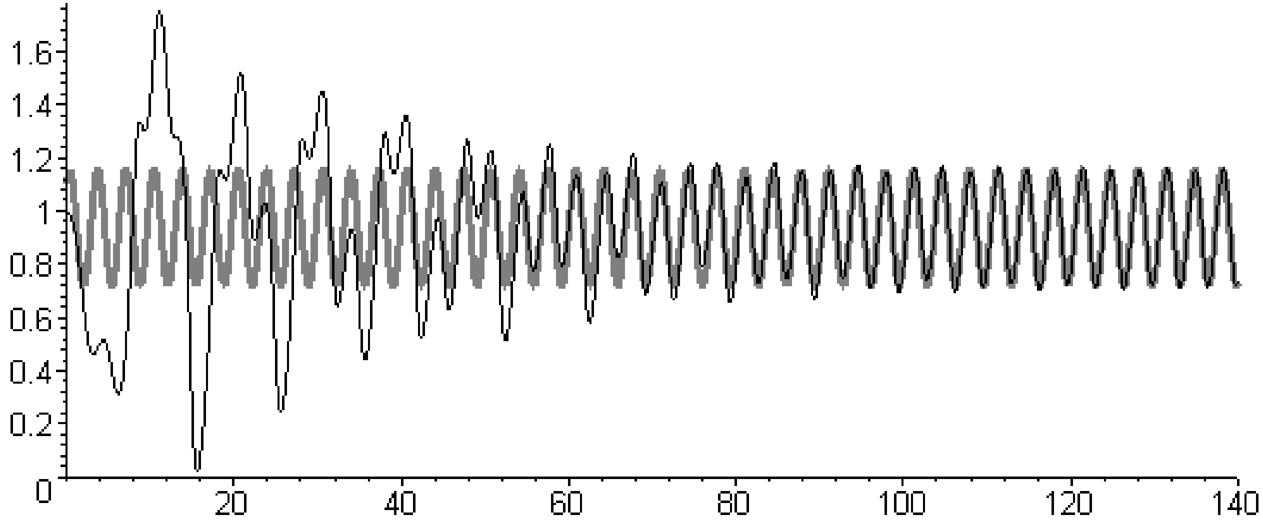
$$\boldsymbol{D} = \boldsymbol{D}_v + s_2\boldsymbol{D}_{s_1} + P\boldsymbol{D}_{s_1} + s_4\boldsymbol{D}_{s_3} + Q\boldsymbol{D}_{s_3}$$

Expression by trigonometric functions

$$1.110+0.262t-0.303t^{2}-0.153t^{3}+0.088t^{4}+0.027t^{5}-0.103t^{6}-0.002t^{7}+0.001t^{8}+\ldots$$

$$\begin{vmatrix} 1 & \rho & \rho^{2} \\ 1.1104 & -0.6064 & 2.1204 \\ -0.6064 & 2.1204 & -7.4147 \end{vmatrix}=0$$

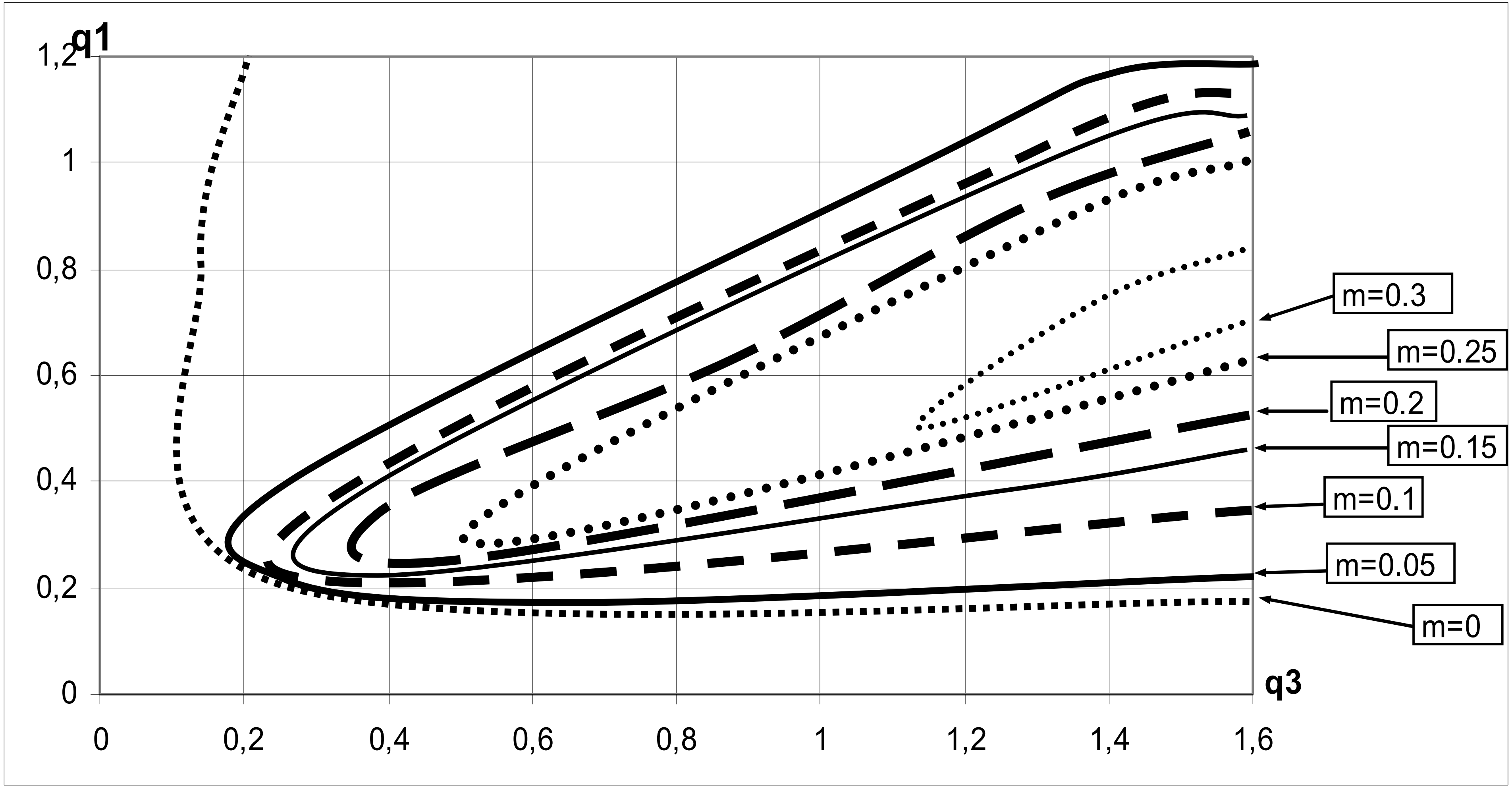
$$\begin{vmatrix} 1 & \hat{\rho} \\ 0.2622 & -0.9169 \end{vmatrix}=0$$



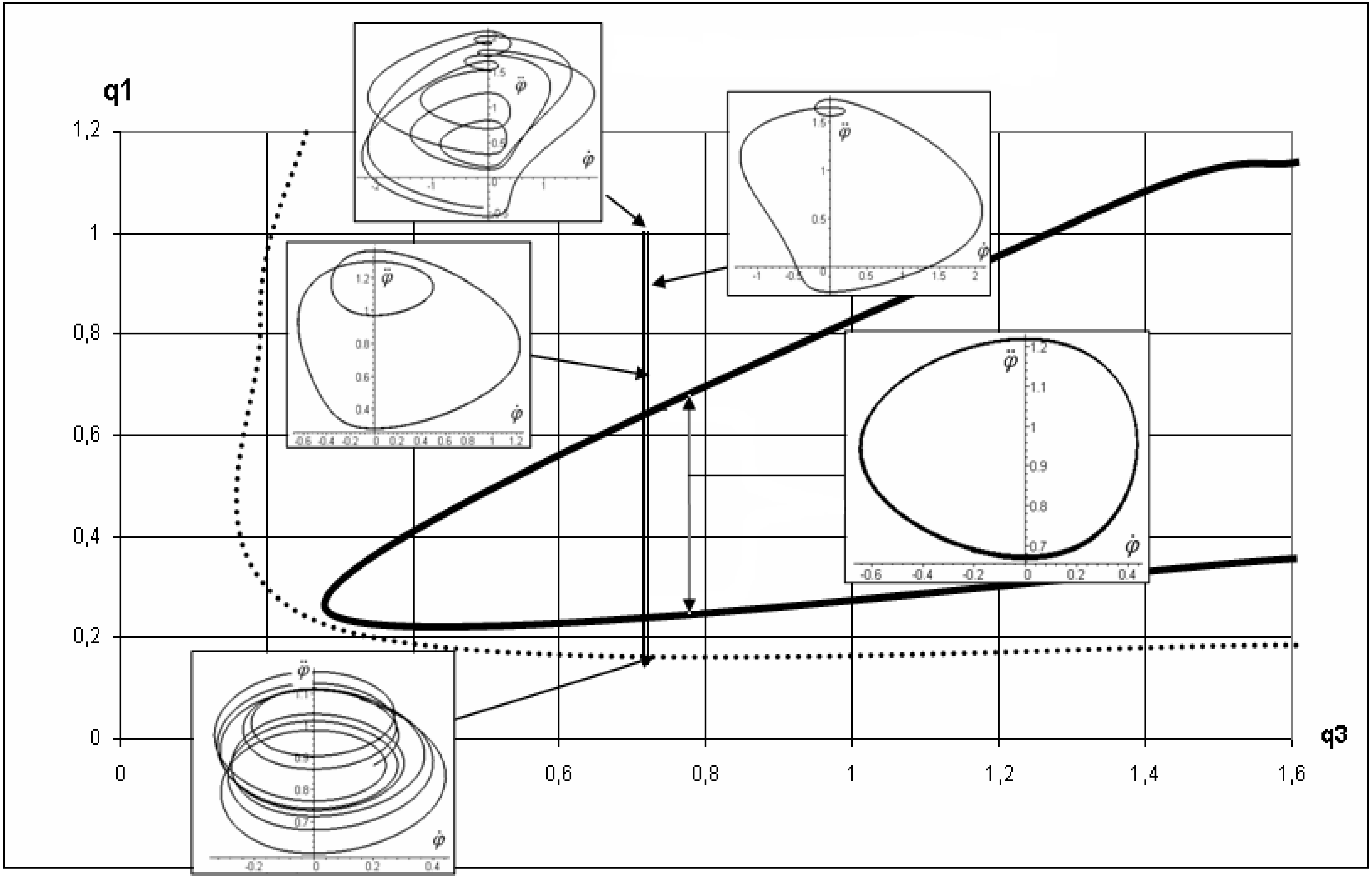
$$\lambda_0=0.937, \lambda_1=0{,}1734,$$
$$\mu_0=0{,}001, \mu_1=1.869,$$
$$\overline{\lambda}_0=0.1402, \overline{\mu}_0=1.871$$

