Information Geometry of Matrices and Mean

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Outline

– Generalization of means.

  general concept of the mean

  extension to more variables

  extension to matrices

  difficulties with combining these ideas together

– Means in quantum information geometry.

– More geometry related to the means.

  examples

For strictly positive numbers x, y

arithmetic mean

$$M_a(x,y) = \frac{x+y}{2}$$

geometric mean

$$M_g(x,y) = \sqrt{xy}$$

harmonic mean

$$M_h(x,y) = \frac{2}{\frac{1}{x} + \frac{1}{y}}$$

Well-known inequality

$$M_h(x,y) \leq M_g(x,y) \leq M_a(x,y)$$

Generalizations

Some natural questions related to means:

Is the function

$$M_{\log}(x,y) = \frac{x-y}{\log x - \log y}$$

is a mean? (*logarithmic mean*)

For more variables we have the intuition

$$M_a(x,y,z) = \frac{x+y+z}{3} \qquad M_g(x,y,z) = \sqrt[3]{xyz}$$

How to generalize the logarithmic mean to more variables?

For matrices we have the intuition

$$M_a(X,Y) = \frac{1}{2}(X+Y).$$

But to compute the geometric mean of matrices?

What is the logarithmic mean of three matrices???

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$x<y \;\Rightarrow\; x<M(x,y)<y$	$f(\cdot>1)>1,\; f(0<\cdot<1)<1$
$x<x_0, y<y_0 \Rightarrow M(x,y)<M(x_0,y_0)$	f increasing
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We have

$$\text{means} = \left\{ f \in C(\mathbb{R}^+, \mathbb{R}^+) \;\middle|\; \begin{array}{c} f \text{ increasing} \\ f(1) = 1 \\ \forall t \in \mathbb{R}^+ : \; f(t) = t f(t^{-1}) \end{array} \right\}$$

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geometric mean: $f(t) = \sqrt{t}$

logarithmic mean: $f(t) = \dfrac{t-1}{\log t}$

Definition: A mean m of three variables is said to be of *type 1 invariant* with respect to M if

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Theorem: To each M there exists a unique m which is type 1 invariant with respect to M.

Proof:

Define $x_0 := x$, $y_0 := y$, $z_0 := z$ and iterate

$$x_{n+1} := M(y_n, z_n) \quad y_{n+1} := M(z_n, x_n) \quad z_{n+1} := M(x_n, y_n)$$

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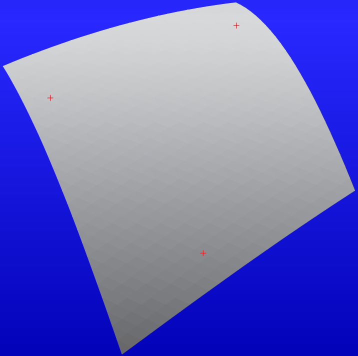

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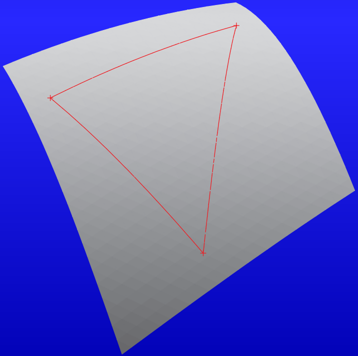

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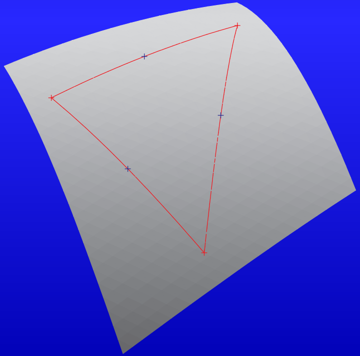

