Bifurcation and discretization in integrodifference equations

Christian Pötzsche

Alpen-Adria Universität Klagenfurt
christian.poetzsche@aau.at

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Farkas Miklós Alkalmazott Analízis Szeminárium

Joint work with Christian Aarset

Motivation: Scalar difference equations

Step 1: Temporal evolution of the total biomass of a population

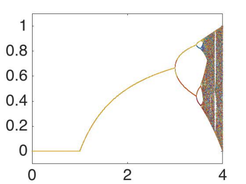
$$\boxed{u_{t+1} = g(u_t)}$$

in one single habitat with nonoverlapping generations

$g : \mathbb{R} \to \mathbb{R}$ depends on the growth rate and carrying capacity

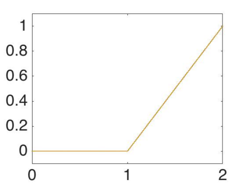
logistic

$g(u) := ru(1-u)$



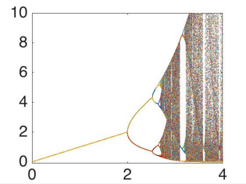
Beverton-Holt

$g(u) := \frac{ru}{1+u}$



Ricker

$g(u) := ue^{r-u}$



Motivation: Spatial difference equations

Growth rate and carrying capacity are different in every point of the habitat

Step 2: Temporal evolution of a sedentary population

$$u_{t+1}(x) = g(x, u_t(x)) \quad \text{for all } x \in \Omega \qquad (S)$$

over the habitat $\Omega \subset \mathbb{R}^\kappa$, $\kappa = 1, 2, 3$

$g : \Omega \times \mathbb{R} \to \mathbb{R}$ describes the evolution depending on the position $x \in \Omega$ in the habitat

Perspectives

- $x \in \Omega$ is a parameter, or
- (S) is an equation in a function space

$$u_{t+1} = \mathcal{G}(u_t)$$

Motivation: Dispersal

Populations disperse!

Step 3: Temporal evolution of a dispersive population

$$\boxed{u_{t+1}(x) = \int_\Omega k(x,y)g(y,u_t(y))\,\mathrm{d}y} \quad \text{for all } x \in \Omega \qquad (H)$$

in a habitat $\Omega \subseteq \mathbb{R}^\kappa$, $\kappa = 1, 2, 3$

Interpretation of (H)

- dispersal kernel $k(x,y) \geq 0$ indicate probability to move from x to $y \in \Omega$
- (Hammerstein) integrodifference equation
  State space: $u_t \in C(\overline{\Omega})$ or $L^p(\Omega)$, $1 \leq p \leq \infty$

Motivation: Types of kernels

Integrodifference equations are flexible

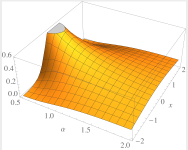
- convolution type

$$k(x,y) = k_0(|x-y|)$$

- flexibility via various kernels

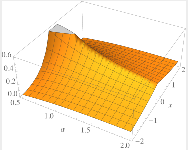
Gauß

$$k_0(x) := \frac{1}{\sqrt{2\pi a^2}} \exp\left(-\frac{1}{2a^2}x^2\right)$$



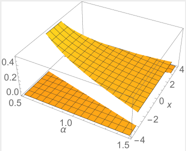
Laplace

$$k_0(x) := \frac{1}{2a} \exp\left(-\frac{1}{a}|x|\right)$$



top hat (finite radius dispersal)

$$k_0(x) := \frac{1}{4a}\chi_{[-a,a]}(x)$$



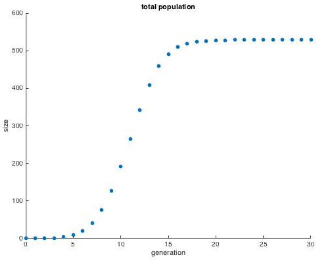
Motivation

Gauß kernel $a = 1$

Beverton-Holt equation

$$u_{t+1} = \int_\Omega k(\cdot, y)\frac{ru_t(y)}{1+u_t(y)}\,\mathrm{d}y$$

Total population



Global convergence

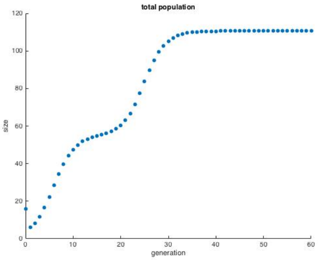
Motivation

Gauß kernel $a = 1.4$

Beverton-Holt equation

$$u_{t+1} = \int_\Omega k(\cdot, y) \frac{ra(y)u_t(y)}{1+u_t(y)}\,\mathrm{d}y$$

Total population



Global convergence

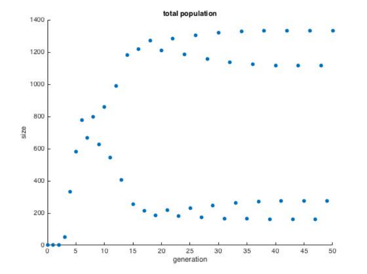
Motivation

Gauß kernel

Ricker equation

$$u_{t+1} = \int_\Omega k(\cdot, y) u_t(y) e^{r(1-u_t(y))}\,\mathrm{d}y$$

Total population



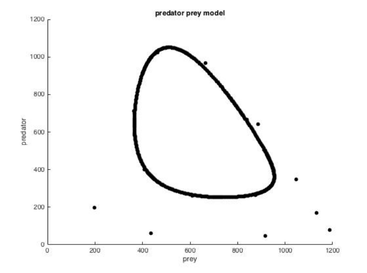
Period doubling cascade

Motivation

Gauß kernel

Predator-prey model

Total population



Invariant circle

Motivation: Punchline

Integrodifference equations

$$u_{t+1} = \int_\Omega k_t(\cdot, y) g_t(u_t(y))\, \mathrm{d}y \qquad (I)$$

Interesting infinite-dimensional discrete dynamical systems:

- M. Kot, W. Schaffer[a], 1986
- D. Hardin, P. Takáč, G. Webb[b], 1988
- S. Day, O. Junge, K. Mischaikow[c], 2004

[a] *Discrete-time growth dispersal models*, Math. Biosci. 80

[b] *A comparison of dispersal strategies for survival of spatially heterogeneous populations*, SIAM J. Appl. Math. 48(6)

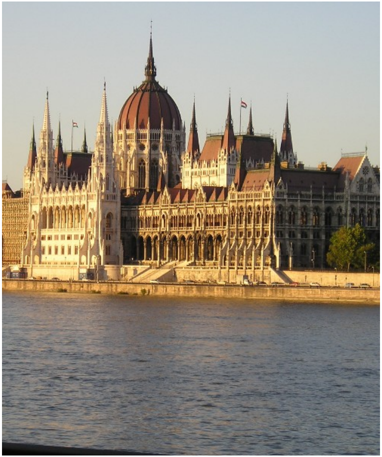
[c] *A rigerous numerical method for the global dynamics of infinite-dimensional discrete dynamical systems*, SIAM J. Appl. Dyn. Syst. 3(2)

1 A flexible class of integrodifference equations

Goals

Introduce a sufficiently rich class of integrodifference equations

1. equations of Urysohn- and Hammerstein-type
2. growth-dispersal and dispersal-growth equations
3. relevant special cases (modelling, discretization)



1 A flexible class of integrodifference equations

A general IDE model

An integrodifference equation (IDE for short) is a difference equation

$$\boxed{u_{t+1} = \mathcal{F}_t(u_t)} \qquad (I)$$

whose right-hand side $\mathcal{F}_t : X \to X$ involves an integral operator on an ambient function space X.

Given a measure space $(\Omega, \mathfrak{A}, \mu)$ with $\mu(\Omega) < \infty$ consider

$$\mathcal{F}_t(u)(x) := G_t\left(x, u(x), \int_\Omega f_t(x, y, u(y))\,\mathrm{d}\mu(y)\right) \quad \text{for all } t \in \mathbb{Z},\ x \in \Omega \qquad (*)$$

1 A flexible class of integrodifference equations

State space $X = C(\Omega)$

General form $(*)$ (nonlinear Urysohn equations), Ω compact metric space

1. Well-posed under continuity and Carathéodory conditions on f_t
2. Smooth under corresponding conditions on $D_{(2,3)}G_t$, $D_3 f_t$
3. Complete continuity

$$u_{t+1}(x) = G_t\left(x, \int_\Omega f_t(x, y, u_t(y))\,\mathrm{d}\mu(y)\right)$$

or set contraction

State space $X = L^p(\Omega)$

Hammerstein equations

$$u_{t+1} = \int_\Omega k_t(\cdot, y) g_t(y, u_t(y))\,\mathrm{d}\mu(y)$$

1. Well-posed under growth conditions on g_t and Hille-Tamarkin conditions on k_t
2. Smooth under growth conditions on $D_2 g_t$
3. Complete continuity

1 A flexible class of integrodifference equations

$$\mathcal{F}_t(u)(x) = G_t\left(x, u(x), \int_\Omega f_t(x, y, u(y))\,\mathrm{d}\mu(y)\right)$$

Example (Urysohn and Hammerstein equations)

Suppose $\Omega \subset \mathbb{R}^\kappa$ is compact, $\mu = \lambda_\kappa$ (Lebesgue measure)

- Urysohn equations: $u_{t+1} = \int_\Omega f_t(\cdot, y, u_t(y))\,\mathrm{d}y$
- Hammerstein equations: $u_{t+1} = \int_\Omega k_t(\cdot, y) g_t(y, u_t(y))\,\mathrm{d}y$
- dispersal-growth equations: $u_{t+1} = g_t\left(\cdot, \int_\Omega k_t(\cdot, y) u_t(y)\,\mathrm{d}y\right)$

1 A flexible class of integrodifference equations

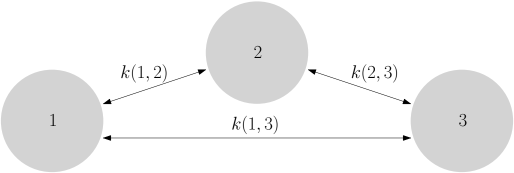
$$\mathcal{F}_t(u)(x) = \int_\Omega k_t(x,y) g_t(y, u(y))\, \mathrm{d}\mu(y)$$

Example (metapopulation models)