$$\widetilde{\varphi}'(t)=0.937\cos(0.001t)+0.1734\cos(1.869t)+0.1402\sin(1.871t)$$

Steady regimes existence domains



Types of solutions



Parallel algorithm of investigation

$\Delta x = \max|\Delta x(t)|, \quad \Delta y = \max|\Delta y(t)|, \quad \Delta\varphi = \max|\Delta\varphi(t)|$ - loss-functions

after substitution of $\tilde{x}(t), \tilde{y}(t)$ and $\tilde{\varphi}(t)$ in specifying systems of the models

Non autonomous system						Autonomous system				
	Δx	Δy	Δφ	Seq., s	Par., s		Δx	Δφ	Seq., s	Par., s
N=6	$3\cdot10^{-5}$	$3\cdot10^{-5}$	$6\cdot10^{-8}$	250-400	80-130	**N=8**	$3\cdot10^{-5}$	$4\cdot10^{-5}$	50-68	20-24
N=7	$2\cdot10^{-7}$	$2\cdot10^{-7}$	$6\cdot10^{-8}$	1000-1270	300-370	**N=10**	$2\cdot10^{-7}$	$3\cdot10^{-7}$	128-150	55-60
N=8	$5\cdot10^{-8}$	$5\cdot10^{-8}$	$6\cdot10^{-8}$	1580 -1800	560-610	**N=12**	$5\cdot10^{-8}$	$6\cdot10^{-8}$	246-320	113-158

Papers

1. *Ragulskis, Kazimieras; Navickas, Zenonas; Bikulčienė, Liepa. Transformation of vibrating process to the rotational motion // Vibroengineering 2004 : proceedings of 5th International Conference, October 14-15, 2004, Kaunas, Lithuania. Kaunas: Technologija. ISSN 1392-8716. 2004, p. 44-48.*
2. *Ragulskis, Minvydas Kazys; Navickas, Zenonas; Bikulčienė, Liepa. Transformation of self vibration to rotational motion // Journal of Vibroengineering = Vibromechanika. Vilnius: Vibromechanika. ISSN 1392-8716. 2005, Vol. 7, no. 2, p. 15-19.*
3. *Ragulskis, Kazimieras; Navickas, Zenonas; Bikulčienė, Liepa. Energy transformation using vibrating system with Duffing terms // Vibroengineering 2006 : proceedings of the 6th International Conference, October 12-14, 2006, Kaunas, Lithuania / Lithuanian Academy of Sciences, IFToMM National Committee, Kaunas University of Technology. Kaunas: Technologija. ISSN 1822-1262. 2006, p. 39-42.*
4. *Ragulskis, Kazimieras; Navickas, Zenonas; Bikulčienė, Liepa. Transformation of self vibration to rotational motion in system with nonlinear stiffness // Vibroengineering 2008 : proceedings of the 7th International Conference, October 9-11, 2008, Kaunas University of Technology, Lithuania / Lithuanian Academy of Sciences, Kaunas University of Technology, IFToMM National Committee. Kaunas: Technologija. ISSN 1822-1262. 2008, p. 120-123.*

Expressions of solutions of partial differential equations using algebraic operators and algebraic convolution

Operator expression for the solution of a partial differential equation

Differential equation with boundary conditions:

$$\frac{\partial^k \omega}{\partial x^k} - \left(P \frac{\partial^m \omega}{\partial s^m} + Q \frac{\partial^n \omega}{\partial t^n} \right) = f_k(x; s, t)$$

$$\left. \frac{\partial^r \omega}{\partial x^r} \right|_{x=0} = f_r(s, t)$$

$$\omega = G\left(f_0(s,t) + f_1(s,t)\frac{x}{1!} + \cdots + f_{k-1}(s,t)\frac{x^{k-1}}{(k-1)!} + L_x^k f_k(x; s, t) \right)$$

Example

$$P := 1,\ Q := s,\ f_0 := (s+t)^3,\ f_1 := s^2 tx$$

$$\boxed{\frac{\partial \omega}{\partial x} - \frac{\partial^2 \omega}{\partial s^2} - s\frac{\partial^3 \omega}{\partial t^3} = s^2 tx,\ \omega(0;s,t) = (s+t)^3}$$

$$\omega = \sum_{j=0}^{+\infty} \left(L_x\left(D_s^2 + sD_t^3\right)\right)^j \left((s+t)^3 + L_x\left(s^2 tx\right)\right) = (s+t)^3 + s^2 t\frac{x^2}{2} +$$

$$+ L_x\left(D_s^2 + sD_t^3\right)\left((s+t)^3 + s^2 t\frac{x^2}{2}\right) = (s+t)^3 + 6(2s+t)x + s^2 t\frac{x^2}{2} + t\frac{x^3}{3} =$$

$$= (s+t)^3 + 6(2s+t)x + \frac{1}{2}s^2 tx^2 + \frac{1}{3}tx^3$$

Formulas

Differential equation

$$\frac{\partial^k w}{\partial x^k} - a^m \frac{\partial^m w}{\partial s^m} - b^n \frac{\partial^n w}{\partial t^n} = \sum_{\alpha,\beta=0}^{+\infty} \gamma_{k\alpha\beta} \frac{s^\alpha}{\alpha!} \frac{t^\beta}{\beta!}$$

boundary conditions

$$\left.\frac{\partial^r w}{\partial x^r}\right|_{x=0} = \sum_{\alpha,\beta=0}^{k} \gamma_{r\alpha\beta} \frac{s^\alpha}{\alpha!} \frac{t^\beta}{\beta!}, r = 0,1,\ldots,k-1$$

Solution

$$w = \sum_{r=0}^{k} L_x^r \sum_{\alpha,\beta=0}^{+\infty} \gamma_{r\alpha\beta} \left(\frac{\bar{s}^\alpha}{\alpha!}\right) * \left(\frac{\bar{t}^\beta}{\beta!}\right) = \sum_{r=0}^{k} \sum_{\alpha,\beta=0}^{+\infty} \gamma_{r\alpha\beta} \sum_{j,l=0}^{+\infty} \binom{j}{l} \frac{a^{ml} s^{\alpha-ml}}{(\alpha-ml)!} \cdot \frac{b^{n(j-l)} t^{\beta-n(l-l)}}{(\beta-n(j-l))!} \cdot \frac{x^{kj+r}}{(kj+r)!}$$

Example

Differential equation

$$\frac{\partial\omega}{\partial x}-4\frac{\partial\omega}{\partial s}+9\frac{\partial^2\omega}{\partial t^2}=1+\frac{st^2}{2},\omega(0;s,t)=2s-t$$