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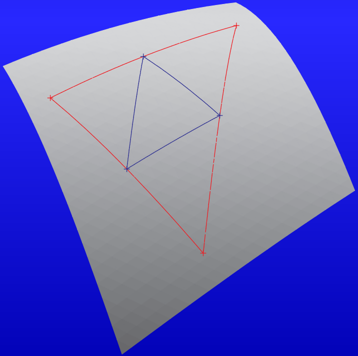

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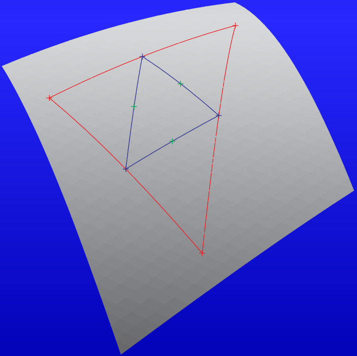

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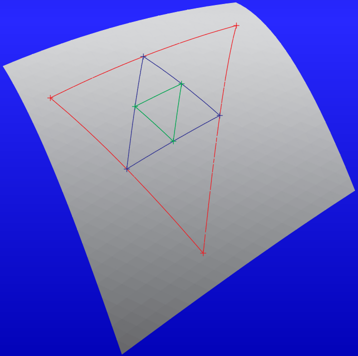

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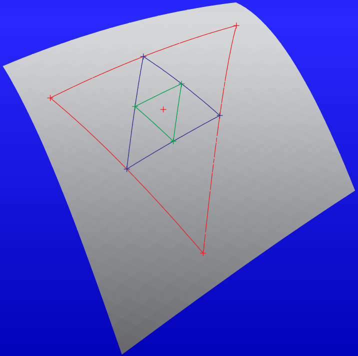

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Example:

$$\frac{\frac{x+y}{2} + \frac{y+z}{2} + \frac{z+x}{2}}{3} = \frac{x+y+z}{3}$$

$$\sqrt[3]{\sqrt{xy} \times \sqrt{yz} \times \sqrt{zx}} = \sqrt[3]{xyz}$$

Logarithmic mean $L(x,y) = \dfrac{x-y}{\log x - \log y}$ with three variables:

$$U_0(x,y,z) = \sqrt{\frac{1}{2} \times \frac{(x-z)(y-z)(x-y)}{x \log \frac{y}{z} + y \log \frac{z}{x} + z \log \frac{x}{y}}}$$

$$U_1(x,y,z) = \frac{1}{2} \times \frac{(y-z)(x-z)(x-y)}{x(y-z)\log x + y(z-x)\log y + z(x-y)\log z}$$

Conjecture:

$$U_0(x,y,z) \leq L_3(x,y,z) \leq U_1(x,y,z)$$

Numerical example $x = 1$, $y = 2$, $z = 3$:

$$\sim 1.8644 \leq \sim 1.8791 \leq \sim 1.9111$$

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Define means on $n \times n$, positive definite matrices $\mathcal{M}_n$:

$$X \in \mathcal{M}_n \iff X = X^*, \begin{cases} \langle v, Xv \rangle > 0 \ \forall 0 \neq v \in \mathbb{R}^n, \mathbb{C}^n \\ \text{every eigenvalue of } X \text{ is positive} \end{cases}$$

We write $X \leq Y$ if $Y - X \in \mathcal{M}_n$.

How to compute $f(X)$:

– $X \in \mathcal{M}_n$ can be diagonalized by some unitary matrix U, that is $X = UDU^*$

– X can be written as $X = \sum_{i=1}^{n} \lambda_i E_i$, where $(\lambda_i)_{i=1,\ldots,n}$ are the eigenvalues and $(E_i)_{i=1,\ldots,n}$ are the corresponding projections

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f is operator monotone if $X \leq Y$ then $f(X) \leq f(Y)$.

