Pension models for mathematicians

András Simonovits

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Contents

Introduction

On pension models

- I presume you know what an economic model is: the simplified by logical description of economic activities
- Certain listeners pay contributions, others receive benefits
- What can I say in 1 hour about simple pension models to mathematicians?
- Simple vs. complex and theoretical vs. practical
- Main message: pension modeling is interesting and important

What is the use of simple pension models?

- They are indispensable in education
- They can be checked more easily than their complex counterparts
- Exam: A.S. Edlin–P. Jaraca-Mandic (2007): Erratum, JPE: the yield is not 220, but only 113 bln UDS (Pigou-taxes)
- Cause: a typing error in the program: $-1.09E-07$ was mistaken for $-1.09E-17$

Order of magnitudes

- In modern states, the public pensions amount to 5-10-15% of the GDP (USA, HU, IT, resp.)
- In Anglo-saxon countries, private pensions are important (5% of the GDP), especially for the higher earners
- Pension system are necessary, because
  • no more large families
  • short-sighted workers do not save enough for their old-age
  • old-age poverty is to be avoided even in countries which tolerate other poverty (e.g., USA)
  • It is difficult to prescribe indexed unisex life annuities for older people (cf. UK, 2015), even if it is efficient

Principles

- The pension system be sustainable and adequate
- The system should combine efficiency and fairness
- The democratic competition among parties should remain between rational bounds

Structure of the lecture

- Short-sighted worker vs paternalist government
- Due to population aging, the longitudinal and the cross-sectional equilibria are different
- The more a man earns, the longer he lives, therefore a proportional benefit redistributes from the poor to the rich.
- The later one retires, the longer he lives, because he is healthier $\Rightarrow$ constrained incentives are needed
- Fragmented work careers vs seniority pensions (e.g., Women40)

Short-sighted individual, paternalistic government

Basic scheme/1

- Short-sighted individual cares too little for old-age
- Discounted utility function

$$U(s) = \log(1 - s) + \delta\mu \log(\rho s)$$

- where $s =$ saving, $\mu =$ years spent in retirement/years spent in work, $\rho =$ interest factor
- Individual optimum

$$U'(s) = -\frac{1}{1 - s} + \delta\mu\frac{1}{\tau} = 0 \Rightarrow s^{\mathrm{o}} = \frac{\delta\mu}{1 + \delta\mu}$$

- Example: for $\mu = 1/2 = \delta$, $s^{\mathrm{o}} = 1/5$, ρ is indifferent

Basic scheme/2

- A paternalistic government compensates for individual shortsightedness by eliminating discounting: $\delta = 1$ constrained saving rate τ:

$$V(\tau) = \log(1 - \tau) + 1 \cdot \mu \log(\tau/\mu)$$

- Paternalistic optimum

$$V'(\tau) = -\frac{1}{1-\tau} + \mu\frac{1}{\tau} = 0 \Rightarrow \tau^* = \frac{\mu}{1+\mu} > s^{\mathrm{o}}.$$

- Numerical example: $\tau^* = 1/3 > s^{\mathrm{o}} = 1/5$

Other advantages

- It is easier to achieve income redistribution in a pension system than explicitly
- It naturally provides unisex and indexed life annuity
- It can redistribute between different cohorts

Cohort model

Basic problem for the individual

- She was born in year t
- she started working at age Q, her earnings (in real terms): $w_Q, \ldots, w_{R-1}$
- She paid contributions according to rate τ_a
- She retires at age R, pensions: $b_R, \ldots, b_{D-1}$
- If she dies, her widower and children may inherit survivor's pensions
- Basic issue: $(w_Q, \ldots, w_{R-1}) \Rightarrow (b_R, \ldots, b_{D-1})$?
- Public or private, unfunded vs. funded, mandatory or voluntary? (Combinations)
- Longitudinal vs. cross section?

Macromodels

- Types $i = 1, \ldots, I$, a frequency f_i , age a:
- $q_{i,a} > q_{i,a+1}$ survival probability, depend on calendar time t
- wage–pension-path

$$w_{i,Q}, \ldots, w_{i,R_i-1}, \qquad b_{i,R_i}, \ldots, b_{i,D_i-1}$$

- Contribution = Benefits

$$\tau \sum_{i=1}^{I} \sum_{a=Q}^{R_i-1} f_i q_{i,a} w_{i,a} = \sum_{i=1}^{I} \sum_{a=R_i}^{D_i-1} f_i q_{i,a} b_{i,a}$$

Macromodels (cont.-1)