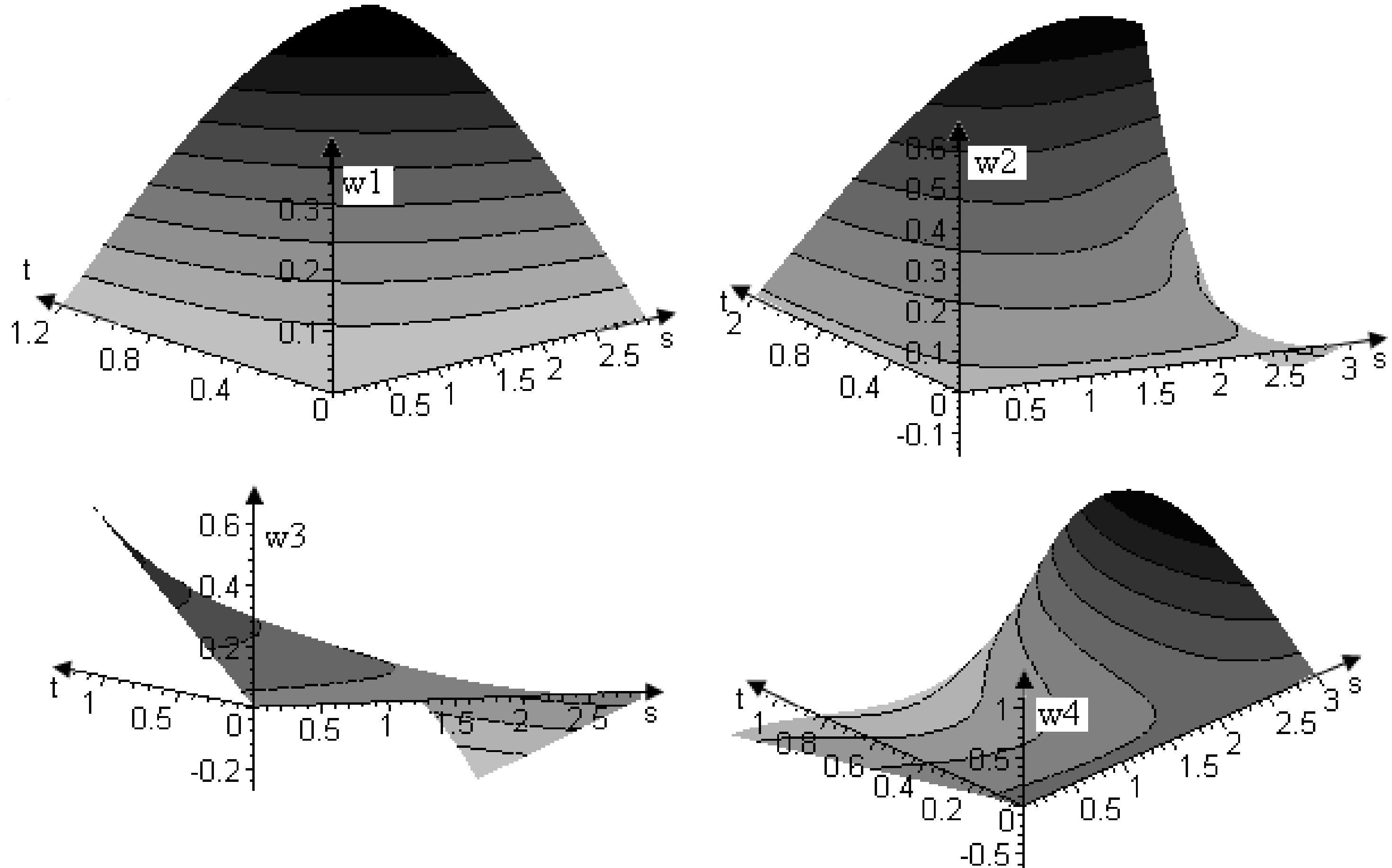
$$\boxed{\omega=(2\hat{s}-\hat{t})+L_x\left(\hat{1}+2\hat{s}*\frac{\widehat{t^2}}{2!}\right)}$$

$$\hat{1}=1,\ \hat{s}=s-4x,\hat{t}=t,\ \frac{\widehat{t^2}}{2!}=\frac{t^2}{2}-9x,$$

$$\hat{s}*\frac{\widehat{t^2}}{2!}=\frac{st^2}{2}+(2t^2-9s)x-36x^2,$$

$$\boxed{\omega=(2s-t)+\left(9+\frac{st^2}{2}\right)x+(2t^2-9s)\frac{x^2}{2}-12x^3.}$$

Approximations of solutions



Bikulčienė, Liepa; Navickas, Zenonas.. Expressions of solutions of linear partial differential equations using algebraic operators and algebraic convolution // Numerical Analysis and Its Applications : 4th International Conference, NAA 2008, June 16-20, 2008, Lozenetz, Bulgaria : revised selected papers /S. Margenov, L. G. Vulkov, J. Wasniewski (Eds.). Berlin, Heidelberg: Springer-Verlag, 2009. (Lecture Notes in Computer Science, Vol. 5434, ISSN 0302-9743). p. 208-215.

Critic to Exp-function method and criterium for exact solutions

Introduction to the problem

The Exp-function method was proposed in 2006 by J.H. He [1]. The Exp-function method was applied to finding exact solutions of many differential equations. With the help of symbolic computation, the said method provides a powerful mathematical tool for solving of high-dimensional nonlinear evolutions in mathematical physics.

Does there exist any analytical criterion determining whether an exact solution of a differential equation (ordinary or partial) can be found by the Exp-function method?

The object was to find and construct such criterion.

[1] Ji-Huan He and Xu-Hong Wu. Exp-function method for nonlinear wave equations. Chaos, Solitons & Fractals. Volume 30, Issue 3, November 2006, Pages 700-708.

Applied Mathematics and Computation 211 (2009) 522–530

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How far one can go with the Exp-function method?

Zenonas Navickas [a], Minvydas Ragulskis [b,*]

An analytical criterion determining if a solution of a differential equation can be expressed in an analytical form comprising exponential functions is developed. The employment of this criterion does not only give an answer to the above-stated question but gives the structure of the solution so that one does not have to guess what the form of the solution is. The load of symbolic calculations is brought before the structure of the solution is identified. This is in contrary to the Exp-function type methods where the structure of the solution is first guessed, and then symbolic calculations are exploited for the identification of parameters.

Case when an H-rank exists

If all roots of Hankel's characteristic equation distinct, i.e.
$\rho_k(s,t) \neq \rho_l(s,t)$, for $k \neq l$,then

$$y(x;s,t) = \sum_{j=0}^{+\infty} \frac{(x-a)^j}{j!} \left(\mu_1(s,t)\rho_1^j(s,t) + \cdots + \mu_m(s,t)\rho_m^j(s,t)\right) =$$

$$= \sum_{r=1}^{m} \mu_r(s,t) \exp((x-a)\rho_r(s,t))$$

If some roots of the characteristic equation are multiple, then

$$y(x;s,t) = \sum_{r=1}^{l} Q_r(x;s,t) \exp((x-a)\rho_r(s,t)),$$

where $Q_r(x;s,t)$, $r = 1,2,\ldots,l$, ($l < m$) are polynomials.

Example 1 – H-rank exists for the original equation

$$y'' + ay' + by = 0\ ;\ a,b \in R\ ;\ y(0;s,t) = s\ ;\ y'_x(x;s,t)\big|_{x=0} = t\ .$$

Then $p_j(s,t) = \left(tD_s - (bs+at)D_t\right)^j s\ ,\ j = 0,1,2,\ldots\ .$

Then the characteristic equation takes the form:

$$\begin{vmatrix} s & t & -(bs+at) \\ t & -(bs+at) & abs + a^2t - bt \\ 1 & \rho & \rho^2 \end{vmatrix} = 0\ .$$

Thus, $\rho_{1,2} = \frac{1}{2}\left(-a \pm \sqrt{a^2 - 4b}\right)$ and

$$y(x;s,t) = \frac{t - \rho_2 s}{\sqrt{a^2 - 4b}} \exp(\rho_1 x) - \frac{t - \rho_1 s}{\sqrt{a^2 - 4b}} \exp(\rho_2 x).$$

If H-rank does not exist

In this case, it is useful to follow the He's method:

$\exp(x) := z\ ;\ \ x = \ln(z);\ \ x \in R\ ;\ \ z > 0\ .$

If all roots $\rho_r(s,t,\exp(a))$, $r \in \{1,2,\ldots,m\}$, are distinct, then

$$\omega(z;s,t) = \sum_{r=1}^{m} \frac{\mu_r(s,t,\exp(a))}{1-\rho_r(s,t,\exp(a))(x-\exp(a))}.$$

Thus, $\omega(z;s,t) = \dfrac{\sum_{l=1}^{n_1} a_l z^l}{\sum_{l=1}^{n_2} b_l z^l}$ and $y(x;s,t) = \dfrac{\sum_{l=1}^{n_1} a_l \exp(lx)}{\sum_{l=1}^{n_2} b_l \exp(lx)}$

Example 2 – H-rank does not exist for the original equation but exists for the image equation

The coefficients of differential equation $y' = 1 - y^2$; $y(0;s) = s$

$\left(\left(\left(1 - s^2\right) D_s \right)^j s;\, j \in Z_0 \right)$ do not have an H-rank.

The image equation reads:

$$\omega_z' = \frac{1}{z}\left(1 - \omega^2\right);\ \omega(1;s) = s\ .$$

The coefficients of the image equation

$$\hat{p}_j(s,a) = \left(\frac{1}{a}\left(1 - s^2\right) D_s \right)^j s\ ;\ j \in Z_0 \text{ have an } H\text{-rank};$$

$$Hr\left(\frac{1}{j!}\hat{p}_j(s,a);\, j \in Z_0 \right) = 3\ .$$

Example 2 – H-rank does not exist for the original equation but exists for the image equation

Using roots of the characteristic algebraic equation

$$\hat{p}_0(s,a)=1\,;\ \frac{\hat{p}_j(s,a)}{j!}=\mu_2\rho_2^j+\mu_3\rho_3^j\,;$$

as $\rho_1^0=0^0=1\,;\ \rho_1^j=0\,,\ j=1,2,\ldots.:$

$$\sum_{j=0}^{+\infty}\frac{\hat{p}(s,a)}{j!}(z-a)^j=\frac{(1+s)(z-a)^2-a^2(1-s)}{(1+s)(z-a)^2+a^2(1-s)}\,.$$

The solution of the image differential equation is

$$\omega(z;s)=\frac{(1+s)z^2-(1-s)}{(1+s)z^2+(1+s)}\,.$$

Then, the solution of the original differential equation reads:

$$y(x;s)=\frac{(1+s)\exp(2x)-(1-s)}{(1+s)\exp(2x)+(1-s)}=\frac{(1+s)\exp x-(1-s)\exp(-x)}{(1+s)\exp x+(1-s)\exp(-x)}$$

