A Geometric Model for Shape Evolution of Ooid Particles: Uniqueness of Equilibrium Shapes

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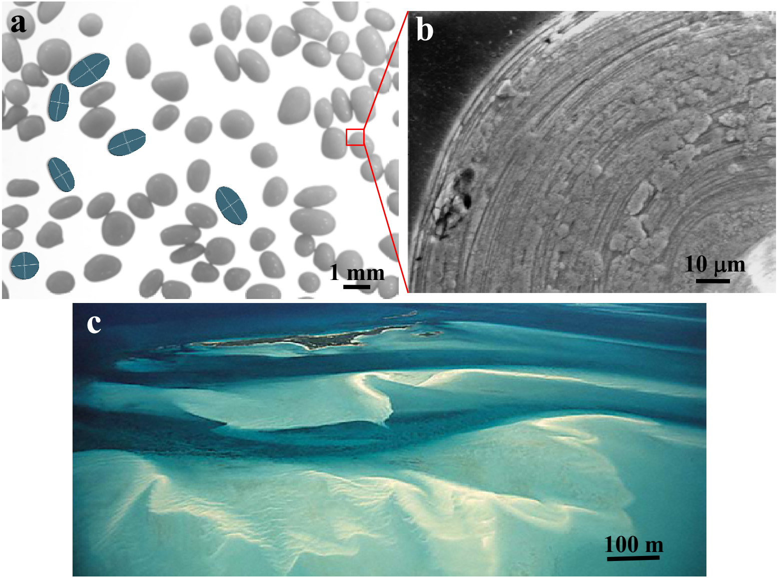
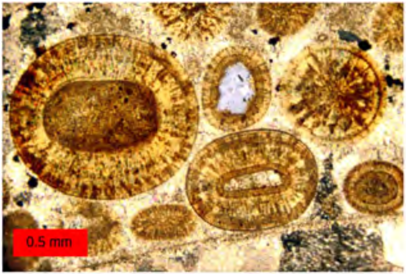
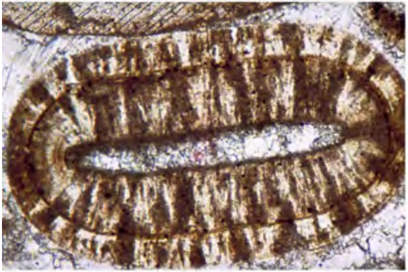
Motivation



Motivation



Ooid sections (source: wikipedia)



Midjurrassic Limestone, Central Atlas Region

Oolith sand, Great Sand Lake, Stansbury

Outline of the talk

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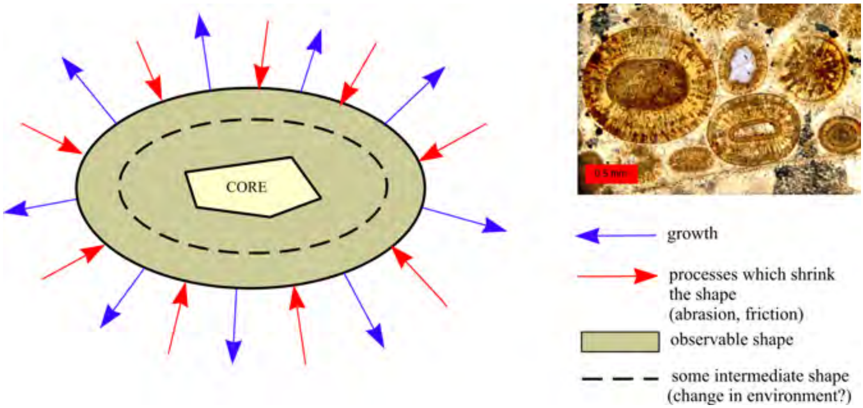
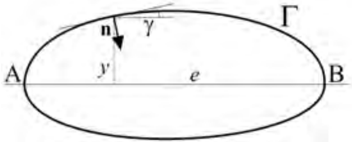
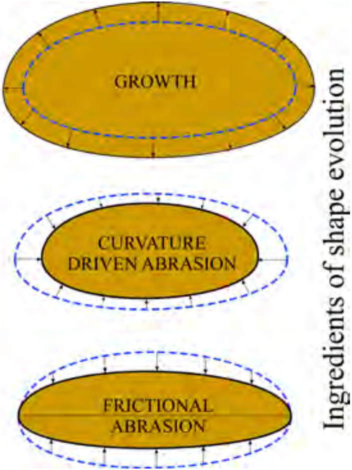
The physical intuition



