Optimal strong stability preserving time-stepping methods with upwind- and downwind-biased operators

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in memory of Willem Hundsdorfer
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Outline

1. Background
   - Spatial and temporal discretizations
   - SSP LMMs and RK methods
2. SSP methods with downwind-biased operators
3. Perturbed and additive linear multistep methods
   - Perturbed linear multistep methods
   - Linear multistep methods for additive problems
4. Perturbed Runge–Kutta methods
   - Implicit methods
5. Conclusion

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Nonlinear Stability for hyperbolic problems

Hyperbolic conservation laws:

$$\frac{\partial \boldsymbol{U}(\boldsymbol{x}, t)}{\partial t} + \nabla \cdot \boldsymbol{\mathcal{F}}\left(\boldsymbol{U}(\boldsymbol{x}, t)\right) = 0$$

Semi-discrete problem (IVP):

$$\begin{aligned} \boldsymbol{u}'(t) &= \boldsymbol{F}(\boldsymbol{u}(t), t), \\ \boldsymbol{u}(t_0) &= \boldsymbol{u}_0 \end{aligned}$$

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Exact solution usually satisfies **nonlinear or strong stability** properties:

- monotonicity: $\|\boldsymbol{u}(t)\| \leq \|\boldsymbol{u}(t-h)\|$;
- contractivity: $\|\boldsymbol{u}(t) - \tilde{\boldsymbol{u}}(t)\| \leq \|\boldsymbol{u}(t-h) - \tilde{\boldsymbol{u}}(t-h)\|$;
- positivity: $\boldsymbol{u}(t) \geq 0$ if $\boldsymbol{u}(t_0) \geq 0$;

These qualitative properties should be also maintained by the numerical solution.

Spatial discretizations

One way to guarantee that the numerical solution is strongly stable is to require stability in the total variation (TV) semi-norm:

$$\|\boldsymbol{u}^n\|_{\text{TV}} \leq \|\boldsymbol{u}^{n-1}\|_{\text{TV}},$$

where $\|\boldsymbol{u}\|_{\text{TV}} = \sum_j |\boldsymbol{u}_{j+1} - \boldsymbol{u}_j|$.

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Total variation diminishing (TVD) methods introduced by Harten (1983) and further analyzed by many others (Hirsch, van Leer, Roe, Sweby, Laney, Toro).

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Weighted essentially nonoscillatory (WENO) spatial discretizations provide good resolution around discontinuities and higher order of accuracy at smooth regions (Zhang & Shu, 2011).

Finite volume methods

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Finite volume integration over a cell $C_i = (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$:

$$\frac{d}{dt}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \boldsymbol{U}(x,t)dx = -\left(\boldsymbol{\mathcal{F}}\Big(\boldsymbol{U}(x_{i+\frac{1}{2}},t)\Big) - \boldsymbol{\mathcal{F}}\Big(\boldsymbol{U}(x_{i-\frac{1}{2}},t)\Big)\right)$$

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Semi-discrete system of ODEs:

$$\boldsymbol{u}_i'(t) = -\frac{1}{\Delta x}\left(\boldsymbol{F}_{i+\frac{1}{2}} - \boldsymbol{F}_{i-\frac{1}{2}}\right),$$

$$\boldsymbol{u}_i(t) \approx \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \boldsymbol{U}(x,t)dx, \qquad \boldsymbol{F}_{i\pm\frac{1}{2}} \approx \boldsymbol{\mathcal{F}}\Big(\boldsymbol{U}(x_{i\pm\frac{1}{2}},t)\Big)$$

Finite volume methods (cont.)

Let $\boldsymbol{u}_{i-1}$ and $\boldsymbol{u}_i$ be the left and right cell averages across interface $x_{i-\frac{1}{2}}$.
Solving the local Riemann problem gives a solution $\bar{\boldsymbol{u}}(\boldsymbol{u}_{i-1}, \boldsymbol{u}_i)$, hence

$$\boldsymbol{F}_{i-\frac{1}{2}} := \mathcal{F}\Big(\bar{\boldsymbol{u}}(\boldsymbol{u}_{i-1}, \boldsymbol{u}_i)\Big).$$

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TVD[1]**:** $\bar{\boldsymbol{u}}\Big(\boldsymbol{u}^L_{i-\frac{1}{2}}, \boldsymbol{u}^R_{i-\frac{1}{2}}\Big)$; $\quad \boldsymbol{u}^L_{i-\frac{1}{2}} = \boldsymbol{u}_{i-1} + \frac{\Delta x}{2}\boldsymbol{\sigma}_{i-1}, \quad \boldsymbol{u}^R_{i-\frac{1}{2}} = \boldsymbol{u}_i - \frac{\Delta x}{2}\boldsymbol{\sigma}_i.$

WENO[2]**:** $\bar{\boldsymbol{u}}\Big(\boldsymbol{u}^-_{i-\frac{1}{2}}, \boldsymbol{u}^+_{i-\frac{1}{2}}\Big)$; $\quad \boldsymbol{u}^-_{i-\frac{1}{2}}$, $\boldsymbol{u}^+_{i-\frac{1}{2}}$ are WENO reconstructions,

e.g., $\boldsymbol{u}^-_{i-\frac{1}{2}} = \omega_1 \boldsymbol{u}^{(1)}_{i-\frac{1}{2}} + \omega_2 \boldsymbol{u}^{(2)}_{i-\frac{1}{2}} + \omega_3 \boldsymbol{u}^{(3)}_{i-\frac{1}{2}}$ over $\{I_{i-3}, I_{i-2}, I_{i-1}, I_i, I_{i+1}\}$.