- At an abstract level: we have to solve a multivariate high-order difference equation numerically

$$x_{t+1} = A_{1,t}x_t + \cdots + A_{r,t}X_{t-r} + b_t, \qquad t = 0, 1, \ldots,$$

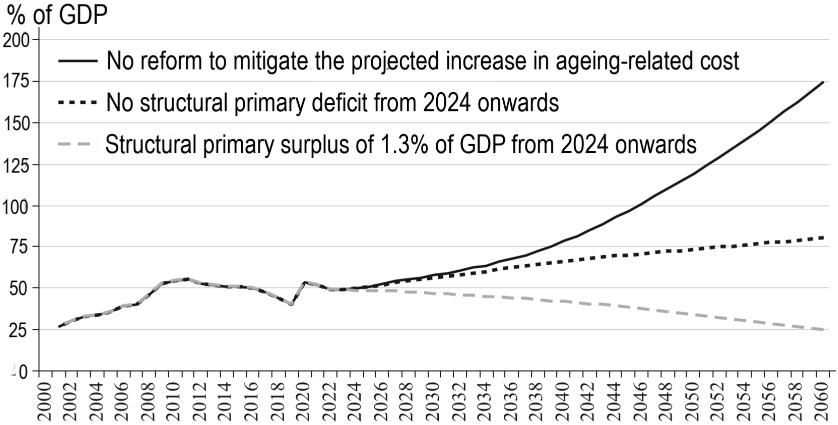
where x_t is the system's n-dimensional state vector, $x_0, \ldots, x_{-r+1}$ are initial states, r the order of lags, say 100 and $A_{1,t}, \ldots, A_{r,t}$ matrices.

- Scenarios: various demographic and economic scenarios, i.e. pension reforms

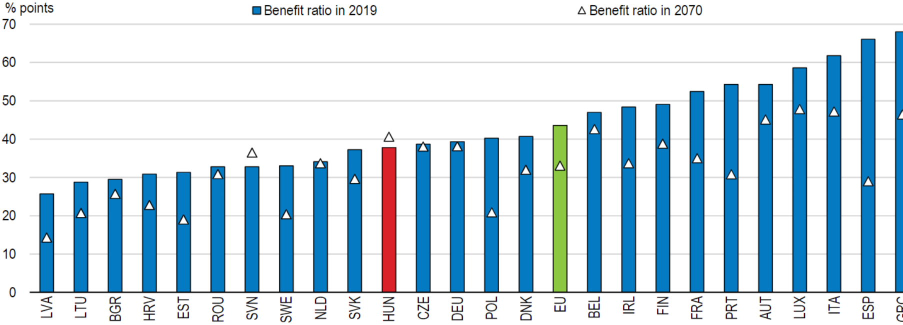
Macromodels (cont.-2)

- Auerbach–Kotlikoff (1987) ... Hans Fehr at al. (2000) ...: optimizer types with racional expectations
- In HU: no optimization
  NYIKA (2010) (Holtzer, ed.)
- Bajkó–Maknics–Tóth–Vékás (2015): Corvinus
- Freudenberg–Berki–Reiff (2016): H National Bank model
- OECD (2024): Strengthening the Hungarian Pension System

Forecasts for OECD

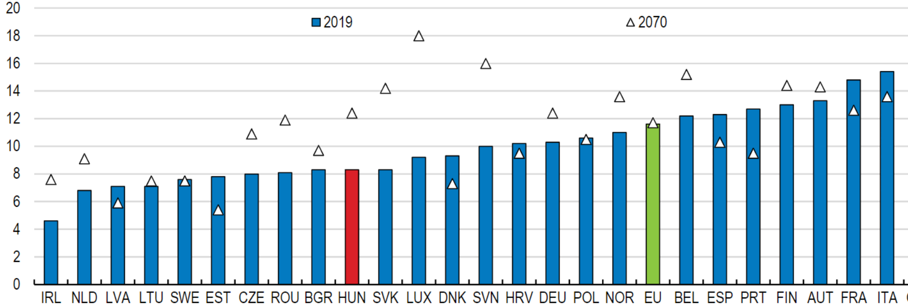


Current vs. planned benefit-ratios



Note: The benefit ratio is the ratio between the average pension and the average wage, both measured in gross terms.
Source: 2021 Ageing Report (European Commission, 2021, p. 84[29]).