Notations



The shape: a curve Γ with a unique maximal diameter AB

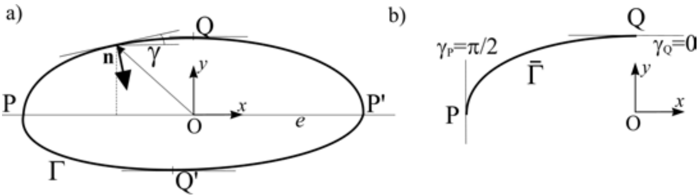


Notations

We consider a geometric, non-local PDE to model the ooid shapes:

$$\Gamma_t = c_3\left(-1 + c_1 A\kappa + c_2 A y \cos\gamma\right)\mathbf{n}, \qquad (1)$$

here the curvature κ and the direction γ depend on the spatial derivatives of Γ. The exact form depend on the parametrization of Γ.



Steady-state solutions (if they exist) fulfill:

$$-1 + c_1 A\kappa + c_2 A y \cos\gamma = 0, \qquad (2)$$

Propositions

Proposition 1

Any smooth, convex, steady state solution Γ^* of eq. (1) with positive parameters ($c_1 > 0$ and $c_2 \geq 0$) embedded in $\mathbb{R}^2$ possesses D2 symmetry.

Proposition 2

Smooth, convex, steady state solutions of eq. (1) are uniquely determined by c_1 and c_2, and for any positive values of the two parameters there exists a Γ^* curve.

Outline of the talk

The local equation

Let restrict ourselves on the part of the solution curve between $0 \leq \gamma \leq \pi/2$. We choose a non-trivial parametrization: we write the curve respect to y:

$$-1 + c_1 A\kappa(y) + c_2 Ay\cos(\gamma(y)) = 0. \tag{3}$$

From this point $()'$ refers to derivation respect to y. In this Section we handle A as just a real number (i.e. we do not consider, that this is the area enclosed by the curve.) In this case, the solutions of the equation are determined by the triple $(c_1,\ c_2,A)$. Observe, that both c_1 and c_2 is multiplied by A, thus any solution fulfills

$$-1 + \hat{c}_1\kappa(y) + \hat{c}_2 y\cos(\gamma(y)) = 0, \tag{4}$$

where $\hat{c}_1 = c_1 A$ and $\hat{c}_2 = c_2 A$.

The local equation

$$\kappa(y) = -\frac{\mathrm{d}\gamma}{\mathrm{d}s} = -\frac{\mathrm{d}\gamma}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}s} = -\gamma'(y)\sin(\gamma(y)). \tag{5}$$

Note, that the negative sign relates to the fact, that by definition $\gamma(y)$ is decreasing between points P and Q (Fig. 8. b)). By the virtue (5) eq. (4) takes the following form, which is a first order ODE:

$$-1 - \hat{c}_1\gamma'(y)\sin(\gamma(y)) + \hat{c}_2 y\cos(\gamma(y)) = 0. \tag{6}$$