<table>
<tr><td colspan="3">Given a problem, use the operator method to construct the solution in the form of a series $y(x;s,t)=\sum_{j=0}^{+\infty} p_j(s,t)\frac{(x-a)^j}{j!}$.</td></tr>
<tr><td colspan="3">Does the H-rank of the series of coefficients $\left(p_j(s,t);\, j\in Z_0\right)$ exist?</td></tr>
<tr><td>YES</td><td colspan="2">$Hr\left(p_j(s,t);\, j\in Z_0\right)=m$. Find roots $\rho_k(s,t)$, $k=1,2,\ldots,m$
Solution is expressed as a sum of exponential functions.</td></tr>
<tr><td rowspan="5">NO</td><td colspan="2">H-rank does not exist. Change the variable $\exp(x)=z$.</td></tr>
<tr><td colspan="2">Construct the solution in the form of a series
$\omega(z;s,t,a)=\sum_{j=0}^{+\infty}\frac{(z-\exp(a))^j}{j!}\hat{p}_j(s,t,\exp(a))$.</td></tr>
<tr><td colspan="2">Does the H-rank of the series of coefficients exist?</td></tr>
<tr><td>YES</td><td>Find roots $\rho_r(s,t,\exp(a))$; $r=1,2,\ldots,m$.
Exact solution is expressed as a ratio of sums of exponential functions.</td></tr>
<tr><td>NO</td><td>Exact solution cannot be expressed in the form comprising only exponential functions.</td></tr>
</table>

Commun Nonlinear Sci Numer Simulat 15 (2010) 3874–3886

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

Be careful with the Exp-function method – Additional remarks

Zenonas Navickas [a,1], Minvydas Ragulskis [b,*], Liepa Bikulciene [a,1]

Applied Mathematics and Computation 216 (2010) 2380–2393

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Generalization of Exp-function and other standard function methods

Zenonas Navickas [a], Liepa Bikulciene [a], Minvydas Ragulskis [b,*]

Special solutions of Huxley and Liouville’s differential equations

The Huxley's equation

Hodkin and Huxley presented the results of electrophysiological experiments in which they investigated the flow of electric current thougt the surface membrane of the giant nerve fiber of a squid. Huxley equation is a nonlinear partial differential equation of second order of the form

$$\boxed{u_t = u_{xx} + u(k-u)(u-1)}$$

This equation is an evolution equation that describes the nerve propagation in biology. From this equation molecular properties can be calculated. It also gives a description of the behavior of the miosin heads. This equation has many fascinating phenomena such as bursting oscilation, interpsike, bifurcation and chaos.

The Huxley equation

Huxley equation is a core mathematical framework for modern biophysically based neural modeling. It is often useful to obtain a generalized solitary solution for fully understanding its physical meanings. There are many methods to solve this equation: the traditional approaches to this task are variational iteration method, the homotopy perturbation method, Adomian's decomposition method and the tanh method; however, many methods may sometimes fail or the solution procedure becomes complicated as degree of nonlinearity increases. The Exp-function method, proposed by He and Wu [1] seemed to be most promising for that purpose. Zhou [2] obtained solutions of Huxley equation using this method.

[2] X.W.Zhou, Exp-function method for solving Huxley equation, Mathematical Problems in Engineering, Volume 2008, Article ID:538489.

The Huxley equation

The **Soliton model** in neuroscience is a recently developed model that attempts to explain how signals are conducted within neurons. It proposes that the signals travel along the cell's membrane in the form of certain kinds of sound (or density) pulses known as **solitons**. As such the model presents a direct challenge to the widely accepted Hodgkin-Huxley model [3] which proposes that signals travel as action potentials: voltage-gated ion channels in the membrane open and allow ions to rush into the cell, thereby leading to the opening of other nearby ion channels and thus propagating the signal in an essentially electrical manner.

[3] Hodgkin, A., and Huxley, A. (1952): A quantitative description of membrane current and its application to conduction and excitation in nerve. J. Physiol. 117:500–544.

They received the 1963 Nobel Prize in Physiology or Medicine for this work

The Huxley equation

Using the wave variable $\eta = wx + vt$ was obtained

$$-vu' + w^2u'' + u(k-u)(u-1) = 0$$ or

$$\boxed{y'' + by' = 2a^2\left(y^3 + \alpha y^2 + \beta y + \gamma\right)}$$

$$a, b, \alpha, \beta, \gamma \in R$$

The solitary solution of the Huxley equation, produced by the Exp-function method, does not satisfy the original differential equation for all initial conditions. The alternative operator-based method to derive the solitary solution of the Huxleys equation and have identified the region (in the parameter plane) of the initial conditions, where this solution does exist.

Solving of the Huxley equation

Solutions of Huxley equation

$$\boxed{y'' + by' = 2a^2\left(y^3 + \alpha y^2 + \beta y + \gamma\right)} \qquad (1)$$

$$a, b, \alpha, \beta, \gamma \in R$$

$$\boxed{y(x) = \frac{A_1 e^{\lambda_1 (x-v)} + A_2 e^{\lambda_2 (x-v)}}{B_1 e^{\lambda_1 (x-v)} + B_2 e^{\lambda_2 (x-v)}}} \qquad (2)$$

A_1, A_2, B_1, B_2 depends on Cauchy conditions

Solving of the Huxley equation

Theorem

Equation (1) has (2) type solution when necessary and sufficient condition

$$y^3+\alpha y^2+\beta y+\gamma=(y-y_1)(y-y_2)\left(y-\frac{1}{2}\left(y_1+y_2-\frac{b}{a}\right)\right) \qquad (3)$$

is satisfied. Besides,

$$y(x)=\frac{y_2(s-y_1)e^{y_1(x-v)}-y_1(s-y_2)e^{y_2(x-v)}}{(s-y_1)e^{y_1(x-v)}-(s-y_2)e^{y_2(x-v)}} \qquad (4)$$

when $y_1\neq y_2$

(3) holds true if and only if

$$\left((b-2a\alpha)(b+a\alpha)+9\beta a^2\right)(b+a\alpha)=27a^3\gamma$$

Remarks

1. Special solution (4) when s is chose free satisfies such second Cauchy condition $t = y'(x)\big|_{x=v} = a(s-y_1)(s-y_2)$

and this condition can't be taken free.

2. If in (1) equation b=0, then we have Mekery equation

$$y'' = 2a^2\left(y^3 + \alpha y^2 + \beta y + \gamma\right)$$

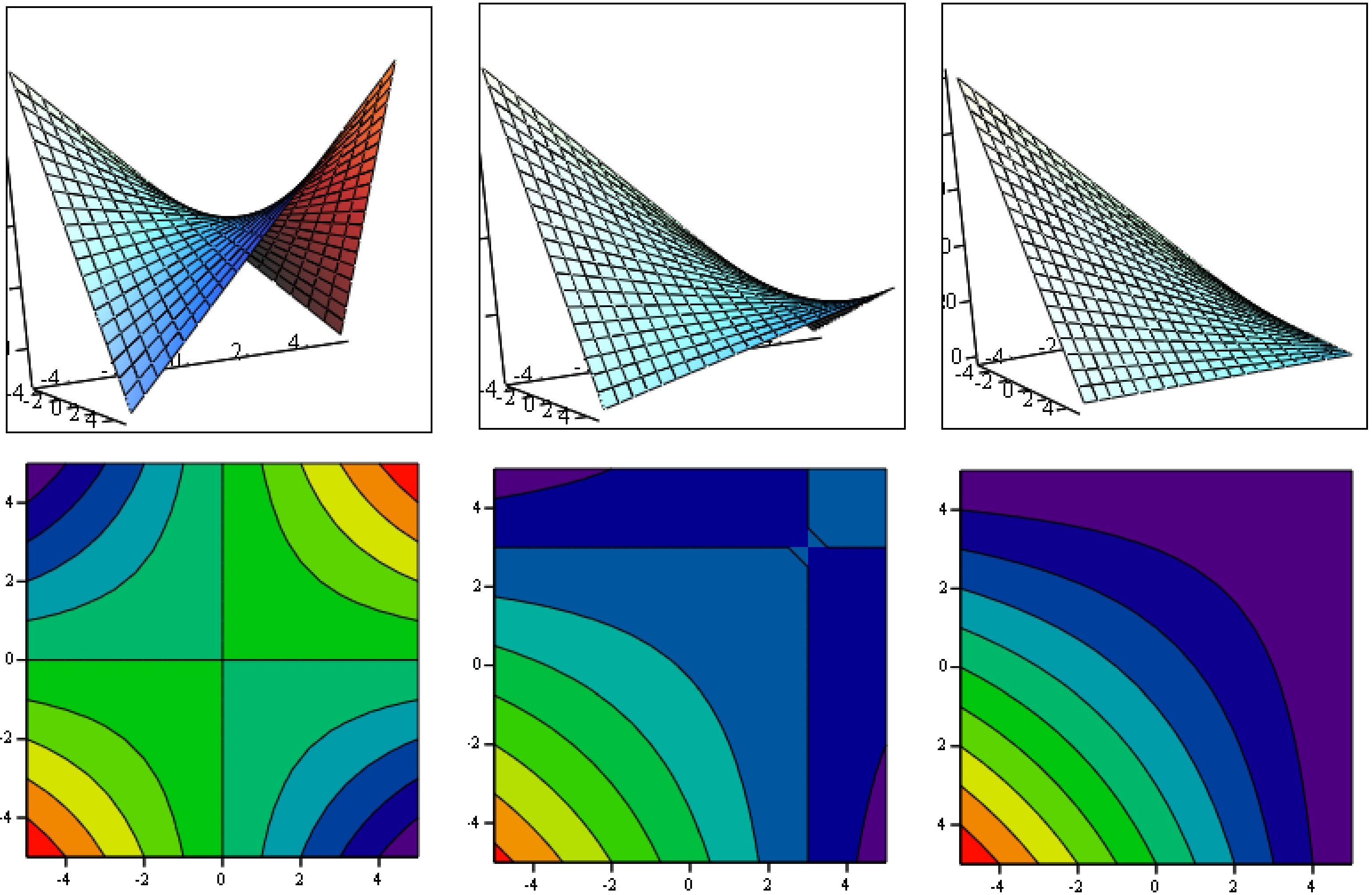
with adequate special solutions and conditions.