M is a *mean of matrices* if for every $X, Y \in \mathcal{M}_n$

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Problems with means

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– More variables : if more $= 3$ ✓

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if more $= 4$:

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– General mean ✓

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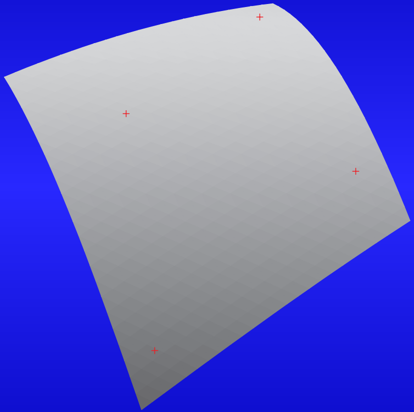
if more = 4 :



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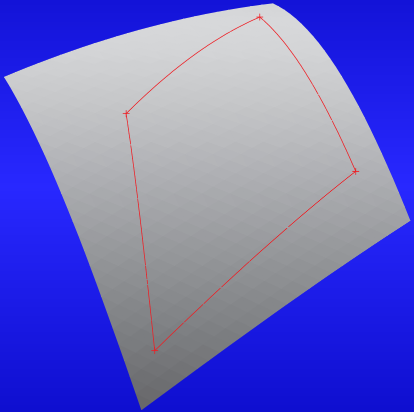

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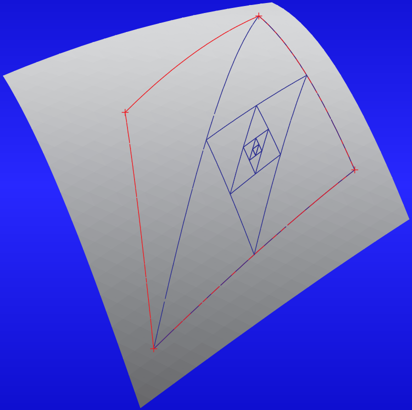

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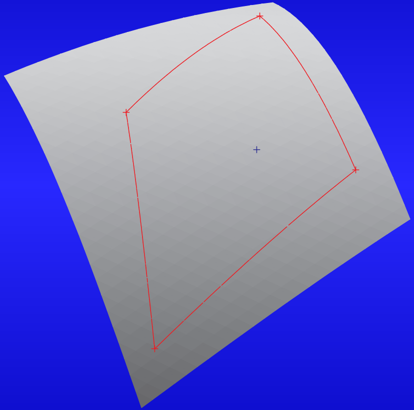

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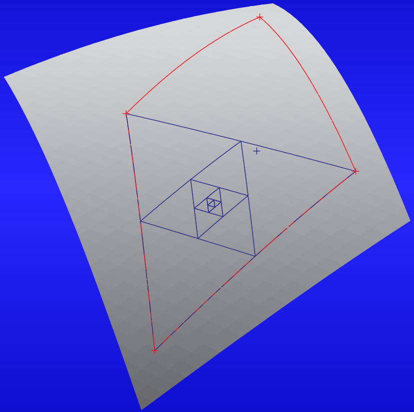

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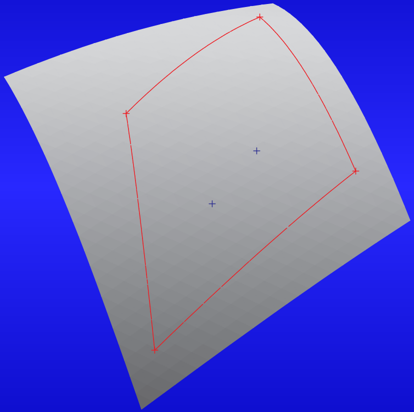

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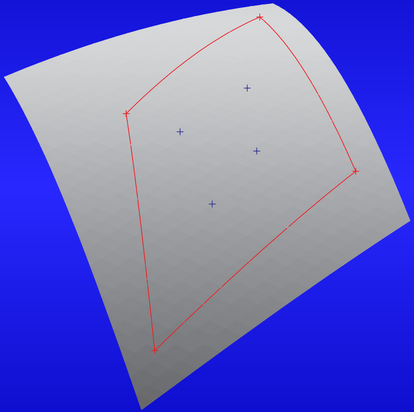