From now on this equation is called *local*. There exist a closed-form solution for the local equation:

$$\gamma(y) = \arccos\left(\frac{\sqrt{\pi}\,\mathrm{erf}\left(\sqrt{\frac{\hat{c}_2}{2\hat{c}_1}}yi\right) - Ci\sqrt{\frac{2\hat{c}_2}{\hat{c}_1}}}{\hat{c}_1 i\exp\left(\frac{\hat{c}_2 y^2}{2\hat{c}_1}\right)\sqrt{\frac{2\hat{c}_2}{\hat{c}_1}}}\right), \tag{7}$$

The solution

and consequently

$$\kappa(y) = \frac{1}{\hat{c}_1} + \frac{\sqrt{\pi}}{\hat{c}_1}\sqrt{\frac{\hat{c}_2}{2\hat{c}_1}} \exp\left(-\frac{\hat{c}_2 y^2}{2\hat{c}_1}\right) \mathsf{erf}\left(\sqrt{\frac{\hat{c}_2}{2\hat{c}_1}} yi\right) yi. \qquad (8)$$

$$q := \sqrt{\frac{\hat{c}_2}{2\hat{c}_1}}, \qquad (9)$$

whence the solution (keeping $C = 0$) in (8) can be reformulated as

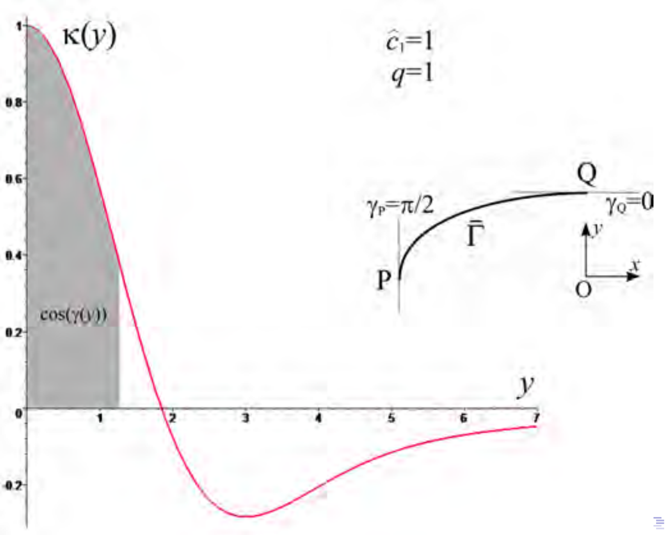
$$\kappa(y) = \frac{1}{\hat{c}_1}\left(1 + \sqrt{\pi}\frac{\mathsf{erf}(qyi)}{\exp(q^2y^2)} qyi\right). \qquad (10)$$

The solution

The following properties of $\kappa(y)$ can be settled:

1. $\kappa(y)$ is real ($\mathbb{R}^+ \cup \{0\} \rightarrow \mathbb{R}$).
2. $\kappa(y)$ is continuous.
3. $\kappa(0)$ is positive and equals $\hat{c}_1^{-1}$.
4. $\kappa(y)$ has a maximum at $y = 0$.
5. $\kappa(y) \rightarrow 0$ as $y \rightarrow \infty$.
6. There is exactly one point, denoted to y_0, where $\kappa(y)$ vanishes and y_0 solely depends on q.
7. There is no local extrema for $\kappa(y)$ between $0 < y < y_0$, thus it is monotonic in this range.

The graph of $\kappa(y)$



Realization of the shape

To realize a steady state shape Γ^* we need $\gamma(y)$ itself. By the virtue of (5)

$$\gamma(y) = \arccos\left(\int_0^y \kappa(\eta)\mathrm{d}\eta\right). \tag{11}$$

As we have seen, $\kappa(0)$ depends solely on $\hat{c}_1$ and for fixed q the value of y_0 is fixed, too. This leads to the conclusion that for any fixed q there exists a $\hat{c}_{1,\text{crit}}$ critical value at which

$$\int_0^{y_0} \kappa(\eta)\mathrm{d}\eta = 1. \tag{12}$$

For further convenience for a fixed q we define the set

$$\chi_q : \{\hat{c}_1 \quad | \quad 0 < \hat{c}_1 \leq \hat{c}_{1,\text{crit}}\} \tag{13}$$