With $\Omega = \{1, \dots, n\}$ and the counting measure μ consider

$$u_{t+1}(i) = \sum_{j=1}^{n} k_t(i,j) g_t(j, u_t(j)) \quad \text{for all } i \in \{1, \dots, n\},$$

where $k_t(i,j)$ yields the probability to move from patch i to j



1 A flexible class of integrodifference equations

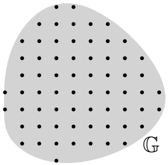
$$\mathcal{F}_t(u)(x) = G_t\left(x, u(x), \int_\Omega f_t(x, y, u(y))\,\mathrm{d}\mu(y)\right)$$

Example (Nyström discretizations)

On a grid $\Omega = \mathbb{G}$ the measure $\mu(\Omega') := \sum_{\eta\in\Omega'} w_\eta$ yields the recursions

$$u_{t+1}(\xi) = G_t\left(\xi, u(\xi), \sum_{\eta\in\mathbb{G}} \omega_\eta f_t(\cdot, \eta, u_t(\eta))\right) \quad \text{for all } \xi \in \mathbb{G}.$$

They realize quadrature/cubature rules with weights $\omega_\eta \geq 0$ and nodes $\eta \in \mathbb{G}$



2 Bifurcation from simple multipliers

Goals

Periodic solutions of periodic IDEs bifurcate into periodic solutions:

1. lifted map vs. period map (cyclic maps instead of compositions)
2. crossing curve bifurcations

2 Bifurcation from simple multipliers

A parameter-dependent integrodifference equation

$$\boxed{u_{t+1} = \mathcal{F}_t(u_t, \alpha)} \qquad (I_\alpha)$$

with right-hand side $\mathcal{F}_t : X \times A \to X$ on an ambient function space X and a real parameter space $A \subseteq \mathbb{R}$ is assumed to be θ_0-periodic, i.e.

$$\mathcal{F}_t = \mathcal{F}_{t+\theta_0} \quad \text{for all } t \in \mathbb{Z}.$$

Theorem (characterization of periodic solutions)

If θ is a multiple of θ_0, then the following are equivalent:

1. $\phi = (\phi_t)_{t\in\mathbb{Z}}$ *is a θ-periodic solution of* (I_α)
2. $\phi_0 \in X$ *is a fixed-point of the period map* $\Pi_\theta = \mathcal{F}_{\theta-1} \circ \ldots \circ \mathcal{F}_0$
3. $\hat{\phi} = (\phi_1, \ldots, \phi_\theta)$ *is a zero of the lifted map*

$$G(\hat{\phi}, \alpha) := \begin{pmatrix} \phi_1 - \mathcal{F}_0(\phi_\theta, \alpha) \\ \phi_2 - \mathcal{F}_1(\phi_1, \alpha) \\ \vdots \\ \phi_\theta - \mathcal{F}_{\theta-1}(\phi_{\theta-1}, \alpha) \end{pmatrix} \in X^\theta$$

2 Bifurcation from simple multipliers

Period map (for Urysohn equations):

$$\begin{aligned}\Pi_\theta(u) &= \mathcal{F}_\theta \circ \ldots \circ \mathcal{F}_1(u) \\ &= \int_\Omega f_\theta\left(\cdot, y_\theta, \cdots \int_\Omega f_2\left(y_3, y_2, \int_\Omega f_1(y_2, y_1, u(y_1))\,\mathrm{d}y_1\right)\mathrm{d}y_2 \cdots\right)\mathrm{d}y_\theta\end{aligned}$$

Period map vs. lifted map

Applied to IDEs (I_α) the lifted map G

1. avoids to evaluate multiple integrals (and an application of cubature rules over high-dimensional domains)
2. yields cyclic as opposed to product eigenvalue problems, which have better numerical stability properties (Kressner[a], 2006)

[a] *The periodic QR algorithm is a disguised QR algorithm*, Linear Algebra and its Applications 417(2–3)

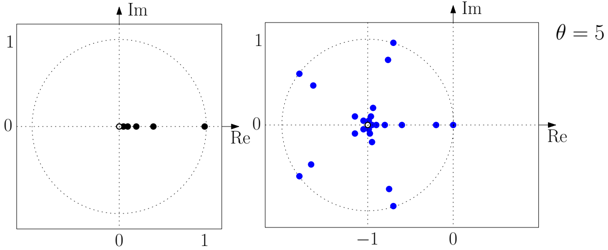
2 Bifurcation from simple multipliers

Structure of the (point) spectrum

If $\theta \in \mathbb{N}$ is a multiple of θ_0, then

$$\sigma_p(D_1 G(\hat{\phi}^*, \alpha^*)) = \left\{\lambda - 1 \in \mathbb{C} : \lambda^\theta \in \sigma_p(\Xi_\theta(\alpha^*))\right\}$$

with $\Xi_\theta(\alpha^*) := D_1 \mathcal{F}_{\theta-1}(\phi^*_{\theta-1}, \alpha^*) \cdots D_1 \mathcal{F}_0(\phi^*_0, \alpha^*)$



2 Bifurcation from simple multipliers

Duality pairings (Kress[a], 2014)

[a] *Linear Integral Equations*, Springer

Two Banach spaces X and Y together with a (nondegenerate) bilinear form $\langle \cdot, \cdot \rangle : Y \times X \to \mathbb{R}$ are called a duality pairing $\langle Y, X \rangle$. Given $T \in L(X)$, the dual operator $T' \in L(Y)$ is determined via

$$\langle y, Tx \rangle = \langle T' y, x \rangle \quad \text{for all } x \in X,\ y \in Y.$$