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[1] Sweby, P. K., *High resolution schemes using flux limiters for hyperbolic conservation laws*, *SIAM J. Numer. Anal.* 21.5 (1984), pp. 995–1011.

[2] Shu, C.-W., *High order weighted essentially nonoscillatory schemes for convection dominated problems*, *SIAM Rev.* 51.1 (2009), pp. 82–126.

Need for SSP methods

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- First use the method of lines to obtain a semi-discretization that is strongly stable in a certain norm (or semi-norm or convex functional) with forward Euler under a time-step $h \leq h_{FE}$.

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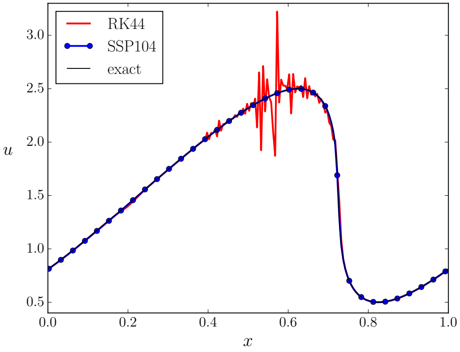
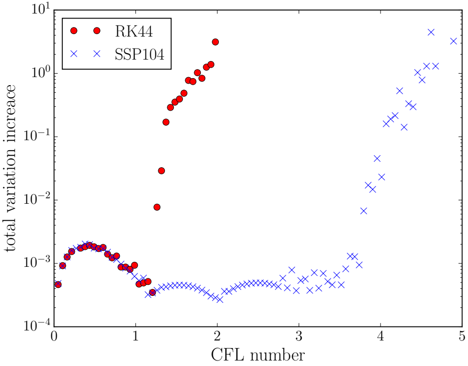
- First use the method of lines to obtain a semi-discretization that is strongly stable in a certain norm (or semi-norm or convex functional) with forward Euler under a time-step $h \le h_{FE}$.
- Then try to find a higher order time discretization that maintains the strong stability in the same norm but under a different (relaxed) time-step restriction

$$\mathbf{h} \le \mathcal{C}\ \mathbf{h}_{\mathbf{FE}}.$$

Example 1



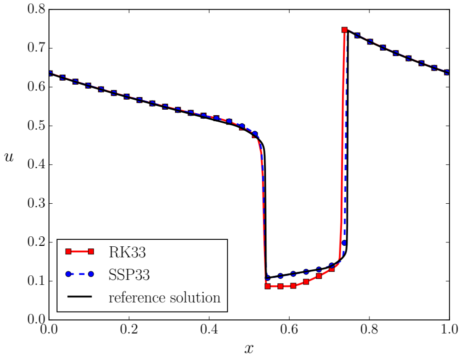
Burgers' equation, $T = 0.16$,
CFL number $\nu = 2.2$.



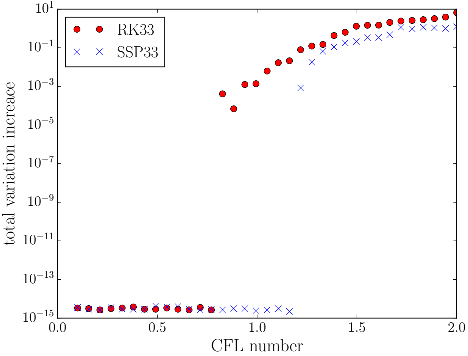
Maximum rise in total variation.

Usually first-order methods are non-oscillatory and satisfy those properties; however, high-order time integrators can lead to spurious overshots.

Example 2



Buckley–Leverett equation, $T = 0.5$, CFL number $\nu = 1.2$.



Maximum rise in total variation.

Historical background

ODEs

$\boldsymbol{u}' = \boldsymbol{L}\boldsymbol{u}$

- positivity (Bolley & Crouzeix, 1978)
- contractivity (Spijker, 1983)
- **absolute monotonicity**,
  optimal RK methods
  (Kraaijevanger, 1986)
- contractive LMMs (Sand, 1986)

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$u' = F(u)$

- **contractivity $\Leftrightarrow$ absolute monotonicity** (Kraaijevanger, 1991)
- LMMs (Lenferink, 1989,1991)
- RK positivity (Horváth, 1997,1998)

PDEs

SSP(TVD) time-discretizations

- TVD (Shu, 1988)
- ENO/WENO (Shu & Osher, 1988,1998)

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Optimal SSP methods

- RK methods & bounds (Gottlieb et.al, 1988-2003, Ruuth & Spiteri, 2002-2006)
- TVB RK methods (Ferracina & Spijker, 2005)
- TVD/TVB LMMs (Hundsdorfer & Ruuth, 2005-2007)

Historical background (cont.)

absolute monotonicity $\Leftrightarrow$ optimal Shu–Osher representation

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For a given RK method, the largest step-size bound over all Shu–Osher representations corresponds to the radius of absolute monotonicity of the method.