Realization of the shape

As y provides a possible parametrization of the curve segment $\bar{\Gamma}$, the unique closed form solution in (7) can be realized as a unique curve in $\mathbb{R}^2$. Finally we prove uniqueness for Γ^* itself. So far we know that for proper $\hat{c}_1$ and $\hat{c}_2$ the curve segment $\bar{\Gamma}^*$ is uniquely determined. Note, that $\bar{\Gamma}^*$ has vertical tangent at P and horizontal tangent at Q. As we consider smooth shapes the only way to glue the $\bar{\Gamma}^*$ curve-segments to form a closed, non-intersecting curve are reflections along the x and y axes. It clearly hints to that a smooth steady state shape must possess D2 symmetry.

Outline of the talk

A map between the parameter spaces

We turn to investigate steady state solutions of the non-local equation:

$$-1 + c_1 A\kappa + c_2 Ay\cos\gamma = 0, \tag{14}$$

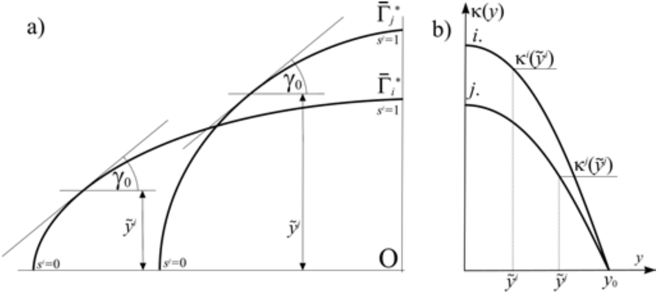
Let us assign $(\hat{c}_1, \hat{c}_2)$ and (c_1, c_2) if they result in an identical curve as a steady state solution of the proper model. In this sense we can talk about a *mapping between the parameter spaces*. Observe, that

$$\sqrt{\frac{1}{2}\frac{\hat{c}_2}{A}\frac{A}{\hat{c}_1}} = q = \sqrt{\frac{c_2}{2c_1}} \tag{15}$$

holds, implying that q is invariant under the above mentioned map. Instead of $\hat{c}_2$ and c_2 we use q as one of the parameters in the problem. Based on the previous section, in the local model only $\hat{c}_1 \in \chi_q$ can result in a smooth curve. Let us formally define the map F at a fixed value of q as:

$$F: \quad \chi_q \to \mathbb{R}^+ \tag{16}$$
$$\hat{c_1} \to c_1.$$

A map between the parameter spaces



Goal: obtain some properties of the $c_1(\hat{c}_1)$ function at a fixed value of q.
Choose $\hat{c}_1^j = (1+\varepsilon)\hat{c}_1^i$, where $\varepsilon > 0$.

A map between the parameter spaces

We choose two points along $\bar{\Gamma}_i^*$ and $\bar{\Gamma}_j^*$, one for each, such way that their tangent direction, γ_0 is identical, the $\tilde{(.)}$ sign refers to any quantity evaluated at these points. As $\gamma(y)$ is monotonic along $\bar{\Gamma}$, the position of the two points is well-defined.For our two curves we see, that

$$\int_0^{\tilde{y}^i} \kappa^i(\eta)\mathrm{d}\eta = \cos(\gamma_0) = \int_0^{\tilde{y}^j} \kappa^j(\eta)\mathrm{d}\eta \tag{17}$$

must hold, which implies $\tilde{y}^i < \tilde{y}^j$. By the properties of $\kappa(y)$ it is easy to see, that curvatures at the chosen point-pair fulfill

$$\kappa^i(\tilde{y}^i) > (1+\varepsilon)\kappa^j(\tilde{y}^j), \tag{18}$$

From this observation and the positivity of all the involved quantities we conclude, that

$$\frac{\tilde{y}^j}{(1+\varepsilon)\kappa^j(\tilde{y}^j)} > \frac{\tilde{y}^i}{\kappa^j(\tilde{y}^i)}. \tag{19}$$