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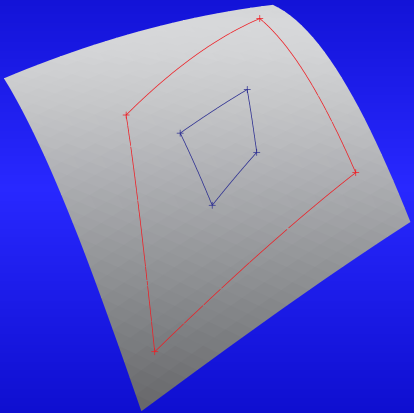

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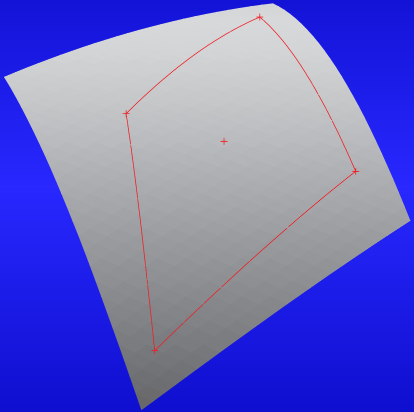

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General mean of more matrices:

Problems with means

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+? : Conjecture: $\|x_{n+1} - y_{n+1}\| \leq \|x_n - y_n\|$. $(Petz)$
- : $x_0 := 0.01$, $y_0 := 0.02$, $z_0 := 1$

$$f(x) = \frac{500x}{999x+1} + \frac{500x}{x+999}$$

$$\|x_{n+1} - y_{n+1}\| \approx 0.02669 > 0.02 = \|x_n - y_n\|.$$

Contradiction!

Means in Quantum Info. Geometry

In the classical case: uniqueness of the Fisher information.

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Open set of distributions on $X_n = \{1, \ldots, n\}$

$$\mathcal{P}_n = \left\{ (p_1, \ldots, p_n) \,\middle|\, 0 < p_i < 1, \sum_{i=1}^{n} p_i = 1 \right\}.$$

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Theorem (*Čencov:*) Assume that for every $n \in \mathbb{N}$ the pair $(\mathcal{P}_n, g_n)$ is a Riemannian-manifold. If for every Markovian map $\kappa : X_n \times X_m \to \mathbb{R}$
the following monotonicity property holds

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then the family of metrics $(g_n)_{n \in \mathbb{N}}$ is unique up to a positive real number.

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$$g_{T(D)}(T(X), T(X)) = \left\langle T(X), \mathbf{J}_{T(D)}^{-1}(T(X)) \right\rangle$$

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monotonicity:

$$T^*\mathbf{J}_{T(D)}^{-1} T \leq \mathbf{J}_D^{-1}$$

$$T\mathbf{J}_D T^* \leq \mathbf{J}_{T(D)}$$

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M is a mean!

Theorem (*Petz:*) Assume that for every $n \in \mathbb{N}$ the pair $(\mathcal{M}_n, g_n)$ is a Riemannian-manifold. If for every stochastic map T
the following monotonicity property holds

$$g_{T(D)}(T(X), T(X)) \leq g_D(X, X) \; \forall D \in \mathcal{M}_n, \forall X \in \mathrm{T}_p\mathcal{M}_n \; ,$$

then there exists an operator monotone function $f : \mathbb{R}^+ \to \mathbb{R}$ with the property $f(x) = x f(x^{-1})$, such that

$$g_D(X, Y) = \mathrm{Tr}\left(X \left(R_{n,D}^{\frac{1}{2}} f(L_{n,D} R_{n,D}^{-1}) R_{n,D}^{\frac{1}{2}} \right)^{-1}(Y) \right) .$$