Example

1. $\langle X', X \rangle$ with the duality pairing $\langle x', x \rangle = x'(x)$
2. $\langle L^p(\Omega), L^{p'}(\Omega) \rangle$ with $\langle u, v \rangle := \int_\Omega u(y)v(y)\,\mathrm{d}\mu(y)$ and $\frac{1}{p} + \frac{1}{p'} = 1$, $p > 1$
3. $\langle C(\Omega), C(\Omega) \rangle$ with the bilinear form $\langle u, v \rangle := \int_\Omega u(y)v(y)\,\mathrm{d}\mu(y)$

2 Bifurcation from simple multipliers

$$u_{t+1} = \mathcal{F}_t(u_t, \alpha^*), \qquad \mathcal{F}_{t+\theta_0} = \mathcal{F}_t \qquad (I_{\alpha^*})$$

Assumptions

1. ϕ^* is a θ_1-periodic solution of (I_{α^*})
2. 1 is a simple Floquet multiplier: There exists nonzero $\xi_0^* \in C(\Omega)$ with $\Xi_\theta(\alpha^*)\xi_0^* = \xi_0^*$ obtained from the cyclic eigenvalue problem

$$\begin{pmatrix} D_1\mathcal{F}_0(\phi_0^*, \alpha^*)\xi_0^* \\ D_1\mathcal{F}_1(\phi_1^*, \alpha^*)\xi_1^* \\ \vdots \\ D_1\mathcal{F}_{\theta-1}(\phi_{\theta-1}^*, \alpha^*)\xi_{\theta-1}^* \end{pmatrix} = 1 \begin{pmatrix} \xi_1^* \\ \xi_2^* \\ \vdots \\ \xi_0^* \end{pmatrix}$$

and choose $\eta_0^* \in C(\Omega)$ such that

$$N(\Xi_\theta(\alpha^*)' - I_{C(\Omega)}) = R(\Xi_\theta(\alpha^*) - I_{C(\Omega)})^\perp = \operatorname{span}\{\eta_0^*\}$$

Bilinear form $\langle\hat{\phi}, \hat{\psi}\rangle_\theta := \sum_{t=0}^{\theta-1} \int_\Omega \langle\phi_t(x), \psi_t(x)\rangle \, \mathrm{d}\mu(x)$

2 Bifurcation from simple multipliers

Theorem (crossing curve bifurcation)

Let $\theta = \operatorname{lcm}\{\theta_0, \theta_1\}$. *If* $D_2\mathcal{F}_t(\phi_t^*, \alpha^*) = 0$, *the transversality condition*

$$g_{11} := \sum_{t=0}^{\theta-1} \int_\Omega \langle \eta_{t+1}^*(x), [D_1 D_2 \mathcal{F}_t(\phi_t^*, \alpha^*)\xi_t^*](x)\rangle \, \mathrm{d}\mu(x) \neq 0$$

and

$$\sum_{t=0}^{\theta-1} \int_\Omega \langle \eta_{t+1}^*(x), [D_2^2 \mathcal{F}_t(\phi_t^*, \alpha^*)](x)\rangle \, \mathrm{d}\mu(x) = 0$$

hold, then the θ*-periodic solution* ϕ^* *of an IDE* (I_{α^*}) *bifurcates at* α^* *as follows:* (ϕ^*, α^*) *is the intersection of two branches* Γ_1, Γ_2 *of* θ*-periodic solutions and every* θ*-periodic solution of* (I_α) *in* $B_\varepsilon(\phi^*)$ *is captured by* Γ_1 *or* Γ_2.

Period doubling: Apply theorem to the equation with period 2θ

Proof.

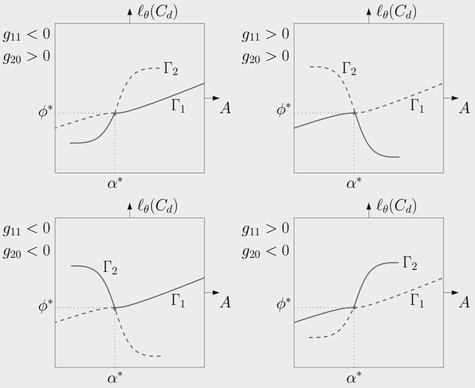
Apply Morse lemma (instead of implicit function theorem) to the reduced equation after the Lyapunov-Schmidt reduction. □

2 Bifurcation from simple multipliers

Corollary (transcritical bifurcation)

Under the additional assumption

$$g_{20} := \sum_{t=0}^{\theta-1} \int_\Omega \langle \eta^*_{t+1}(x), [D_1^2 \mathcal{F}_t(\phi^*_t, \alpha^*)(\xi^*_t)^2](x) \rangle \, \mathrm{d}\mu(x) \neq 0$$



2 Bifurcation from simple multipliers

Corollary (pitchfork bifurcation)

Suppose $\psi \in \ell_\theta(C_d)$ is the uniquely determined solution of the cyclic Fredholm equations of the second kind