A map between the parameter spaces

We switch to the parametrization of $\bar{\Gamma}$ respect to the tangent direction γ. Based on eq. (5) we see, that the $\bar{A}$ area under $\bar{\Gamma}$ can be computed as

$$\bar{A} = \int_{\frac{\pi}{2}}^{0} -\frac{\mathrm{d}s}{\mathrm{d}\gamma}\cos(\gamma)y(\gamma)\mathrm{d}\gamma = \int_{\frac{\pi}{2}}^{0} \frac{y(\gamma)}{\kappa(\gamma)}\cos(\gamma)\mathrm{d}\gamma. \tag{20}$$

As we have demonstrated in (19), the argument of the integral in the RHS of (20) is smaller for $\bar{\Gamma}_i^*$ than for $\bar{\Gamma}_j^*$, and this holds for any $\gamma \in (0, \pi/2)$, whence we conclude

$$\frac{1}{1+\varepsilon}\bar{A}^j = \int_{\frac{\pi}{2}}^{0} \frac{y^j(\gamma)}{(1+\varepsilon)\kappa^j(\gamma)}\cos(\gamma)\mathrm{d}\gamma > \int_{\frac{\pi}{2}}^{0} \frac{y^i(\gamma)}{\kappa^i(\gamma)}\cos(\gamma)\mathrm{d}\gamma = \bar{A}^i. \tag{21}$$

Recall, that $\hat{c}_1^j = (1+\varepsilon)\hat{c}_1^i$, hence

$$\frac{\bar{A}^j}{\hat{c}_1^j} > \frac{\bar{A}^i}{\hat{c}_1^i}. \tag{22}$$

A map between the parameter spaces

As a steady state Γ^* curve possesses D2 symmetry $A = 4\bar{A}$ holds so we are left with the conclusion that

$$c_1^i = \frac{\hat{c}_1^i}{A^i} > \frac{\hat{c}_1^j}{A^j} = c_1^j, \tag{23}$$

which is exactly the monotonicity of the $c_1(\hat{c}_1)$ function. This proves that F is injective, as different elements in χ_q cannot be mapped to identical values. It is also worthy to note, that for all $\hat{c}_1 \in \chi_q$ the area is obviously positive thus $c_1(\hat{c}_1)$ is a positive, monotonously decreasing, continuous function.

A map between the parameter spaces

To prove surjectivity we have to investigate the limits of $c_1(\hat{c}_1)$. First we investigate the limit as $\hat{c}_1 \to 0$ Recall that the curvature along $\bar{\Gamma}$ is maximal at point P and $\kappa(0) = \hat{c}_1^{-1}$ here. Curvature is the reciprocal of the r radius of its osculating circle. It provides an estimate on the area under the curve via $\bar{A} > 0.25 r^2 \pi = 0.25 \hat{c}_1^2 \pi$. Similarly, at point Q the curvature is minimal which fact yields the following inequality:

$$\frac{\hat{c}_1}{\pi \left(\frac{\hat{c}_1}{1 - 2q^2 \hat{c}_1 y(1)} \right)^2} < \frac{\hat{c}_1}{A(\hat{c}_1)} < \frac{\hat{c}_1}{\pi \hat{c}_1^2} \tag{24}$$