3. Special solution (4) is also the solution of Riccati equation

$$y' = a(y-y_1)(y-y_2)$$

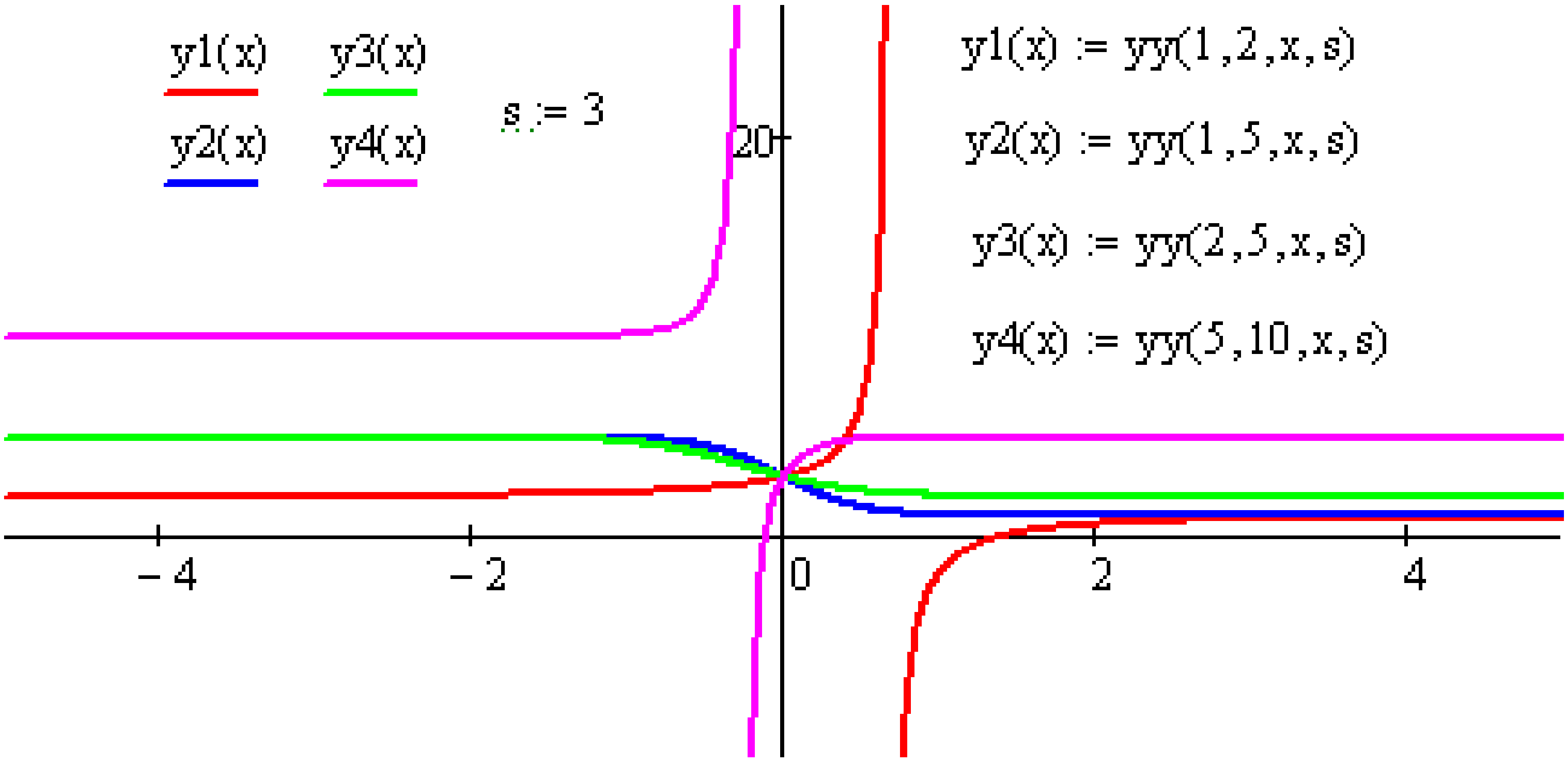
The graphical representation of initial conditions at s=0, s=3 and s=5