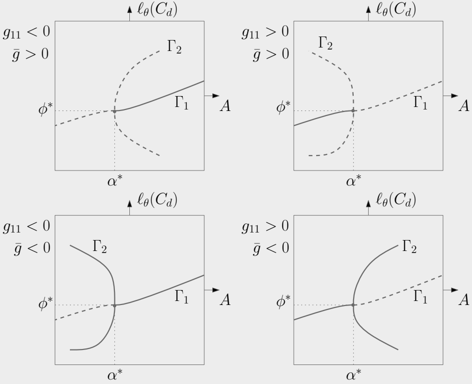
$$\begin{cases} \psi_0 = D_1\mathcal{F}_{\theta-1}(\phi^*_{\theta-1},\alpha^*)\psi_{\theta-1} + D_1^2\mathcal{F}_{\theta-1}(\phi^*_{\theta-1},\alpha^*)(\xi^*_{\theta-1})^2, \\ \psi_1 = D_1\mathcal{F}_0(\phi^*_0,\alpha^*)\psi_0 + D_1^2\mathcal{F}_0(\phi^*_0,\alpha^*)(\xi^*_0)^2, \\ \quad \vdots \\ \psi_{\theta-1} = D_1\mathcal{F}_{\theta-2}(\phi^*_{\theta-2},\alpha^*)\psi_{\theta-2} + D_1^2\mathcal{F}_{\theta-2}(\phi^*_{\theta-2},\alpha^*)(\xi^*_{\theta-2})^2, \\ 0 = \sum_{t=0}^{\theta-1}\int_\Omega \langle \eta^*_{t+1}(x), \psi_t(x)\rangle \,\mathrm{d}\mu(x). \end{cases} \tag{1}$$

Under the additional assumptions $g_{20} = 0$,

$$\begin{aligned} \bar{g} := &\sum_{t=0}^{\theta-1}\int_\Omega \langle \eta^*_{t+1}(x), [D_1^3\mathcal{F}_t(\phi^*_t,\alpha^*)(\xi^*_t)^3](x)\rangle \,\mathrm{d}\mu(x) \\ &+ 3\sum_{t=0}^{\theta-1}\int_\Omega \langle \eta^*_{t+1}(x), [D_1^2\mathcal{F}_t(\phi^*_t,\alpha^*)\xi^*_t\psi_t](x)\rangle \,\mathrm{d}\mu(x) \neq 0 \end{aligned}$$

2 Bifurcation from simple multipliers

Corollary (pitchfork bifurcation)



In both cases, an exchange of stability principle holds.

2 Bifurcation from simple multipliers

Numerical aspects

- Numerical bifurcation theory
  - Solution branches (pseudo-arclength continuation with Newton method as corrector)
  - eigenpairs (`Matlab` solver for linear and eigenvalue equations)
- Integral operators and inner products
  - ... Part 3

2 Bifurcation from simple multipliers

Example (Beverton-Holt equation)

$$u_{t+1} = \alpha_t \int_\Omega k(\cdot, y) \frac{\alpha u_t(y)}{1 + u_t(y)} \, \mathrm{d}y$$

with θ-periodic sequence $(\alpha_t)_{t \in \mathbb{Z}}$ in $(0, \infty)$ and bifurcation parameter $\alpha > 0$

(1) Primary transcritical bifurcation into a branch of globally attractive (Hardin, Takáč, Webb[a], 1988) (w.r.t. $C_+(\Omega)$) θ-periodic solutions

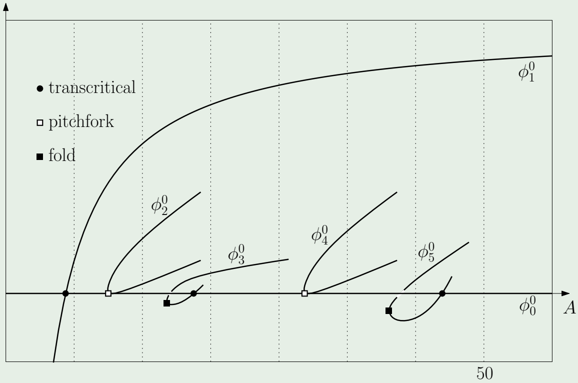
(2) Countable number of bifurcations along 0, which are alternately of transcritical and supercritical pitchfork type

(3) Transcritical bifurcations come from a supercritical fold

[a] *A comparison of dispersal strategies for survival of spatially heterogeneous populations*, SIAM J. Appl. Math. 48(6)

2 Bifurcation from simple multipliers

Example (Beverton-Holt equation)



2 Bifurcation from simple multipliers

Example (Ricker equation)

$$u_{t+1} = \alpha_t \int_\Omega k(\cdot, y)\alpha u_t(y)e^{-u_t(y)}\,\mathrm{d}y$$

with θ-periodic sequence $(\alpha_t)_{t\in\mathbb{Z}}$ in $(0, \infty)$ and bifurcation parameter $\alpha > 0$

(1) Primary transcritical bifurcation into a branch of θ-periodic solutions

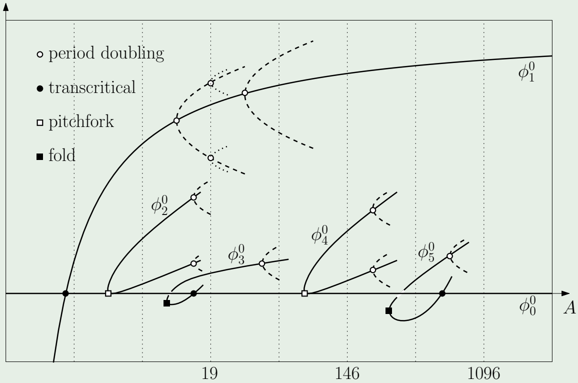
(2) Countable number of bifurcations along 0, which are alternately of transcritical and supercritical pitchfork type