As both the lower and the upper expression in the above inequality approach $+\infty$ as $\hat{c}_1 \to 0$ we conclude

$$\lim_{\hat{c}_1 \to 0} \frac{\hat{c}_1}{A(\hat{c}_1)} = +\infty. \tag{25}$$

A map between the parameter spaces

Finally we investigate the $\hat{c}_1 \to \hat{c}_{\text{crit}}$ limit. As $\hat{c}_{\text{crit}}$ is finite it is enough to investigate the $\bar{A}(\hat{c}_1)$ area in the limit. We consider the already used identity between the curvature and and arch length. Taking again the parametrization respect to γ we write

$$\kappa(\gamma) = -\left(\frac{\mathrm{d}S(\gamma)}{\mathrm{d}\gamma}\right)^{-1}, \tag{26}$$

where $S(\gamma)$ is the arch length between point P and the point with tangent inclination γ. As at $\hat{c}_1 = \hat{c}_{\text{crit}}$ the curvature at point Q vanish we conclude, that

$$\lim_{\gamma \to 0} -\frac{\mathrm{d}S(\gamma)}{\mathrm{d}\gamma} = \lim_{\gamma \to 0} \frac{1}{\kappa(\gamma)} = \infty. \tag{27}$$

A map between the parameter spaces

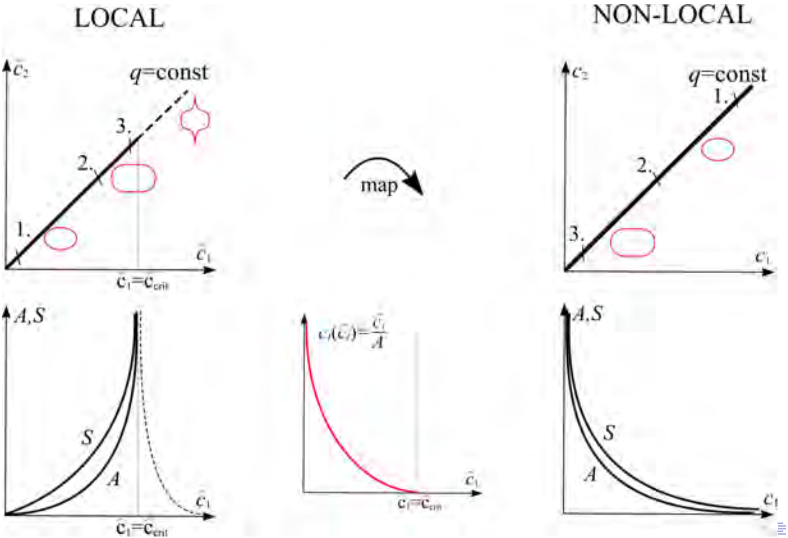
Thus the curve is unbounded. As the area $\bar{A}$ under $\bar{\Gamma}$ can be computed from the arc length (y is finite!) we obtain

$$\lim_{\hat{c}_1 \to \hat{c}_{1,\text{crit}}} S = \lim_{\hat{c}_1 \to \hat{c}_{1,\text{crit}}} A = \infty, \tag{28}$$

which provides the required limit as

$$\lim_{\hat{c}_1 \to \hat{c}_{\text{crit}}} \frac{\hat{c}_1}{A(\hat{c}_1)} = 0. \tag{29}$$