$$t = y'(x)\big|_{x=v} = (s - y_1)(s - y_2)$$

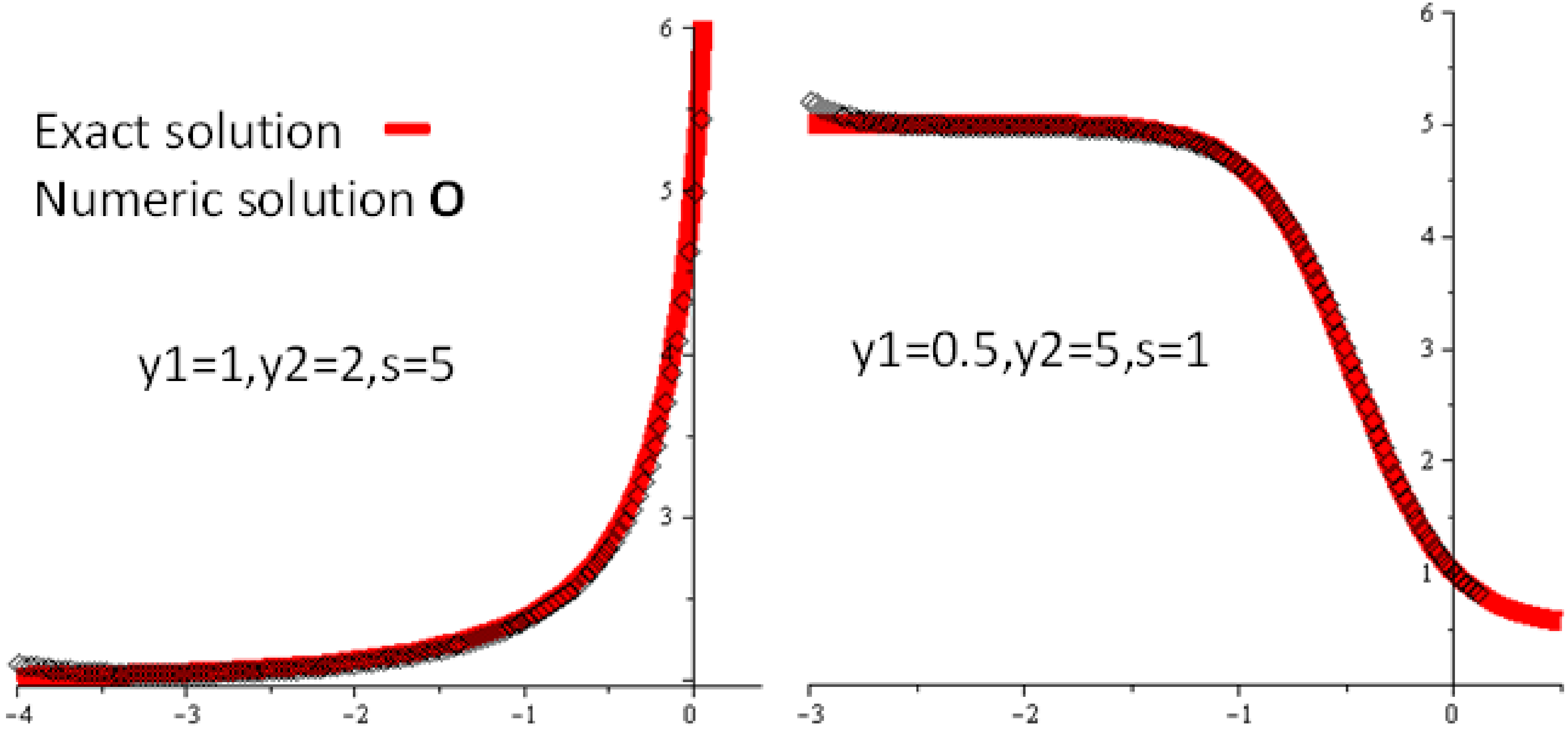


The graphical representation of solutions at s=3, v=0

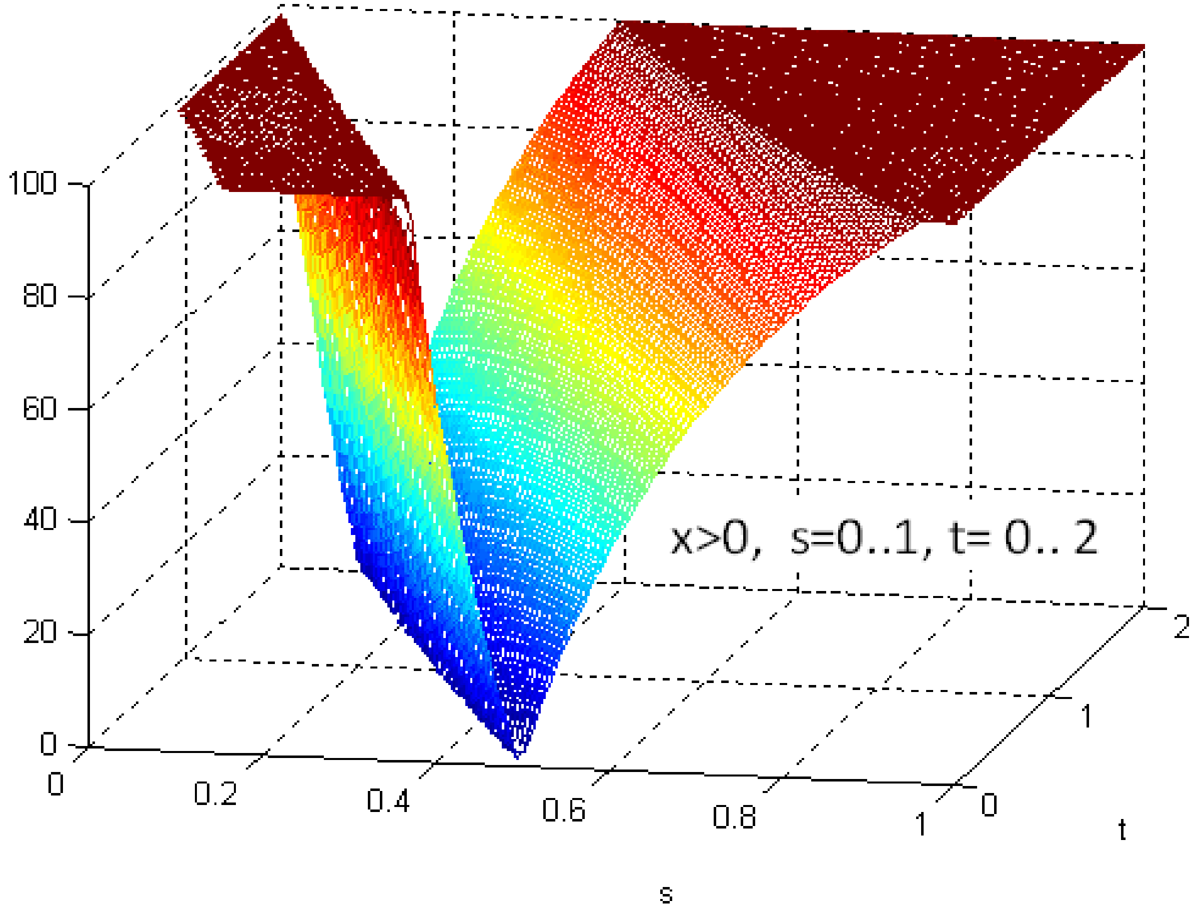
$$y(x) = \frac{y_2(s-y_1)e^{y_1(x-v)} - y_1(s-y_2)e^{y_2(x-v)}}{(s-y_1)e^{y_1(x-v)} - (s-y_2)e^{y_2(x-v)}}$$

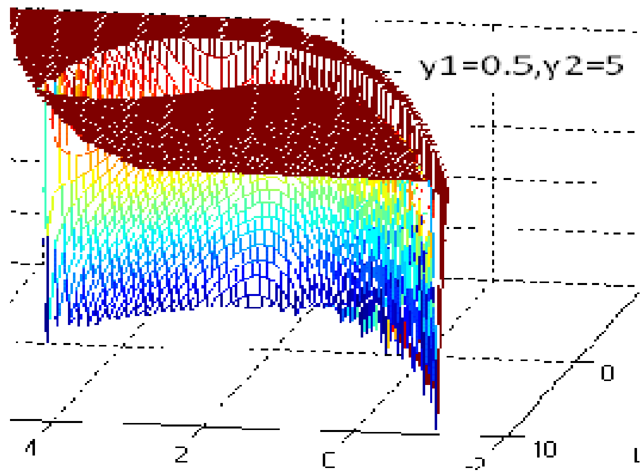


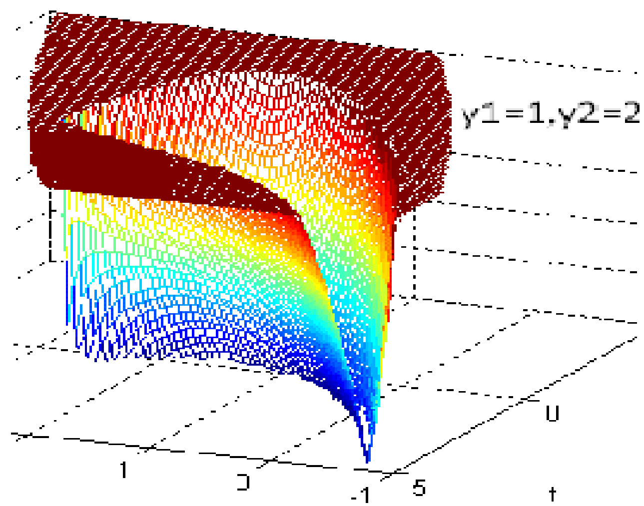
The graphical representation of numerical and exact solutions



Total difference between numerical and exact solutions for different initial conditions







The solution of the Liouville's equation

The Liouville's equation $\frac{\partial^2 u}{\partial\xi\partial\tau} = -\exp(u)$.

Using the transformation $y = \exp(u)$ and $x = k\xi + \Omega\tau + \varphi$,

$k\Omega yy'' - k\Omega(y')^2 + y^3 = 0$. If $k\Omega = \frac{1}{2\gamma}$, then

$yy'' - (y')^2 + 2\gamma y^3 = 0$; $\gamma \neq 0$.

We seek only solutions which can be expressed as a ratio of finite sums of exponential functions. Our motives are determined by the fact that the structure of the solution proposed by X.H. Wu and J.H. He [1], is

$$y = \frac{a_1 \exp(x) + a_0 + a_{-1} \exp(-x)}{\exp(x) + b_0 + b_{-1} \exp(-x)}$$

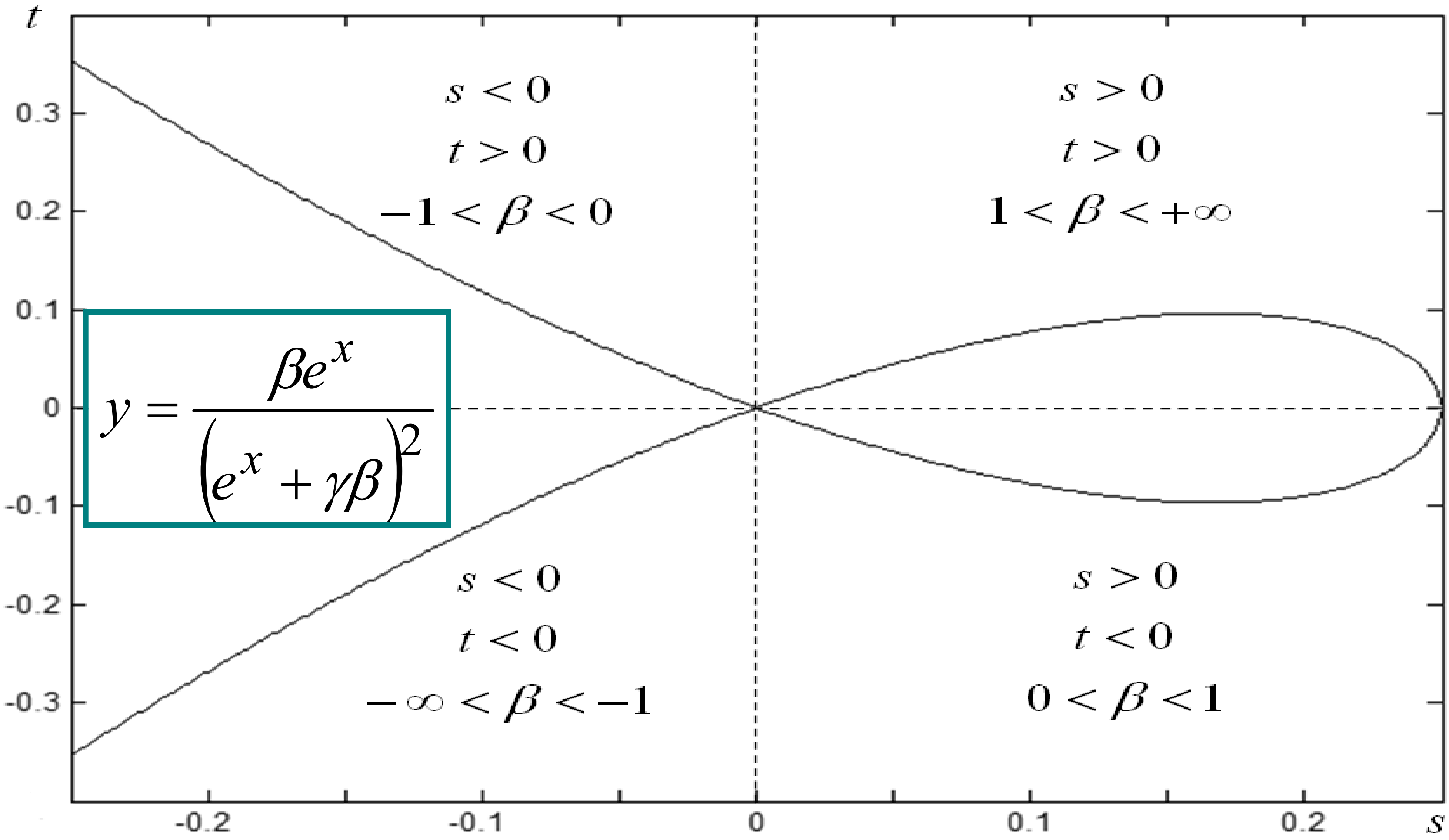
The solution of the Liouville equation

$y(x) = \overline{\omega}(\exp(x)) = \dfrac{\beta \exp(x)}{(\exp(x) + \gamma\beta)^2}$ – partial solutions of the differential