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More recently:
Implicit SSP RK methods, low storage, IMEX SSP methods, multirate, methods, spectral deferred correction methods, multistep multistage methods, multistage multiderivative methods, effective order methods, etc.

Outline

Monotonicity of linear multistep methods

$$\boldsymbol{u}'(t) = \boldsymbol{F}(\boldsymbol{u}(t), t)$$

Assume the upwind-biased operator $\boldsymbol{F}$ approximates $-\nabla \cdot \boldsymbol{\mathcal{F}}(\boldsymbol{U})$ and satisfies the forward Euler (FE) condition

$$\|\boldsymbol{u} + h\boldsymbol{F}(\boldsymbol{u})\| \leq \|\boldsymbol{u}\|, \quad \forall \boldsymbol{u} \in \mathbb{R}^m, \ 0 \leq h \leq h_{\mathsf{FE}}.$$

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A k-step linear multistep method (LMM) with non-negative coefficients

$$\boldsymbol{u}^n = \sum_{j=0}^{k-1} \alpha_j \boldsymbol{u}^{n-k+j} + h \sum_{j=0}^{k} \beta_j \boldsymbol{F}(\boldsymbol{u}^{n-k+j}),$$

is SSP, if it satisfies

$$\|\boldsymbol{u}^n\| \leq \max\Big\{\|\boldsymbol{u}^{n-1}\|, \|\boldsymbol{u}^{n-2}\|, \ldots, \|\boldsymbol{u}^{n-k}\|\Big\},$$

whenever $h \leq \mathcal{C} h_{\mathsf{FE}}$.

The SSP coefficient of the method is $\mathcal{C} = \min_j \frac{\alpha_j}{\beta_j}$.

Monotonicity of Runge–Kutta methods

Again, the main assumption is

$$\|\boldsymbol{u} + h\boldsymbol{F}(\boldsymbol{u})\| \leq \|\boldsymbol{u}\|, \quad \forall \boldsymbol{u} \in \mathbb{R}^m,\ 0 \leq h \leq h_{\mathsf{FE}}.$$

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Butcher form of Runge–Kutta method RK(2,2):

$$\begin{aligned}
\boldsymbol{u}^* &= \boldsymbol{u}^{n-1} + h\boldsymbol{F}(\boldsymbol{u}^{n-1}),\\
\boldsymbol{u}^n &= \boldsymbol{u}^{n-1} + \frac{1}{2}h\boldsymbol{F}(\boldsymbol{u}^{n-1}) + \frac{1}{2}h\boldsymbol{F}(\boldsymbol{u}^*).
\end{aligned}$$

Optimal Shu-Osher representation:

$$\begin{aligned}
\boldsymbol{u}^* &= \boldsymbol{u}^{n-1} + h\boldsymbol{F}(\boldsymbol{u}^{n-1}),\\
\boldsymbol{u}^n &= \frac{1}{2}\boldsymbol{u}^{n-1} + \frac{1}{2}\left(\boldsymbol{u}^* + h\boldsymbol{F}(\boldsymbol{u}^*)\right).
\end{aligned}$$

Monotonicity of Runge–Kutta methods (cont.)

Canonical Shu-Osher form of Runge–Kutta (RK) methods:

$$\begin{aligned} \boldsymbol{y} &= \boldsymbol{v}_r \boldsymbol{u}^{n-1} + \boldsymbol{\alpha}_r \left(\boldsymbol{y} + \frac{h}{r} \boldsymbol{F}(\boldsymbol{y}) \right), \\ \boldsymbol{u}^n &= \boldsymbol{y}_{s+1}, \end{aligned}$$

where $\boldsymbol{v}_r = (\boldsymbol{I} + r\boldsymbol{K})^{-1}\boldsymbol{e}$ and $\boldsymbol{\alpha}_r = r(\boldsymbol{I} + r\boldsymbol{K})^{-1}\boldsymbol{K}$. ($\boldsymbol{K}$: Butcher array)

We call

$$\mathcal{C}(\boldsymbol{K}) = \sup\{r \mid \exists(\boldsymbol{I} + r\boldsymbol{K})^{-1} \text{ and } \boldsymbol{v}_r \geq 0, \boldsymbol{\alpha}_r \geq 0\}$$

the **SSP coefficient** of a given method with coefficients $\boldsymbol{K}$.

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A RK with $\mathcal{C} > 0$ is SSP, if it satisfies $\|\boldsymbol{u}^n\| \leq \|\boldsymbol{u}^{n-1}\|$ whenever $h \leq \mathcal{C} h_{\mathsf{FE}}$.

Bounds and Barriers

Let $\mathcal{C}_{\text{eff}} = \dfrac{\mathcal{C}}{s}$ for RK methods (s : # of stages).

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SSP explicit methods:

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- Linear multistep methods have no order barrier on order but $\mathcal{C} \leq 1$.

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SSP implicit methods:

- Unconditional monotonicity only for implicit Euler method.
- Runge–Kutta methods have order $p \leq 6$ and $\mathcal{C}_{\text{eff}} \leq 2$.
- Linear multistep methods have no order barrier on order but $\mathcal{C} \leq 2$.

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Motivation: Why downwinding is important?