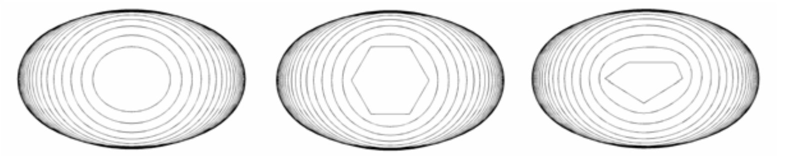
Summary



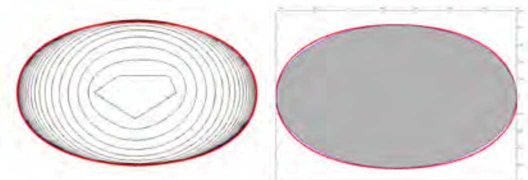
Outline of the talk

Shape evolution in numerics

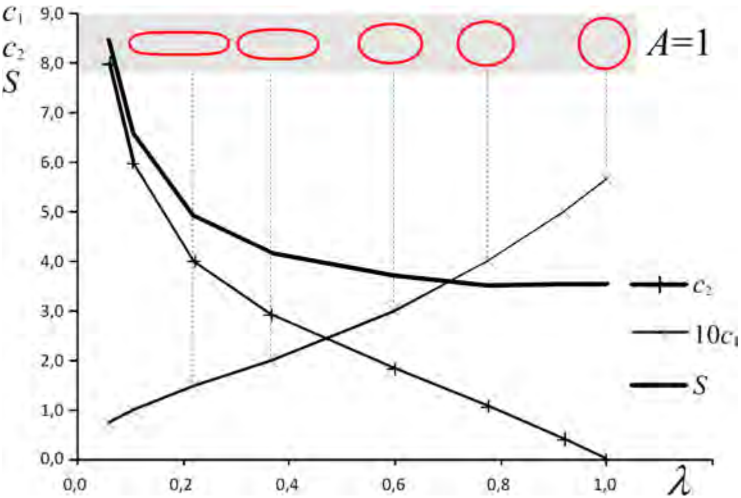
Level-set simulation of the evolution from different initial shapes:



As the problem can be formulated as an IVP, the invariant shape can be computed by minimizing the difference between the measured and expected area of the shape. Observe, that ellipses are not invariant under the flow:



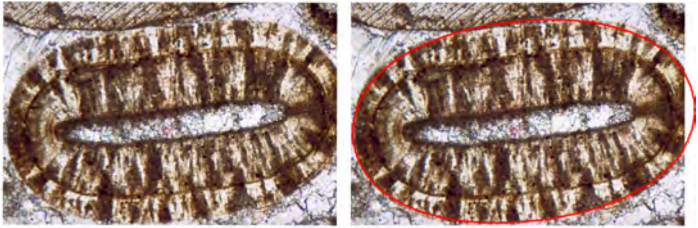
Reconstruction of parameters via the aspect ratio λ



Reconstruction of parameters

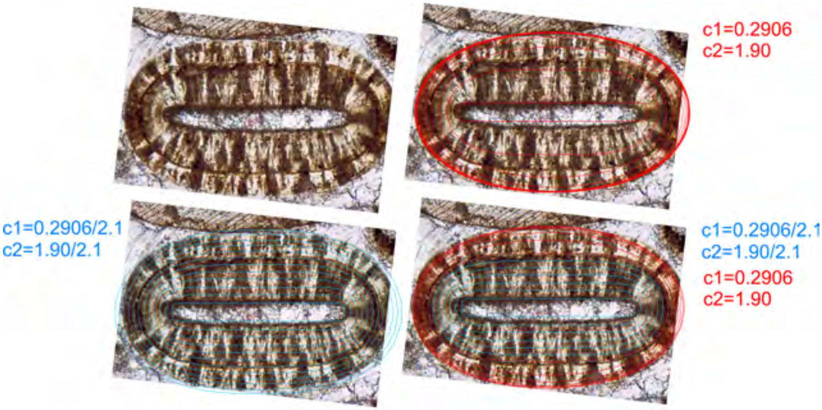
$$c_1 = \frac{1}{A\kappa_0} \qquad c_2 = \frac{1 - c_1 A\kappa(\mathbf{x}_Q)}{Ay(\mathbf{x}_Q)} = c_1 \frac{\kappa_0 - \kappa(\mathbf{x}_Q)}{y(\mathbf{x}_Q)}. \tag{30}$$

It means, having a curve $\Gamma_{observed}$ we need to identify the maximal diameter. Next we determine point Q with a tangent parallel to the maximal diameter. Finally, we measure A, κ_0, $\kappa(\mathbf{x}_Q)$ and $y(\mathbf{x}_Q)$ and calculate c_1 and c_2.



c_1=0.2906 c_2=1.90

Reconstruction of parameters



Thank you for your the attention!

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Ellipse is not an invariant shape