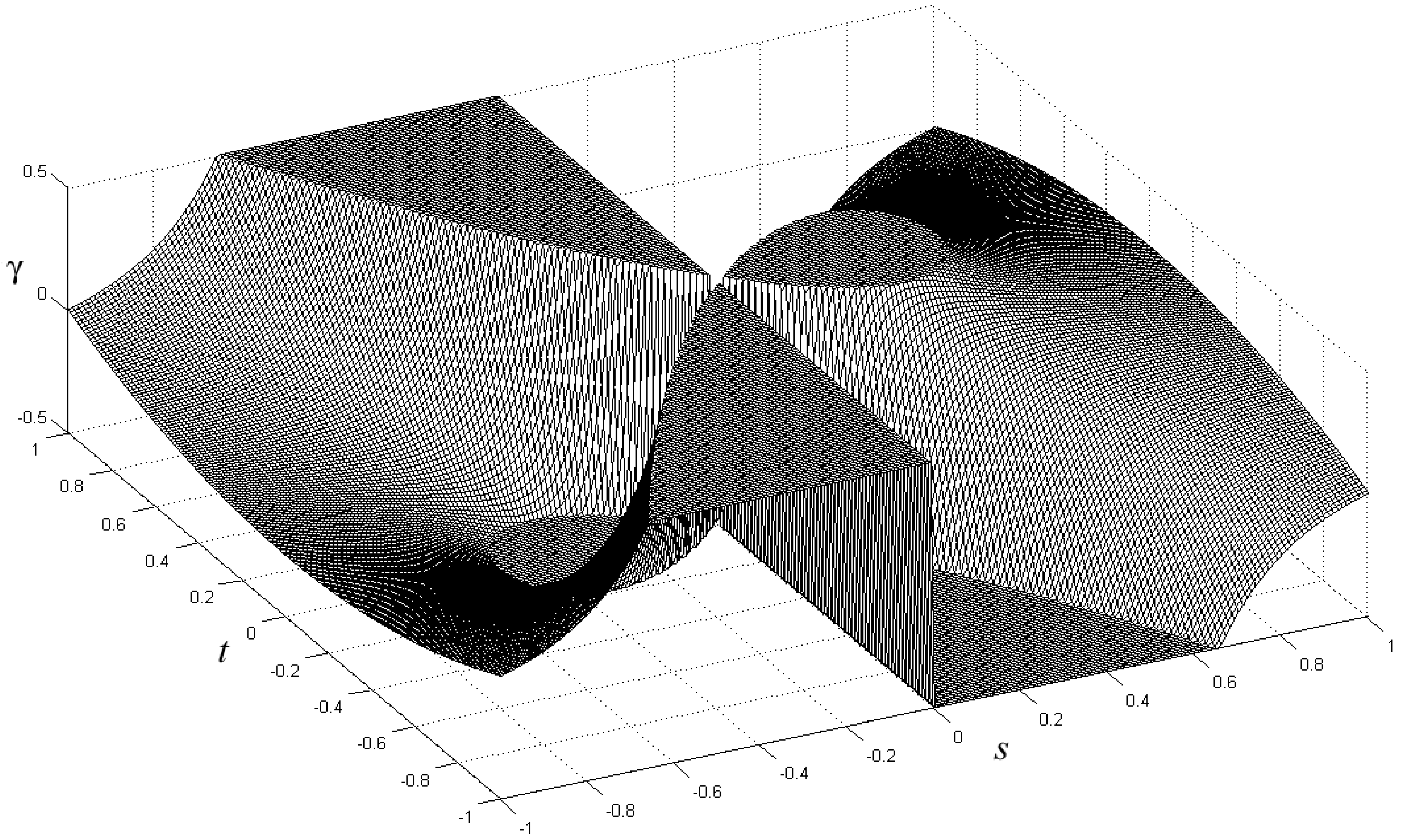
equation when $y(0) = \dfrac{\beta}{(\gamma\beta + 1)^2}$ and $y'_x(0) = \dfrac{\beta(\gamma\beta - 1)}{(\gamma\beta + 1)^3}$ for all β (except $\beta = -1$

The solitary solution of the Liouville's equation, produced by the Exp-function method, does not satisfy the original differential equation for all initial conditions. We have used an alternative operator-based method to derive the solitary solution of the Liouville's equation and have identified the region (in the parameter plane) of the initial conditions, where this solution does exist.

The graphical representation of the constrain linking initial conditions *s* and *t* at γ=1, when $t^2=s^2-4s^3$.



Graphical representation of the surface $-0.5 \leq \gamma(s,t) \leq 0.5$



MATHEMATICAL MODELLING AND ANALYSIS
Volume 16 Number 2, June 2011, 248–259
Doi:10.3846/13926292.2011.579627

www.tandf.co.uk/journals/TMMA
Publisher: Taylor&Francis and VGTU
Online ISSN: 1648-3510
Print ISSN: 1392-6292

Special Solutions of Huxley Differential Equation

Zenonas Navickas, Minvydas Ragulskis
and Liepa Bikulčienė

Computers and Mathematics with Applications 62 (2011) 376–382

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

The solitary solution of the Liouville equation produced by the Exp-function method does not hold for all initial conditions

Minvydas Ragulskis [a,*], Zenonas Navickas [b], Liepa Bikulciene [b]

New results of the research group based on operator method

Commun Nonlinear Sci Numer Simulat 18 (2013) 1374–1389

Contents lists available at SciVerse ScienceDirect

Commun Nonlinear Sci Numer Simulat

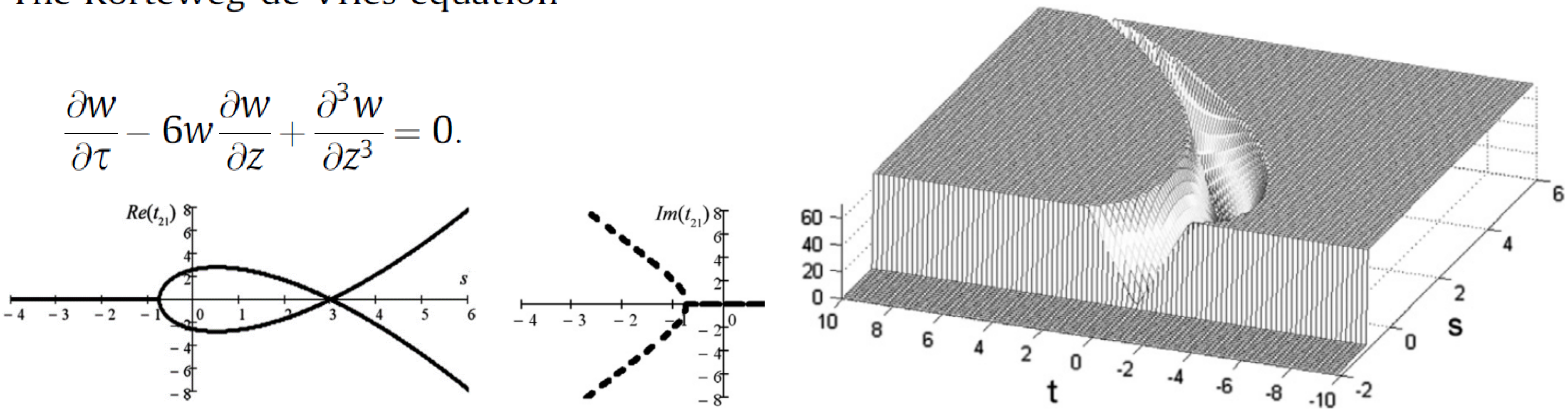
journal homepage: www.elsevier.com/locate/cnsns

Algebraic operator method for the construction of solitary solutions to nonlinear differential equations

Zenonas Navickas [a,1], Liepa Bikulciene [a,2], Maido Rahula [b], Minvydas Ragulskis [c,*]

The Korteweg-de Vries equation

$$\frac{\partial w}{\partial \tau} - 6w\frac{\partial w}{\partial z} + \frac{\partial^3 w}{\partial z^3} = 0.$$



$$y(x;c,s) = y_0 + \frac{4(s-y_0)(y_0-y_3)\exp\left(\sqrt{\frac{2}{3}}\gamma\sqrt{y_0-y_3}(x-c)\right)}{\left(\left(\sqrt{s-y_3}+\sqrt{y_0-y_3}\right)-\left(\sqrt{s-y_3}-\sqrt{y_0-y_3}\right)\exp\left(\sqrt{\frac{2}{3}}\gamma\sqrt{y_0-y_3}(x-c)\right)\right)^2}.$$

Applied Mathematics and Computation 243 (2014) 419–425

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Comments on "A new algorithm for automatic computation of solitary wave solutions to nonlinear partial differential equations based on the Exp-function method"

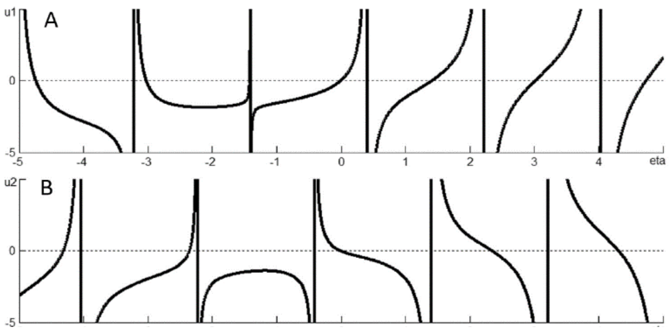
Zenonas Navickas [a], Minvydas Ragulskis [b,*]