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- Optimal methods with downwind-biased operators attain larger SSP coefficient compared to classical SSP methods.
- Optimal implicit perturbed Runge–Kutta methods can attain arbitrarily large SSP coefficients.
- Extends monotonicity analysis for methods applied to additive problems, i.e. $\boldsymbol{F}$ and $\widetilde{\boldsymbol{F}}$ approximate different parts of the PDE.

Example

Consider the LeVeque and Yee problem

$$U_t + f(U)_x = s(U), \quad U(x,0) = U_0(x), \quad x \in \mathbb{R},\ t \geq 0,$$

where $s(U) = -\mu U(U-1)(U-0.5)$, $\mu > 0$.

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Semi-discretization: Let $u_i \approx U(x_i, t)$ and define

$$D_i(u) = -\frac{f(u_i) - f(u_{i-1})}{\Delta x}, \qquad \widetilde{D}_i(u) = -\frac{f(u_{i+1}) - f(u_i)}{\Delta x},$$
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$$S_i(u) = \widetilde{S}_i(u) = s(u_i).$$

Consider the initial value problems (IVPs):

$$u'(t) = F(u(t)), \quad F = D + S, \qquad u'(t) = \widetilde{F}(u(t)), \quad \widetilde{F} = \widetilde{D} + \widetilde{S},$$
$$u(0) = u_0. \qquad u(0) = u_0.$$

Example (cont.)

It can be shown that if $u \in [0, 1]$, then

$$0 \le u + h\,F(u) \le 1 \quad \text{for } 0 \le h \le h_{\mathsf{FE}} = \frac{2\tau}{2 + \mu\tau},$$
$$0 \le u - h\,\widetilde{F}(u) \le 1 \quad \text{for } 0 \le h \le \tilde{h}_{\mathsf{FE}} = \frac{16\tau}{16 + \mu\tau},$$

where $\tau > 0$ is such that

$$0 \le u + h\,D(u) \le 1 \quad \text{for } 0 \le h \le \tau,$$
$$0 \le u - h\,\widetilde{D}(u) \le 1 \quad \text{for } 0 \le h \le \tau.$$

Example (cont.)

$$\text{Let } \xi = \frac{h_{\mathsf{FE}}}{\tilde{h}_{\mathsf{FE}}} = \frac{16 + \mu\tau}{8(2 + \mu\tau)}$$

and apply an optimal explicit perturbed SSP LMM to $u' = F(u)$.

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Then, the numerical solution lies in $[0, 1]$ under a step-size restriction

$$h \le \mathcal{C}(\xi)\, h_{\mathsf{FE}}.$$

This is less strict compared to the "classical" optimal SSP LMM (without downwinding), for all $\xi \in \mathbb{R}^+$.

e.g., choose SSP LMM(3,2) and $\mu\tau = 8/3$. Then, $\xi = 1/2$ and

$$h \le \mathcal{C}(1/2)\, h_{\mathsf{FE}} = 0.3044\tau.$$

On the other hand, without downwinding: $h \le 0.2143\tau$.

Outline

LMMs with downwind-biased operators

In addition to the operator $\boldsymbol{F}$, consider the associated **downwind-biased operator** $\widetilde{\boldsymbol{F}} \approx -\nabla \cdot \mathcal{F}(\boldsymbol{U})$ such that

$$\|\boldsymbol{u} + h\boldsymbol{F}(\boldsymbol{u})\| \leq \|\boldsymbol{u}\|, \quad \forall \boldsymbol{u} \in \mathbb{R}^m,\ 0 \leq h \leq \mathbf{h}_{\mathsf{FE}},$$
$$\|\boldsymbol{u} - h\widetilde{\boldsymbol{F}}(\boldsymbol{u})\| \leq \|\boldsymbol{u}\|, \quad \forall \boldsymbol{u} \in \mathbb{R}^m,\ 0 \leq h \leq \tilde{\mathbf{h}}_{\mathsf{FE}}.$$

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$$\|\boldsymbol{u} - h\widetilde{\boldsymbol{F}}(\boldsymbol{u})\| \leq \|\boldsymbol{u}\|, \quad \forall \boldsymbol{u} \in \mathbb{R}^m,\ 0 \leq h \leq \tilde{\mathbf{h}}_{\mathsf{FE}}.$$

By using both upwind and downwind operators, a k-step linear multistep method (LMM) applied to a semi-discrete problem

$$\begin{aligned} \boldsymbol{u}'(t) &= \boldsymbol{F}(\boldsymbol{u}(t)), \quad t \geq t_0, \\ \boldsymbol{u}(t_0) &= \boldsymbol{u}_0, \end{aligned}$$

takes the form

$$\boldsymbol{u}^n = \sum_{j=0}^{k-1} \alpha_j \boldsymbol{u}^{n-k+j} + h \sum_{j=0}^{k} \left(\bar{\beta}_j \boldsymbol{F}(\boldsymbol{u}^{n-k+j}) + \tilde{\beta}_j \left(\boldsymbol{F}(\boldsymbol{u}^{n-k+j}) - \widetilde{\boldsymbol{F}}(\boldsymbol{u}^{n-k+j}) \right) \right).$$

Perturbed LMMs

Let $\bar{\beta}_j = \beta_j - \tilde{\beta}_j$, then LMMs can be also written in the form

$$\boldsymbol{u}^n = \sum_{j=0}^{k-1} \alpha_j \boldsymbol{u}^{n-k+j} + \sum_{j=0}^{k} \left(\beta_j h \boldsymbol{F}(\boldsymbol{u}^{n-k+j}) - \tilde{\beta}_j h \widetilde{\boldsymbol{F}}(\boldsymbol{u}^{n-k+j}) \right).$$

The above LMMs are referred to as **perturbed LMMs** when applied to $\boldsymbol{u}'(t) = \boldsymbol{F}(\boldsymbol{u}(t))$, where $\boldsymbol{F}$ and $\widetilde{\boldsymbol{F}}$ satisfy the FE condition with different step-size restrictions.