(3) Transcritical bifurcations come from a supercritical fold

(4) Period doublings at all branches; period doubling cascade along the primary branch ϕ_1^0

2 Bifurcation from simple multipliers

Example (Ricker equation)

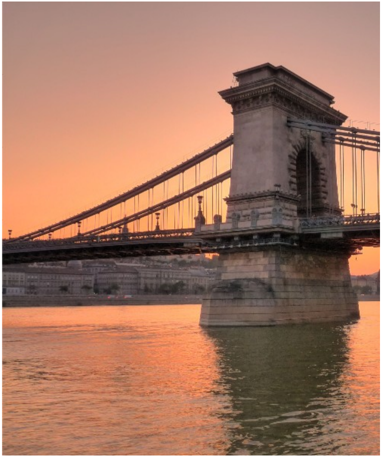


3 Numerical dynamics

Goals

Behavior of hyperbolic solutions under full discretization:

1. Nyström discretizations of IDEs
2. periodic solutions and their saddle-point structure
3. global asymptotic stability



3 Numerical dynamics

Simulations of (I) require spatial discretizations:

$$\boxed{u_{t+1} = \mathcal{F}_t^n(u_t)} \qquad (I_n)$$

Nyström methods

Replace the integral in (I) by a convergent quadrature/cubature rule

$$\mathcal{F}_t^n(u)(\eta_i) := G_t\left(\eta_i, \sum_{j=1}^{q_n} w_j f_t(\eta_i, \eta_j, u(\eta_j))\right) \quad \text{for all } 1 \leq i \leq q_n \qquad (2)$$

with $q_n \in \mathbb{N}$, the nodes $\eta_j \in \Omega$ and weights $w_j \geq 0$

Newton-Cotes (with positive weights), Gauß, etc.

3 Numerical dynamics

What is numerical dynamics?

1. **Persistence**: Is a certain dynamical property of

$$\boxed{u_{t+1} = \mathcal{F}_t(u_t)} \qquad (I)$$

preserved under discretization (I_n) (periodic solution, bifurcation, attractor, etc.)?

2. **Convergence**: Does convergence of this particular property as $n \to \infty$ hold (preserving the convergence rate)?

3. **"Shadowing"**: Can one conclude from properties of the discretization (I_n) to the original equation (I)?

Kloeden & Lorenz^{1986}, Beyn^{1986}, Garay^{1990s}, ...,
Stuart & Humphries^{1998}, ...

3 Numerical dynamics: Intrinsic problem

Given the difference equations

$$u_{t+1} = \mathcal{F}_t(u_t), \qquad \varphi(t;\tau,\cdot) = \mathcal{F}_{t-1} \circ \ldots \circ \mathcal{F}_\tau,$$
$$u_{t+1} = \mathcal{F}_t^n(u_t), \qquad \varphi^n(t;\tau,\cdot) = \mathcal{F}_{t-1}^n \circ \ldots \circ \mathcal{F}_\tau^n$$

define $L := \limsup_{t\to\infty} \operatorname{Lip} \mathcal{F}_t$

Global discretization error

$$\|\varphi(t;\tau,u) - \varphi^n(t;\tau,u)\| \leq C(u)\frac{L^{t-\tau}-1}{L-1}\|\mathcal{F}_t^n(u) - \mathcal{F}_t(u)\| \quad \text{for all } \tau \leq t$$

and $u \in X$ satisfies

- $\lim_{n\to\infty} \|\varphi(t;\tau,u) - \varphi^n(t;\tau,u)\| = 0$ for every fixed $t \geq \tau$
- but the error bound explodes for $t \to \infty$

Classical error estimate are useless to infer asymptotic properties!

3 Numerical dynamics: Problem with Nyström methods

Nyström methods

+ Our abstract bifurcation results apply to Nyström discretizations (choose suitable measure in the integrals) (persistence)
– For convergent quadrature/cubature rules, linear Fredholm integral operators

$$\mathcal{K} \in L(C(\Omega)), \qquad \mathcal{K}v := \int_\Omega k(\cdot, y)v(y)\,\mathrm{d}y$$

feature strong convergence

$$\lim_{n\to\infty} \|\mathcal{K}v - \mathcal{K}^n v\| = 0 \quad \text{for all } v \in C(\Omega),$$

but not uniform convergence. It even holds

$$\|\mathcal{K}\| \leq \|\mathcal{K} - \mathcal{K}^n\| \quad \text{for all } n \in \mathbb{N}.$$

3 Numerical dynamics

On the state space $X = C(\Omega)$ consider an IDE (I) along with a Nyström discretization

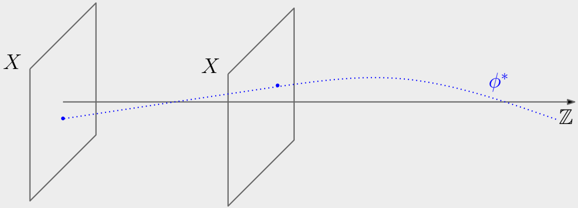
$$\boxed{u_{t+1} = \mathcal{F}_t^n(u_t)} \qquad (I_n)$$

being pointwise convergent, i.e.

$$\|\mathcal{F}_t(u) - \mathcal{F}_t^n(u)\| \leq \omega(\tfrac{1}{n}, \mathcal{F}_t(u)) \quad \text{for all } n \in \mathbb{N}$$

and a function ω satisfying $\lim_{\rho \searrow 0} \omega(\rho, u) = 0$.