Fiscal effects on pension expenditures, HU



Answer from the armchair

- No growth, no real interest rate, no seniority, no complications
- $$w_Q = \cdots = w_{R-1} = w, \quad b_R = \cdots = b_{D-1} = b$$
- Longitudinal equilibrium
$$b = \tau \frac{w(R-Q)}{D-R} = \tau \frac{wS}{T} = \frac{\tau w}{\mu}$$
- If pension $= \gamma\times$ the net wage ($\gamma =$ replacement ratio), then $b = \gamma(1-\tau)w$,
- i.e.,
$$\tau w = \mu\gamma(1-\tau)w \Rightarrow \tau^* = \frac{\gamma\mu}{1+\gamma\mu}$$

Answer from the armchair (cont.)

- Delayed retirement credit

$$\delta(R) = \frac{db(R)}{b(R)dR} = \frac{d\log b(R)}{dR}$$

- $$\log b(R) = \log(\tau w) + \log(R - Q) - \log(D - R)$$

- Expressed as

$$\delta(R) = \frac{d\log(R - Q)}{dR} + \frac{d\log(D - R)}{dR}$$

- Analytical expression

$$\delta(R) = \frac{1}{R - Q} + \frac{1}{D - R}$$

Flexible retirement

Numerical illustration: *dependence* of monthly pension on retirement age for $Q = 25$, $D = 85$ yrs, $v = 1$, $\tau = 0.2$.

Table 1. Pension/Net wage as a function of retirement age

Retirement age R	Benefit/ Net wage $b(R)$	Relative rise $b'(R)/b(R)-1$
62	0.611	–
63	0.656	0.074
64	0.705	0.075
65	0.759	0.077
66	0.819	0.079
67	0.886	0.081
68	0.960	0.084

The impact of the real wage growth and change in inflation are also important, see Women40 later

Complications

- Due to population aging, the longitudinal and the cross sectional equilibria are different
- The more a man earns, the longer he lives, therefore a proportional benefit redistributes from the poor to the rich.
- The later one retires, the longer he lives, because he is healthy $\Rightarrow$ constrained incentives are needed
- Fragmented work careers vs seniority pensions (e.g., Women40)

Population aging

The case of Hungary, 1970-2050

Table 2. Dynamics of age distributions HU (65–)

Year	Children	Elderlies	Dependency
	share		ratio
t	K_t/N_t	P_t/N_t	p_t
1970	0.283	0.131	0.224
2000	0.236	0.146	0.236
2050	0.189	0.262	0.477

Cross sectional equilibrium

- Pay-as-you-go system: Contributions = Benefits

$$\tau M\bar{w} = P\bar{b}$$

- average wage: $\bar{w}$,
- contribution rate: τ
- average pension: $\bar{b}$
- # of workers M,
- # of pensioners: P

Dependency and benefit ratios

- Rearranged

$$\tau = \frac{P\bar{b}}{M\bar{w}} = \frac{P}{M}\frac{\bar{b}}{\bar{w}}$$

- Dependency and benefit ratios

$$\pi = \frac{P}{M}, \qquad \beta = \frac{\bar{b}}{\bar{w}}$$

- Rearranged

$$\tau = \pi\beta.$$

- Stylized example (gross wage):

$$\tau_{US} = 0.3 \cdot 0.4 = 0.12; \quad \tau_{HU} = 0.6 \cdot 0.5 = 0.3 \text{ or } 0.5 \cdot 0.3 = 0.15$$

Hungarian pension system, 1970–

Table 3. Maturing pension system, 1970–2006, HU, %

Year t	Pensions /GDP B_t/Y_t	Entitlement ζ_t	Dependency π_t	Benefit /Wage β_t	Participation μ_t
1970	3.5	66.7	38.7	37.5	91.2
1990	8.8	109.9	41.8	66.2	86.4
1996	8.9	119.2	40.7	58.9	64.0
1996	9.7	131.6	38.3	59.3	58.9
2001	9.3	146.1	33.0	59.1	60.5
2006	10.6	151.7	30.6	62.3	60.3

Longevity gap

Distribution of pensions

Table 4. Distribution of pensions HU 2015, in terms of average net wage

	Female		Male	
quintile	upper	average	upper	average
1	0.493	0.413	0.530	0.437
2	0.610	0.555	0.659	0.591
3	0.696	0.647	0.819	0.733
4	0.869	0.770	1.060	0.930
5	–	1.103	–	1.337

Remark. D. Molnár–Hollósné-Marosi (2015, Table 1–3)

Life expectancy and income

Higher old-age income, longer life expectancy, especially for males

Table 5. Life expectancy–pension HU 2015, (years)

Quintile	Female 60	Male 63
q_1	22.5	14.6
q_2	22.5	15.0
q_3	22.4	15.7
q_4	23.3	17.0
q_5	24.8	18.8
Average	23.0	16.1