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Let $\bar{\beta}_j = \beta_j - \tilde{\beta}_j$, then LMMs can be also written in the form

$$\boldsymbol{u}^n = \sum_{j=0}^{k-1} \alpha_j \boldsymbol{u}^{n-k+j} + \sum_{j=0}^{k} \left(\beta_j h \boldsymbol{F}(\boldsymbol{u}^{n-k+j}) - \tilde{\beta}_j h \widetilde{\boldsymbol{F}}(\boldsymbol{u}^{n-k+j}) \right).$$

The above LMMs are referred to as **perturbed LMMs** when applied to $\boldsymbol{u}'(t) = \boldsymbol{F}(\boldsymbol{u}(t))$, where $\boldsymbol{F}$ and $\widetilde{\boldsymbol{F}}$ satisfy the FE condition with different step-size restrictions.

Next:

- Monotonicity properties
- Step-size bounds for monotonicity
- Optimal SSP perturbed methods

Perturbed SSP LMMs

A perturbed LMM is SSP with threshold factor $(\mathcal{C}, \widetilde{\mathcal{C}})$, if the monotonicity conditions hold

$$\begin{aligned} \beta_j \geq 0, \tilde{\beta}_j \geq 0, \quad j \in \{0, \dots, k\}, \\ \alpha_j - r\beta_j - \tilde{r}\tilde{\beta}_j \geq 0, \quad j \in \{0, \dots, k-1\}, \end{aligned}$$

for all $0 \leq r \leq \mathcal{C}$, $0 \leq \tilde{r} \leq \widetilde{\mathcal{C}}$.

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for all $0 \leq r \leq \mathcal{C}$, $0 \leq \tilde{r} \leq \widetilde{\mathcal{C}}$.

Theorem

Consider an IVP problem for which $\boldsymbol{F}$ and $\widetilde{\boldsymbol{F}}$ satisfy the forward Euler condition for some $h_{\mathrm{FE}} > 0$, $\tilde{h}_{\mathrm{FE}} > 0$. Apply a perturbed SSP LMM with threshold factor $(\mathcal{C}, \widetilde{\mathcal{C}})$. Then the numerical solution satisfies

$$\|\boldsymbol{u}^n\| \leq \max\Big\{\|\boldsymbol{u}^{n-1}\|, \|\boldsymbol{u}^{n-2}\|, \ldots, \|\boldsymbol{u}^{n-k}\|\Big\},$$

under a time-step restriction $h \leq \min\{\mathcal{C}\, h_{\mathrm{FE}}, \widetilde{\mathcal{C}}\, \tilde{h}_{\mathrm{FE}}\}$.

Perturbed SSP LMMs (cont.)

Since $\mathcal{C}$, $\widetilde{\mathcal{C}}$ are continuous functions of the method's coefficients, the maximum step size is achieved when $\mathcal{C} = \widetilde{\mathcal{C}}\,\tilde{h}_{\mathsf{FE}}/\,h_{\mathsf{FE}}$.

For a given number of steps k, order of accuracy p and $\xi := h_{\mathsf{FE}}/\,\tilde{h}_{\mathsf{FE}}$, we want to find the largest possible value $r(\xi)$ for which the monotonicity conditions are satisfied when $\tilde{r} = \xi\, r(\xi)$.

Perturbed SSP LMMs (cont.)

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For a given number of steps k, order of accuracy p and $\xi := h_{\mathsf{FE}} / \tilde{h}_{\mathsf{FE}}$, we want to find the largest possible value $r(\xi)$ for which the monotonicity conditions are satisfied when $\tilde{r} = \xi\, r(\xi)$.

Combining the order conditions and monotonicity constraints we have:

$$
\begin{aligned}
\sum_{j=0}^{k-1} (\gamma_j + r(\beta_j + \xi\tilde{\beta}_j)) j^i + \sum_{j=0}^{k} (\beta_j - \tilde{\beta}_j) i j^{i-1} = k^i, \quad & i \in \{0, \dots, p\}, \\
\beta_j \geq 0, \tilde{\beta}_j \geq 0, \quad & j \in \{0, \dots, k\}, \\
\gamma_j \geq 0, \quad & j \in \{0, \dots, k-1\}.
\end{aligned}
$$

Since the conditions are non-linear only in r we can use bisection and solve a sequence of feasibility linear problems.

Results

In contrast with other optimal SSP methods, now the SSP coefficient depends on the problem, not just the number of steps k and order of accuracy p.

Optimal perturbed SSP LMMs have been found for $k \in \{1, \dots, 40\}$, $p \in \{1, \dots, 15\}$ and for different values of ξ.