Suppose that (I) has a θ_1-periodic solution $\phi^* = (\phi_t^*)_{t\in\mathbb{Z}}$ and set $\theta = \operatorname{lcm}\{\theta_0, \theta_1\}$.



3 Numerical dynamics

Theorem (hyperbolic periodic solutions under Nyström discretization)

If a θ_0-periodic solution $(\phi_t^*)_{t\in\mathbb{Z}}$ to a θ_1-periodic IDE (I) over $\Omega \subset \mathbb{R}^\kappa$ satisfies

$$\sigma(D\mathcal{F}_\theta(\phi_\theta^*)\cdots D\mathcal{F}_1(\phi_1^*)) \cap \mathbb{S}^1 = \emptyset,$$

then there exist $N \in \mathbb{N}$, $C > 0$ so that also (I_n) has a (locally) unique θ-periodic and hyperbolic solution $(\phi_t^n)_{t\in\mathbb{Z}}$; it satisfies

$$\|\phi_t^* - \phi_t^n\| \leq C \sup_{s=1}^{\theta} \omega(\tfrac{1}{n}, \mathcal{F}_s(\phi_s^*)) \quad \text{for all } n \geq N,\ t \in \mathbb{Z}.$$

Proof.

Qualitative implicit function theorem (Weiss[a], 1974) □

[a] *On the approximation of fixed points of nonlinear compact operators*, SIAM J. Numer. Anal. 11(3)

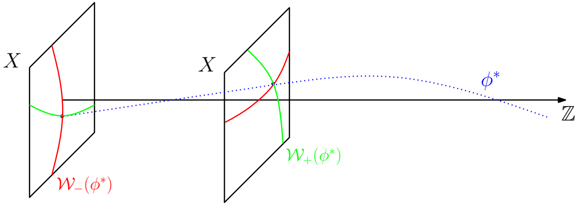
3 Numerical dynamics

The unstable fiber bundle (manifold)

$$\mathcal{W}_-(\phi^*) := \left\{ (\tau, u) \in \mathbb{Z} \times X : \begin{array}{l} \text{there exists a solution} \\ (\psi_t)_{t \leq \tau} \text{ of } (I) \text{ with } \psi_\tau = u, \\ \lim_{t \to -\infty} \|\psi_t - \phi_t^*\| = 0 \end{array} \right\}$$

is locally graph of a function w_- over a finite-dimensional space $\mathcal{V}_-$, i.e.

$$\mathcal{W}_-(\phi^*) = \phi^* + \{(\tau, w_-(\tau, u)) : (\tau, u) \in \mathcal{V}_- \text{ and } \|u\| < \rho\}$$



Similarly for the stable fiber bundle (manifold) $\mathcal{W}_+(\phi^*)$

3 Numerical dynamics

Theorem (unstable fiber bundles under Nyström discretization)

If a θ_0-periodic solution $(\phi_t^*)_{t\in\mathbb{Z}}$ to a θ_1-periodic IDE (I) over $\Omega \subset \mathbb{R}^\kappa$ satisfies

$$\sigma(D\mathcal{F}_\theta(\phi_\theta^*)\cdots D\mathcal{F}_1(\phi_1^*)) \cap \mathbb{S}^1 = \emptyset,$$

then the associate unstable fiber bundle $\mathcal{W}_-$ is θ-periodic in τ. Moreover, there exists a $N \in \mathbb{N}$ so that the following holds for $n \geq N$:

1. *Also the solution ϕ^n to (I_n) has an unstable fiber bundle $\mathcal{W}_-^n(\phi^n)$ being θ-periodic in τ and*

$$\mathcal{W}_-^n(\phi^n) = \phi^n + \{(\tau, w_-^n(\tau, u)) : (\tau, u) \in \mathcal{V}_-^n \text{ and } \|u\| < \rho\}$$

2. $\dim \mathcal{V}_- = \dim \mathcal{V}_-^n < \infty$ *for all $n \geq N$*
3. *The functions describing $\mathcal{W}_-$ and $\mathcal{W}_-^n$ are related by*

$$\sup_{\tau\in\mathbb{Z}} \|w_-(\tau, u) - w_-^n(\tau, u)\| \leq C \sup_{s=1}^{\theta} \omega(\tfrac{1}{n}, \mathcal{F}_t(\phi_s^*)) \quad \text{for all } n \geq N$$

3 Numerical dynamics

Proof.