Examples:

$$f(x)=\frac{2x}{1+x}:\ g_D^{(\mathrm{LA})}(X,Y)=\frac{1}{2}\,\mathrm{Tr}(XD^{-1}Y+YD^{-1}X)$$

$$f(x)=\frac{x-1}{\log x}:\ g_D^{(\mathrm{KM})}(X,Y)=\mathrm{Tr}\int_0^\infty X(D+t)^{-1}Y(D+t)^{-1}\,\mathrm{d}\,t$$

$$f(x)=\frac{1+x}{2}:\ g_D^{(\mathrm{SM})}(X,Y)=\mathrm{Tr}(XZ_{D,Y}),$$

where $Z_{D,Y}$ is the solution of the equation

$$2Y=DZ_{D,Y}+Z_{D,Y}D.$$

We have the inequality

$$g_D^{(\mathrm{SM})}(X,X)\le g_D^{(f)}(X,X)\le g_D^{(\mathrm{LA})}(X,X).$$

Geometry of Means

Geometrical point of view:
Assume that $(\mathcal{M}, g)$ is a Riemannian manifold. Let us define the *mean* of two arbitrary points $X, Y \in \mathcal{M}$:
– Connect X and Y with a geodesic line γ, such that $\gamma(0) = X$ and $\gamma(1) = Y$.
– Then the mean of X and Y is the point $\gamma(1/2)$.

Then we have:

$$\tilde{M}(X,Y) = \tilde{M}(Y,X)$$

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Example:

1. $\mathcal{M} := \mathbb{R}^+$, and $g : \mathcal{M} \to \mathbb{R}^+$ smooth function.

At $p \in \mathcal{M}$ the "scalar product" of the "vectors" $x, y \in \mathbb{R}$ is

$$g_p(x, y) = xyg(p).$$

The equation of the geodesic line $\gamma(t)$

$$\ddot{\gamma}(t) + \frac{g'(\gamma(t))}{2g(\gamma(t))} \left(\dot{\gamma}(t)\right)^2 = 0.$$

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2. Consider the following metric $g(t) = t^{2(p-1)}$: $(p \neq 1)$
The differential equation:

$$\ddot{\gamma}(t) + (p-1)\frac{1}{\gamma(t)}\left(\dot{\gamma}(t)\right)^2 = 0$$

and its solution

$$\begin{cases} \gamma(t) = (C_1 + C_2 t)^{1/p} & \text{if} \quad p \neq 0 \\ \gamma(t) = C_1 C_2^t & \text{if} \quad p = 0 \end{cases}$$

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$$\begin{cases} \gamma(t) = \sqrt[p]{x^p + (y^p - x^p)t} & \text{if} \quad p \neq 0 \\ \gamma(t) = x\left(\dfrac{y}{x}\right)^t & \text{if} \quad p = 0 \end{cases}$$

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(*Power-mean* and Geometric Mean.)

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(Fisher information metric on the space of Gaussian distributions.)

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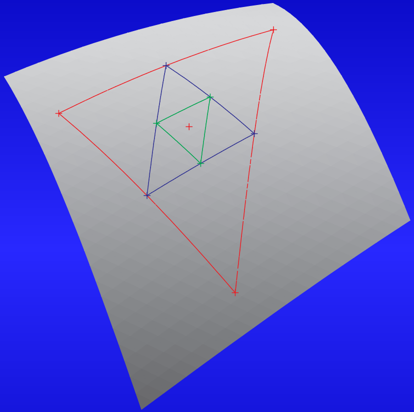

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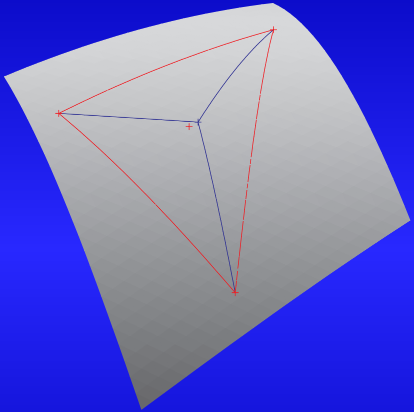