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- Optimal pth-order SSP methods have at most p non-zero coefficients.
- Given k, p, then for any value of ξ the optimal perturbed SSP LMMs attain larger step sizes for monotonicity when compared with other LMMs.

Outline

1. Background
2. SSP methods with downwind-biased operators
3. Perturbed and additive linear multistep methods
   - Perturbed linear multistep methods
   - Linear multistep methods for additive problems
4. Perturbed Runge–Kutta methods
5. Conclusion

Additive linear multistep methods

Now, lets consider linear multistep methods applied to the additive problem

$$\begin{aligned} \boldsymbol{u}'(t) &= \boldsymbol{F}(\boldsymbol{u}(t)) + \widehat{\boldsymbol{F}}(\boldsymbol{u}(t)), \quad t \geq t_0 \\ \boldsymbol{u}(t_0) &= \boldsymbol{u}_0, \end{aligned}$$

and assume that $\boldsymbol{F}$, $\widehat{\boldsymbol{F}}$ satisfy

$$\begin{aligned} \|\boldsymbol{u} + h\boldsymbol{F}(\boldsymbol{u})\| &\leq \|\boldsymbol{u}\|, \quad \forall \boldsymbol{u} \in \mathbb{R}^m,\ 0 \leq h \leq h_{\mathsf{FE}} \\ \|\boldsymbol{u} + h\widehat{\boldsymbol{F}}(\boldsymbol{u})\| &\leq \|\boldsymbol{u}\|, \quad \forall \boldsymbol{u} \in \mathbb{R}^m,\ 0 \leq h \leq \hat{h}_{\mathsf{FE}}. \end{aligned}$$

$$\boldsymbol{u}^n = \sum_{j=0}^{k-1} \alpha_j \boldsymbol{u}^{n-k+j} + \sum_{j=0}^{k} \left(\beta_j h \boldsymbol{F}(\boldsymbol{u}^{n-k+j}) + \hat{\beta}_j h \widehat{\boldsymbol{F}}(\boldsymbol{u}^{n-k+j}) \right)$$

SSP additive LMM methods

An additive linear multistep method has order of accuracy p if

$$\sum_{j=0}^{k-1} \alpha_j j^i + \sum_{j=0}^{k} \beta_j i j^{i-1} = k^i, \quad \sum_{j=0}^{k-1} \alpha_j j^i + \sum_{j=0}^{k} \hat{\beta}_j i j^{i-1} = k^i, \quad i \in \{0, \ldots, p\}.$$

Combining the order conditions and monotonicity constraints we can formulate the feasibility problem:

$$\begin{aligned}
\sum_{j=0}^{k-1} (\gamma_j + r(\beta_j + \xi \hat{\beta}_j)) j^i + \sum_{j=0}^{k} \beta_j i j^{i-1} = k^i, &\quad i \in \{0, \ldots, p\}, \\
\sum_{j=0}^{k} (\beta_j - \hat{\beta}_j) j^{i-1} = 0, &\quad i \in \{0, \ldots, p\}, \\
\beta_j \geq 0, \hat{\beta}_j \geq 0, &\quad j \in \{0, \ldots, k\}, \\
\gamma_j \geq 0, &\quad j \in \{0, \ldots, k-1\}.
\end{aligned}$$

Results

Theorem

For a given $k \geq 1, p \geq 1$ consider the optimal k-step, pth-order ***additive*** *(explicit or implicit) LMM with threshold factor $(\mathcal{C}, \widehat{\mathcal{C}})$.*
Then this method is equivalent to the optimal ***non-additive*** *k-step, pth-order SSP LMM with SSP coefficient $\mathcal{C} + \widehat{\mathcal{C}}$.*

Results

Theorem

*For a given $k \geq 1, p \geq 1$ consider the optimal k-step, pth-order **additive** (explicit or implicit) LMM with threshold factor $(\mathcal{C}, \widehat{\mathcal{C}})$.*
*Then this method is equivalent to the optimal **non-additive** k-step, pth-order SSP LMM with SSP coefficient $\mathcal{C} + \widehat{\mathcal{C}}$.*

It is interesting to consider only SSP IMEX linear multistep methods.

Optimal methods have been found for a range of k,p and for different values of ξ.
But, have small threshold factors; not practically useful.

Instead impose SSP conditions only on the explicit method and maximize $A(\alpha)$ stability region of the implicit method.

Outline

Perturbed Runge–Kutta methods

Assume that

$$\|\boldsymbol{u} + h\boldsymbol{F}(\boldsymbol{u})\| \leq \|\boldsymbol{u}\|, \ \|\boldsymbol{u} - h\widetilde{\boldsymbol{F}}(\boldsymbol{u})\| \leq \|\boldsymbol{u}\| \quad \forall \boldsymbol{u} \in \mathbb{R}^m, \ 0 \leq h \leq h_{\mathsf{FE}}.$$

A downwind-biased (or **perturbed**) Runge–Kutta method takes the form

$$\boldsymbol{y} = v_r \boldsymbol{u}^{n-1} + h\boldsymbol{K}\boldsymbol{F} + h\widetilde{\boldsymbol{K}}(\boldsymbol{F} - \widetilde{\boldsymbol{F}}), \quad \widetilde{\boldsymbol{K}} = \begin{pmatrix} \widetilde{\boldsymbol{A}} & 0 \\ \tilde{\boldsymbol{b}}^{\mathsf{T}} & 0 \end{pmatrix}.$$