Remark. D. Molnár–Hollósné-Marosi (2017, Table 2)

Model/1

- Assumptions:
  contribution length $S = \mathrm{const.}$
- retirement age $R = \mathrm{const.}$,
- span of retirement $T_w = D_w - R$ grows with w
- Progressive pension as a combination of basic and proportional pensions

$$b = \alpha\beta + (1 - \alpha)\beta w$$

Model/2

- Lifetime balance

$$z = \tau S w - T_w b(w)$$

- After substitution

$$z(w) = \tau S w - T_w[\alpha\beta + (1-\alpha)\beta w]$$

- Expected value: ($\mathbf{E}w = 1$ and $T = \mathbf{E}T_w$)

$$0 = \mathbf{E}z(w) = \tau S - \alpha\beta T + (1-\alpha)\beta\mathbf{E}[T_w w]$$

Illustration for a combined pension

Uniform wage distribution

Table 6. Proportional vs combined pension

Wage	LEXP	Proportional		Combined	
		pension	balanced	pension	balanced
w_i	e_i	b_i^A	z_i^A	b_i^C	z_i^C
0.5	17	0.238	0.952	0.366	–1.220
1.0	20	0.476	0.476	0.488	0.244
1.5	23	0.714	–1.429	0.610	0.976

Remark. $Q = 20$, $R = 60$, $\tau = 0.25$, $\alpha = 0.5$.

Fragmented career

Augusztinovics, 2005, Guszti–Köllő, 2007

- In practice, the contributiove period is much shorter than the difference between retirement age and starting age: $S < R - Q$
- The degree of fragmentation varies: $S = \varphi(R - Q)$
- Benefit

$$b(R, \varphi) = \frac{\tau w \varphi(R - Q)}{e_R}$$

where e_R denotes LEXP at age R

- Bilinear approximation

$$b(R, S) = \delta S[1 + \psi(R - R^*)]w,$$

where R^* a normal pensionable age (2022: 65), $\psi = 0.06$/yr.

Women40

- In Hungary, only a longer than critical contributive period allows early retirement, moreover no deduction.
- otherwise no early retirement
- Fragmented career creates negative correlation between R and S – dysfunctional
- In Austria, too but not in Germany or Sweden

Joint distribution of length of contribution and retirement age

Table 7. Joint distribution of length of contribution and retirement age, 2016, Women, HU

Age (year) (year)	Early age $R = 58.6$	NRA $R^* = 63$	Average $\mathbf{E}R = 60.6$
Short $S = 31.4$	0	0,36	0,36
Long $S_{\mathrm{m}} = 41.2$	0.55	0.09	0.64
Average $\mathbf{E}S = 37.8$	0.55	0.45	1.00

Negative correlation between length of contribution and retirement age

Numerically: $p = 0.55$ és $q = 0.36$
Neglecting the variance within categories, the correlation coefficient

$$r(R, S) = -\sqrt{\frac{pq}{(1-p)(1-q)}}$$

With substitution: $r(R, S) = -0.822$. Taking into account internal variance reduces CR to –0.6, and –0.53.

Real wage explosion: 2016–2019

Impact of real wage growth

Table 8. Real growth rates, 2015–2019, HU

Year t	GDP $100(g_t^y - 1)$	Net wage $100(g_t^v - 1)$	Benefits $100(g_t^b - 1)$	Benefit-ratio b_t/v_t
2015	2.9	4.3	3.5	0.668
2016	2.1	7.4	1.4	0.631
2017	4.1	10.2	3.0	0.583
2018	4.0	8.0	2.0	0.551
2019	5.0	7.0	3.0	0.521

Women40 again

- Women40 (2011-) allows HU women to retire after 40 years of entitlements with full pension
- Rigid age limit for others (2016: 63 years; 2022: 65 years)
- Approach 1: unfair to others, because a woman of age 58 received benefits for free for 5.5 years
- Approach 2: unfair to the “favored", because if a woman of age 58 (2016) worked 3 years more, than she would have received annually 37% higher benefits for only a little shorter period
- Lifetime benefit

$$2000 = 20 \times 100 < 18 \times 137 = 2466$$

Conclusions

- Pension modelling is interesting and important
- It is worth making an Atlas of models, where related models are analyzed
- We must improve the model's realism while preserve its simplicity
- Neglected topics
- I assumed that workers know the rules and disciplined to optimize. NO
- I assumed that our simple models show in good directions. NO, e.g., no positive correlation between ret age and length of contr