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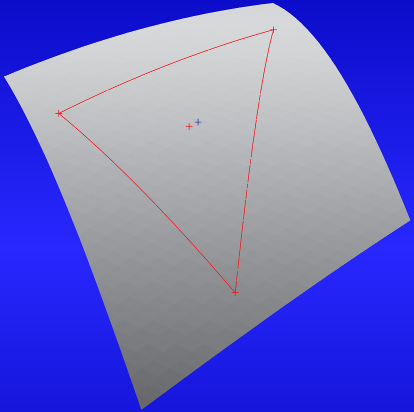

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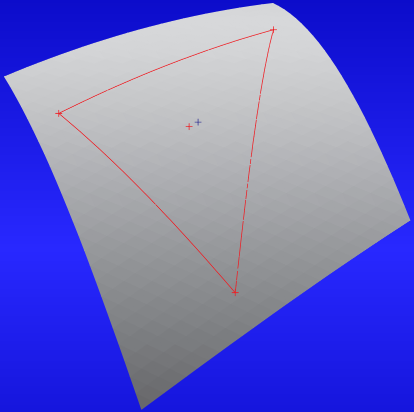

$$\log C^{-1}X + \log C^{-1}Y + \log C^{-1}Z = 0$$

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4. $\mathcal{M} := (\mathcal{M}_n, \mathbb{R}^n)$, and

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Geodesic lines between Gaussian distributions

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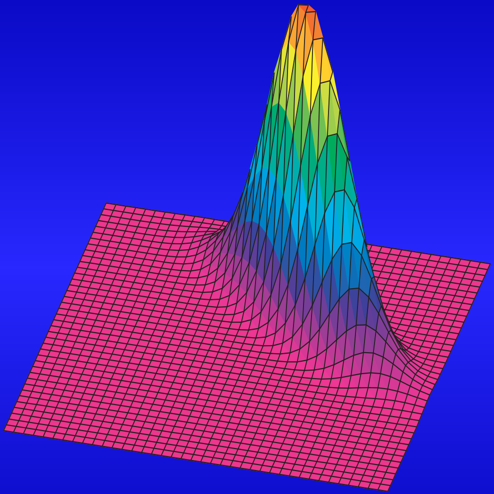

Information Geometry of Matrices and Mean

Attila Andai

Symposium on Information Geometry

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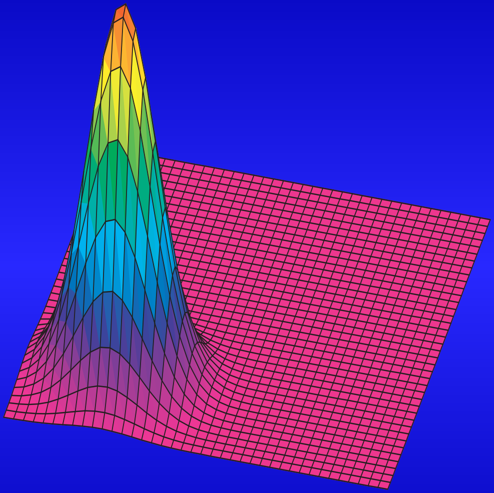

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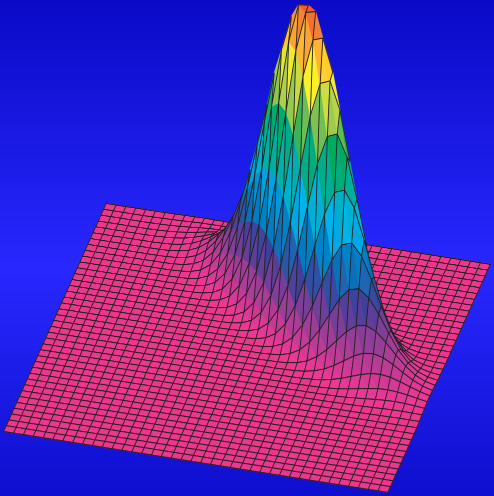