- Implicit function theorem (with metric parameter space) does not apply, since the derivative is merely onto
- Apply a (quantitative version of the) surjective implicit function theorem derived from Weiss[a], 1974

□

[a] *On the approximation of fixed points of nonlinear compact operators*, SIAM J. Numer. Anal. 11(3)

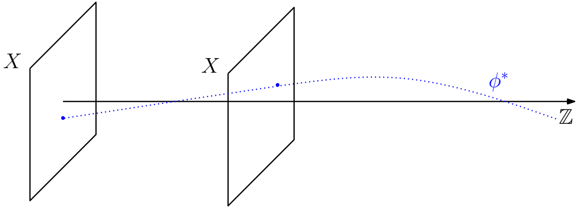
3 Numerical dynamics

On $X = C(\Omega)$ consider an IDE

$$\boxed{u_{t+1} = \mathcal{F}_t(u_t)} \qquad \text{(I)}$$

having a globally attractive solution $\phi^* = (\phi_t^*)_{t\in\mathbb{Z}}$, i.e.

$$\lim_{t\to\infty} \|\varphi(t;\tau,u_\tau) - \phi_t^*\| = 0 \quad \text{for all } (\tau,u_\tau)\in\mathbb{Z}\times X$$



Sufficient conditions

Hardin, Takáč, Webb[1988] (periodic), Krause[2002] (autonomous)

3 Numerical dynamics

Theorem (global attractivity under Nyström discretization)

If a θ_0-periodic solution $(\phi_t^*)_{t\in\mathbb{Z}}$ to a θ_1-periodic IDE (I) over $\Omega \subset \mathbb{R}^\kappa$ satisfies

- ϕ^* *is globally attractive*
- $\sigma(D\mathcal{F}_\theta(\phi_\theta^*)\cdots D\mathcal{F}_1(\phi_1^*)) \subset B_1(0)$
- f_t *is globally bounded,*

then there exist $N \in \mathbb{N}$, $C > 0$ so that also (I_n) has a θ-periodic globally asymptotically stable solution $(\phi_t^n)_{t\in\mathbb{Z}}$; it satisfies

$$\|\phi_t^* - \phi_t^n\| \leq C \sup_{s=1}^{\theta} \omega(\tfrac{1}{n}, \mathcal{F}_s(\phi_s^*)) \quad \text{for all } n \geq N,\ t \in \mathbb{Z}.$$

Proof.

Qualitative version of Smith & Waltman[a], 1999 □

[a] *Perturbation of a globally stable steady state*, Proc. Am. Math. Soc. 127(2)

3 Numerical dynamics

The above constants C depends on

- Lipschitz constants of $\mathcal{F}_t$ near ϕ_t^*
- measure $\lambda_\kappa(\Omega)$ of the domain $\Omega \subset \mathbb{R}^\kappa$
- $\operatorname{dist}_{\mathbb{S}^1} \sigma\big(D\mathcal{F}_\theta(\phi_\theta^*) \cdots D\mathcal{F}_1(\phi_1^*)\big)$
  $\rightsquigarrow$ does not apply to nonhyperbolic solutions or center manifolds

3 Numerical dynamics

Example (Hammerstein IDE)

Let $r_0 := 1$, $r_1 := 5$ and $\Omega = [-1, 1]^2$. The scalar 2-periodic equation

$$u_{t+1}(x) = \frac{\pi^2}{64} \int_\Omega \cos\left(\tfrac{\pi(x_1 - y_1)}{4}\right) \cos\left(\tfrac{\pi(x_2 - y_2)}{4}\right) \left[(1 + r_t)u_t(y) - r_t u_t(y)^2\right] \mathrm{d}y$$

has the globally asymptotically stable 2-periodic solution

$$\phi_t^* : \Omega \to \mathbb{R}, \qquad \phi_t^*(x) := \cos\left(\tfrac{\pi x_1}{4}\right) \cos\left(\tfrac{\pi x_2}{4}\right) \begin{cases} 0.734, & t \text{ is even}, \\ 0.419, & t \text{ is even} \end{cases}$$

3 Numerical dynamics

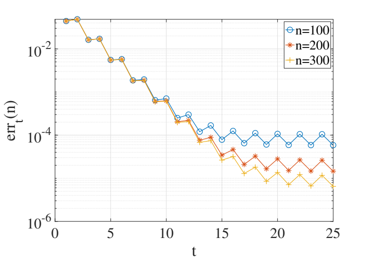
Example (logistic Hammerstein IDE)

For $r_0 := 1$, $r_1 := 5$ and $\Omega = [-1,1]^2$ consider the scalar 2-periodic equation

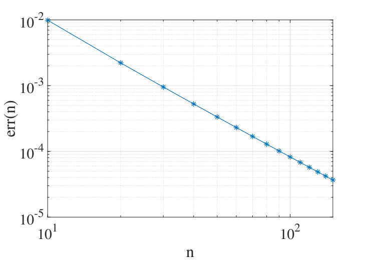
$$u_{t+1}(x) = \frac{\pi^2}{64} \int_\Omega \cos\left(\frac{\pi(x_1-y_1)}{4}\right) \cos\left(\frac{\pi(x_2-y_2)}{4}\right) \left[(1+r_t)u_t(y) - r_t u_t(y)^2\right] \mathrm{d}y.$$

Composite trapezoidal quadrature rule

Error over time



Quadratically decaying error



Conclusions

Integrodifference equations

- IDEs are a flexible tool to model the dispersal and growth of populations with non-overlapping generations. They feature interesting and rich dynamics
- Sufficient criteria for generic bifurcations applicable to a large class of periodic IDEs, including Nyström discretizations or metapopulation models
- Persistence and convergence results in the hyperbolic case, i.e. away from the bifurcations.

Nagyon Szépen Köszönöm!

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