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For example:

$$\begin{aligned} \boldsymbol{y}_1 &= \boldsymbol{u}^{n-1} + \frac{2}{3}h\boldsymbol{F}(\boldsymbol{u}^{n-1}), \\ \boldsymbol{u}^n &= \frac{5}{8}\boldsymbol{u}^{n-1} + \frac{3}{8}\boldsymbol{y}_1 + \frac{3}{4}h\boldsymbol{F}(\boldsymbol{y}_1). \end{aligned}$$

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$$\begin{aligned} \boldsymbol{y}_1 &= \frac{5}{6}\big(\boldsymbol{u}^{n-1} + h\boldsymbol{F}(\boldsymbol{u}^{n-1})\big) + \frac{1}{6}\big(\boldsymbol{u}^{n-1} - h\widetilde{\boldsymbol{F}}(\boldsymbol{u}^{n-1})\big), \\ \boldsymbol{u}^n &= \frac{3}{4}\big(\boldsymbol{y}_1 + h\boldsymbol{F}(\boldsymbol{y}_1)\big) + \frac{1}{4}\big(\boldsymbol{u}^{n-1} - h\widetilde{\boldsymbol{F}}(\boldsymbol{u}^{n-1})\big). \end{aligned}$$

Monotonicity and optimality of perturbed RK

Perturbed Runge–Kutta methods:

- Introduced by Shu & Osher (1988) and further studied by Gottlieb, Ruuth, Spiteri and others.
- Analysis of monotonicity conditions, Shu-Osher representations, and extension to additive problems investigated by Higueras (2005, 2006).
- Algorithms to obtained optimal perturbations and upper bounds on SSP coefficient were developed by Higueras/Ketcheson/Kocsis (2016).
- Formulae for second-order implicit methods with unbounded SSP coefficient (Ketcheson, 2012).

Outline

Implicit perturbed Runge–Kutta methods

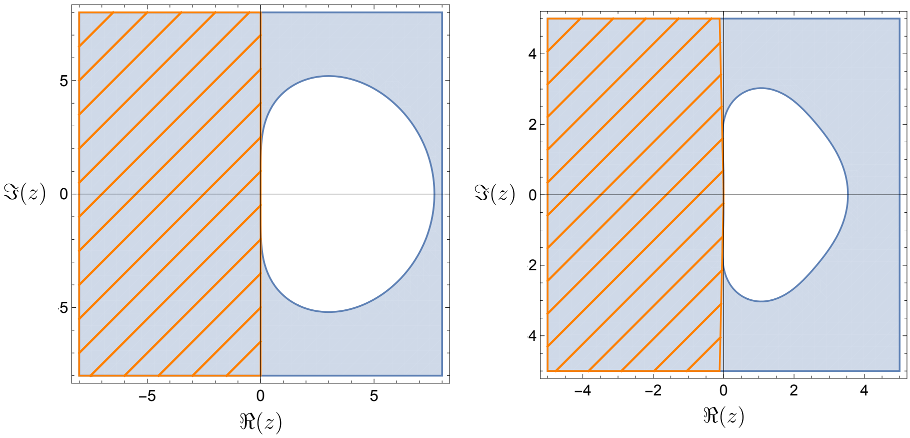
A new three-step, third-order class of implicit perturbed Runge–Kutta methods, with arbitrarily large SSP coefficient $\mathcal{C} = r$:

$$\begin{aligned}
\boldsymbol{y}_1 &= v_1 \boldsymbol{u}^{n-1} + \alpha_{11}\left(\boldsymbol{y}_1 + \frac{h}{r}\boldsymbol{F}(\boldsymbol{y}_1)\right) \\
&\quad + \alpha_{21}\left(\boldsymbol{y}_2 + \frac{h}{r}\boldsymbol{F}(\boldsymbol{y}_2)\right) + \tilde{\alpha}_{13}\left(\boldsymbol{y}_3 - \frac{h}{r}\widetilde{\boldsymbol{F}}(\boldsymbol{y}_3)\right) \\
\boldsymbol{y}_2 &= \alpha_{21}\left(\boldsymbol{y}_1 + \frac{h}{r}\boldsymbol{F}(\boldsymbol{y}_1)\right) + \alpha_{22}\left(\boldsymbol{y}_2 + \frac{h}{r}\boldsymbol{F}(\boldsymbol{y}_2)\right) \\
\boldsymbol{y}_3 &= \boldsymbol{y}_1 + \frac{h}{r}\boldsymbol{F}(\boldsymbol{y}_1) \\
\boldsymbol{u}^n &= \boldsymbol{y}_2 + \frac{h}{r}\boldsymbol{F}(\boldsymbol{y}_2).
\end{aligned}$$

Stability analysis of the underlying method (i.e. when $\widetilde{\boldsymbol{F}} = \boldsymbol{F}$) reveals that:

- if $r = 6$ then the method is A-stable;
- if $r > 6$ then the method is A(α)-stable with $\alpha \geq 88.2302$.