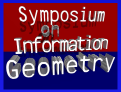

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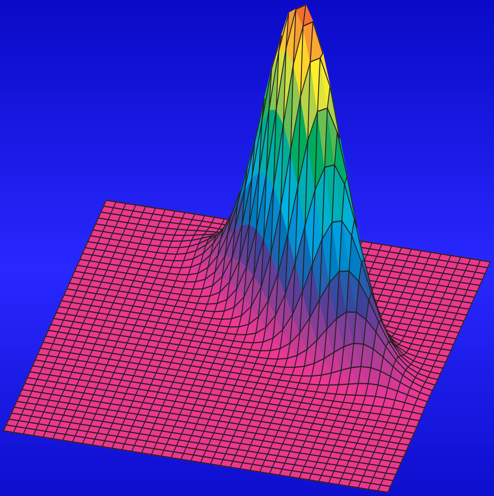

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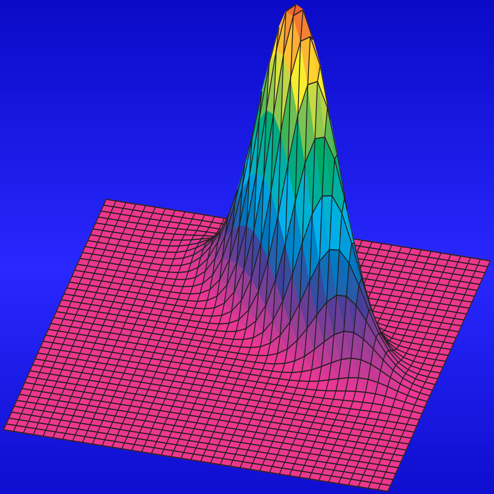

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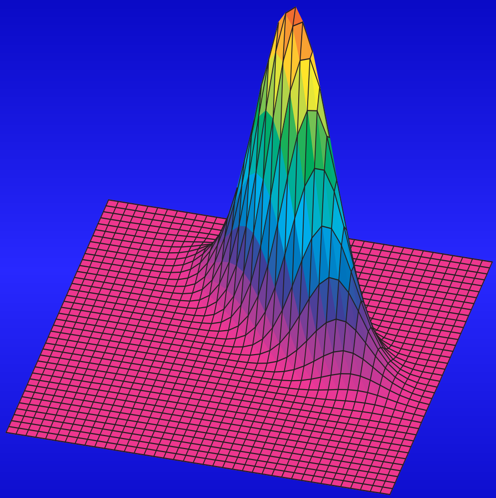

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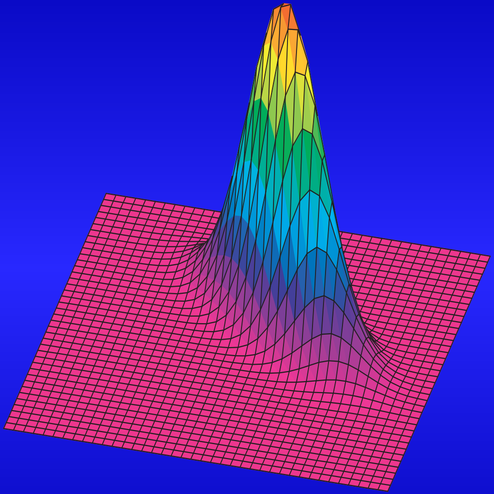