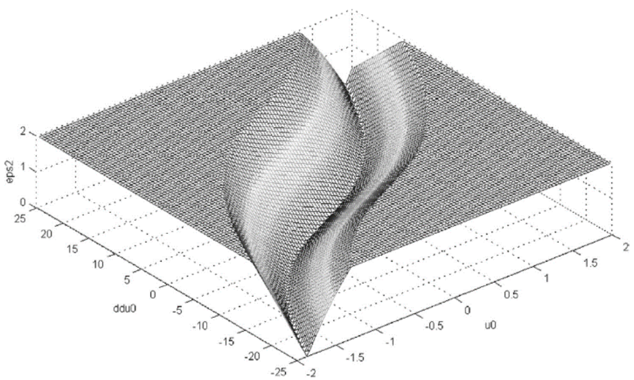
Lei Zhao Equal-Width equation

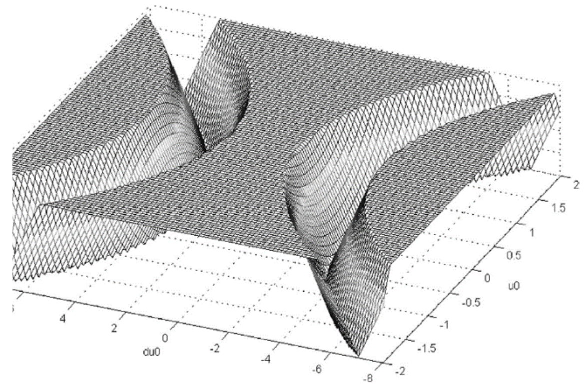
$$\frac{\partial u}{\partial t} + 3u^2\frac{\partial u}{\partial x} - \alpha\frac{\partial^3 u}{\partial^2 x \partial t} = 0.$$

$$u_1(\eta,\eta_0,q,(q^2+3),2q(q^2+3)) = \frac{\sqrt{3}q\cos(\sqrt{3}(\eta-\eta_0)) + 3\sin(\sqrt{3}(\eta-\eta_0))}{\sqrt{3}\cos(\sqrt{3}(\eta-\eta_0)) - q\sin(\sqrt{3}(\eta-\eta_0))};$$

$$u_2(\eta,\eta_0,q,-(q^2+3),2q(q^2+3)) = \frac{\sqrt{3}q\cos(\sqrt{3}(\eta-\eta_0)) - 3\sin(\sqrt{3}(\eta-\eta_0))}{\sqrt{3}\cos(\sqrt{3}(\eta-\eta_0)) + q\sin(\sqrt{3}(\eta-\eta_0))}$$







Computers and Mathematics with Applications 69 (2015) 798–803

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

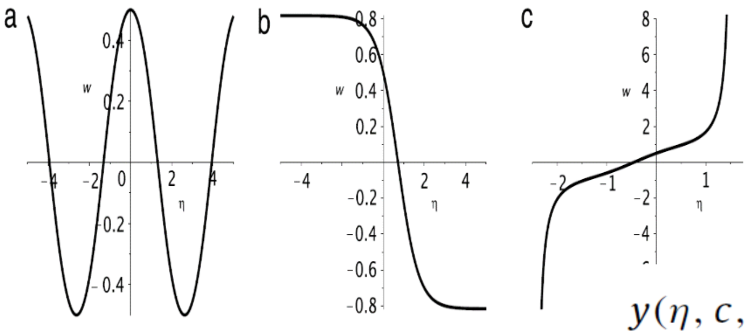
journal homepage: www.elsevier.com/locate/camwa

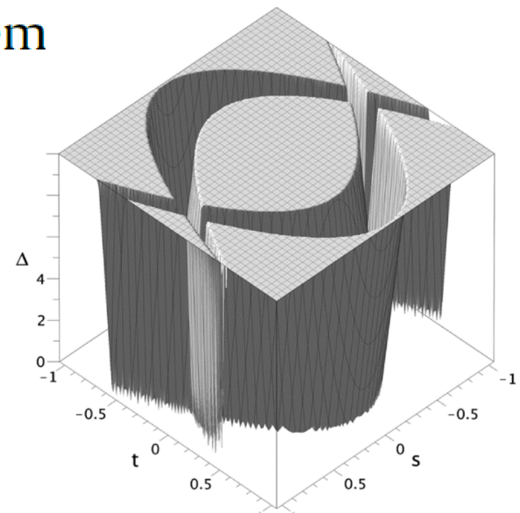
Comments on "The exp-function method and generalized solitary solutions"☆

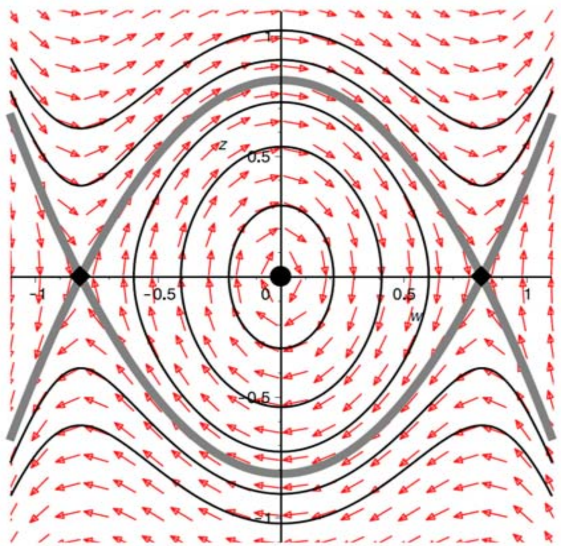
Z. Navickas [a], T. Telksnys [b], M. Ragulskis [b,*]

Chang analyzes the following system

$$u''_{xt} + a v'_x v'_t = 0;$$
$$v'_t + v'''_{xxx} + (v'_x)^3 + b v'_x u'_{xx} = 0.$$







$$y(\eta, c, s) = \frac{y_2(s - y_1)\exp(\alpha y_1(\eta - c)) - y_1(s - y_2)\exp(\alpha y_2(\eta - c))}{(s - y_1)\exp(\alpha y_1(\eta - c)) - (s - y_2)\exp(\alpha y_2(\eta - c))}$$

Applied Mathematics and Computation 283 (2016) 333–338

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Existence of solitary solutions in a class of nonlinear differential equations with polynomial nonlinearity

Z. Navickas [a], M. Ragulskis [b], T. Telksnys [b,*]

$$\frac{\partial^m u}{\partial t^m} + A_{m-1,0}\frac{\partial^{m-1} u}{\partial t^{m-1}} + A_{0,m-1}\frac{\partial^{m-1} u}{\partial z^{m-1}} + \cdots + A_{10}\frac{\partial u}{\partial t} + A_{01}\frac{\partial u}{\partial z} = a_n u^n + \cdots + a_0,$$

$$y_0 = y_0(x) = \sigma\frac{\prod_{j=1}^{l}\left(e^{\eta(x-c)} - y_j\right)}{\prod_{j=1}^{l}\left(e^{\eta(x-c)} - x_j\right)},$$

	(n,m)						
l	(2,1)	(3,2)	(4,3)	(5,4)	(6,5)	(7,6)	(8,7)
1	$\exists$	$\exists^*$	$\exists^*$	$\exists^*$	$\exists^*$	$\exists^*$	$\exists^*$
2	$\nexists$	$\exists^*$	$\exists^*$	$\exists^*$	$\exists^*$	$\exists^*$	$\exists^*$
3	$\nexists$	$\exists^*$	$\exists^*$	$\exists^*$	$\exists^*$	$\nexists$	$\nexists$
4	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$
5	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$

Astrophys Space Sci (2013) 344:281–285
DOI 10.1007/s10509-012-1338-5

LETTER

Comments on "Two exact solutions to the general relativistic Binet's equation"

Zenonas Navickas · Minvydas Ragulskis

Astrophys Space Sci (2016) 361:201
DOI 10.1007/s10509-016-2792-2

CrossMark

ORIGINAL ARTICLE

Solitary solutions to a relativistic two-body problem

R. Marcinkevicius[1] · Z. Navickas[2] · M. Ragulskis[3] · T. Telksnys[3]

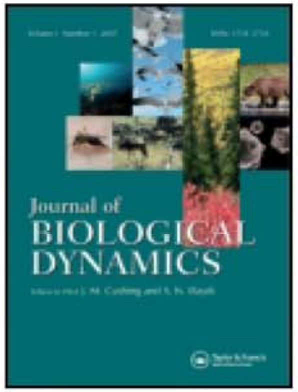
Abouelmagd et al. (2015)
two-body problem perturbed by a small first
Newtonian relativistic term can be described by:

$$\frac{d^2u}{d\theta^2} + u = \frac{1}{h^2} + \varepsilon\left(a_0 + a_1 u + u^2 + \left(\frac{du}{d\theta}\right)^2\right),$$

IMA Journal of Applied Mathematics (2016) **81**, 1163–1190
doi:10.1093/imamat/hxw050

Existence of second order solitary solutions to Riccati differential equations coupled with a multiplicative term

R. MARCINKEVICIUS Z. NAVICKAS T. TELKSNYS AND M. RAGULSKIS



Taylor & Francis
Taylor & Francis Group

Journal of Biological Dynamics

ISSN: 1751-3758 (Print) 1751-3766 (Online) Journal homepage: http://www.tandfonline.com/loi/tjbd20

Direct and inverse relationships between Riccati systems coupled with multiplicative terms

Z. Navickas, R. Vilkas, T. Telksnys & M. Ragulskis

J. Math. Anal. Appl. 448 (2017) 156–170

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Kink solitary solutions to generalized Riccati equations with polynomial coefficients

Z. Navickas[a], M. Ragulskis[a], R. Marcinkevicius[b], T. Telksnys[a,*]

Optik 132 (2017) 223–231

Contents lists available at ScienceDirect

Optik

journal homepage: www.elsevier.de/ijleo

Original research article

Comments on "Soliton solutions to fractional-order nonlinear differential equations based on the exp-function method"

Z. Navickas[b], M. Ragulskis[b], N. Listopadskis[a], T. Telksnys[b,*]

Thanks for Your attention