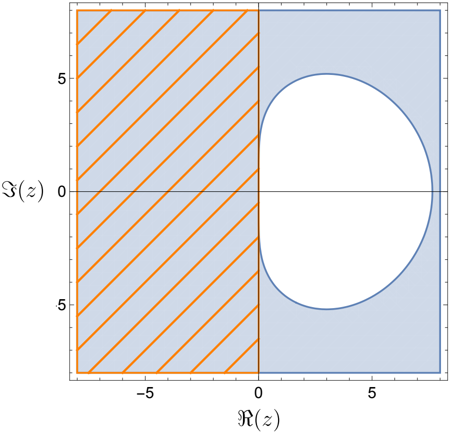
Stability regions



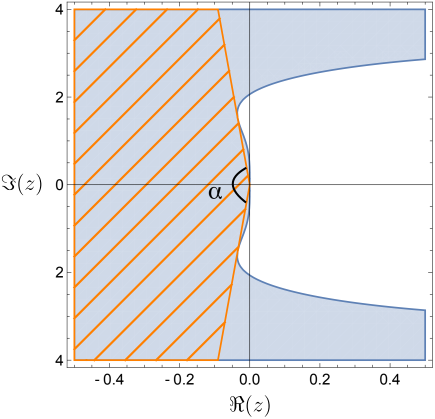
$r = 6$ $r = 12$

Stability regions and A(α)-stability wedges.

Stability regions



$r = 6$



$r = 12$, $\alpha = 88.7015^o$.

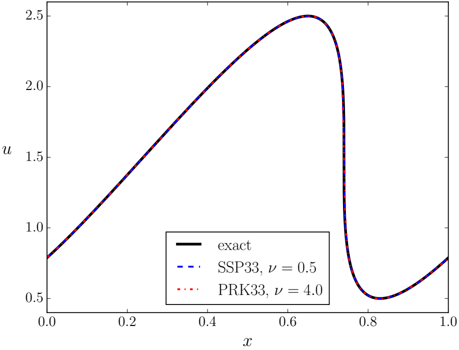
Stability regions and $A(\alpha)$-stability wedges.

Application to 1D problems

Solution of Burgers' equation with 2nd-order TVD spatial discretization.

Time integrators:

- explicit SSP RK$(3,3)$ (SSPRK33)
- implicit perturbed SSPRK$(3,3)$ with $r = 8$ (PRK33)



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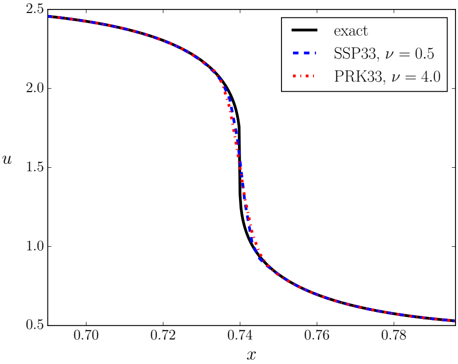

Closeup view of the shock

Outline

Conclusion and future work

Additive SSP linear multistep methods:

- Extended SSP theory of LMMs to problems where upwind and downwind operators have different stiffness properties.
- Analyzed monotonicity properties of perturbed SSP LMMs and construct optimal methods.
- Investigated monotonicity properties of additive linear multistep methods: SSP IMEX methods.

Future work:

- Study asymptotic behavior of SSP coefficient for perturbed methods.
- Perturbed SSP LMMs with variable step size.
- Find optimal IMEX methods:
  (explicit perturbed SSP LMM + $A(\alpha)$-stable implicit LMM).

Conclusion and future work (cont.)

Implicit perturbed SSP Runge–Kutta methods:

- Obtained a third-order implicit RK method with arbitrarily large SSP coefficient.
- Analyzed stability properties.
- Showed good performance with large CFL numbers.

Future work:

- Efficient implementation in relation to Newton iterations required at each step.
- Search for other families of higher order implicit perturbed RK methods.

References

- Hadjimichael, Y. and Ketcheson, D. I., *Strong stability preserving additive linear multistep methods*, (2017), accepted in Math. Comp.
  https://github.com/numerical-mathematics/ssp-almm_RR
- Hadjimichael, Y., Ketcheson, D. I., Lóczi, L., and Németh, A., *Strong Stability Preserving Explicit Linear Multistep Methods with Variable Step Size*, SIAM J. Numer. Anal. 54 (2016), pp. 2799-2832.
  https://github.com/numerical-mathematics/ssp-lmm-vss_RR
- Mandli, K. T., et.al., *Clawpack: building an open source ecosystem for solving hyperbolic PDEs*, PeerJ Computer Science 2:e68 (2016).
- Hadjimichael, Y., Macdonald, C. B., Ketcheson, D. I., and Verner, J. H., *Strong stability preserving explicit Runge-Kutta methods of maximal effective order*, SIAM J. Numer. Anal. 51 (2013), pp. 2149-2165.
  https://bitbucket.org/hadjimy/effective-order-ssprk

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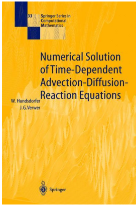
Expert in SSP time discretization methods with a huge contribution to the development of TVD/TVB LMMs and RK methods, IMEX methods, monotone multirate and partitioned RK methods, splitting methods, etc.

Among other research interests he worked on

- Stiff ODEs,
- Time-dependent PDEs,
- Streamer simulations for multiscale dynamical problems



https://www.cwi.nl/news/2017/in-memoriam-willem-hundsdorfer