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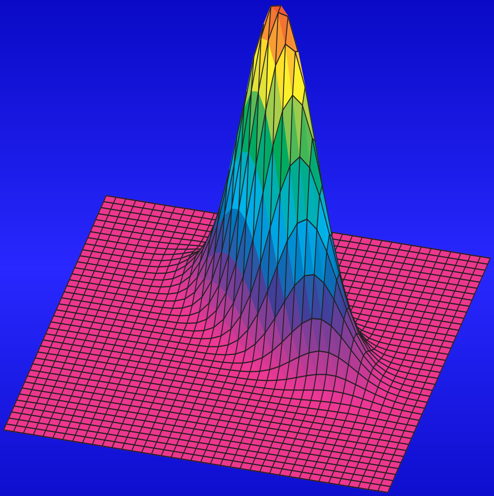

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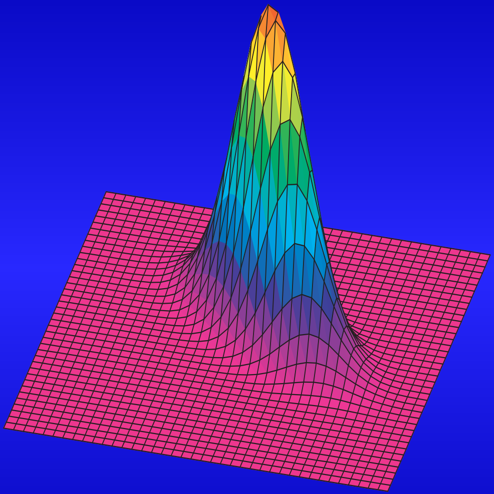

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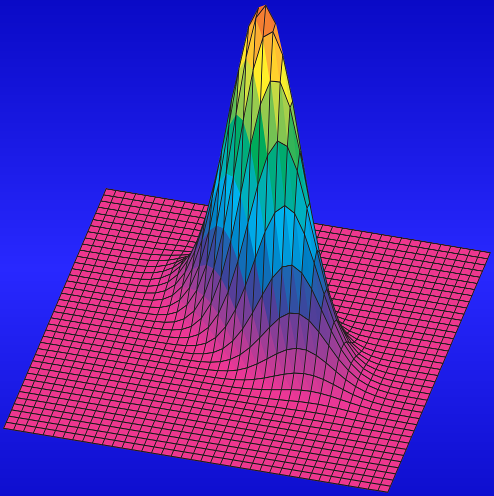

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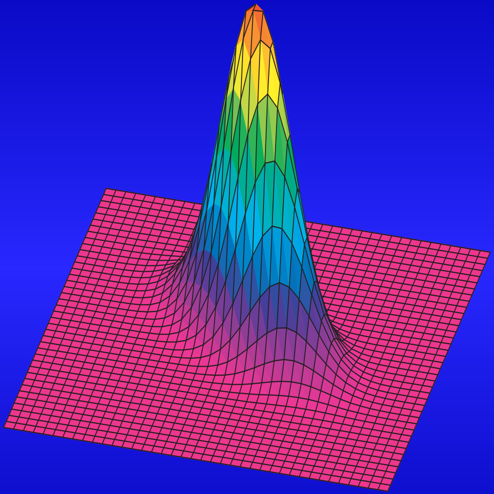

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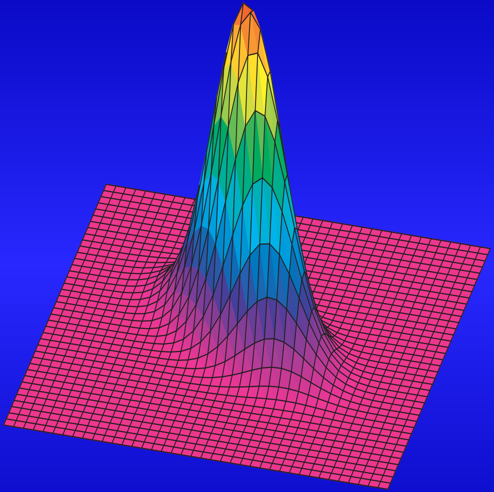

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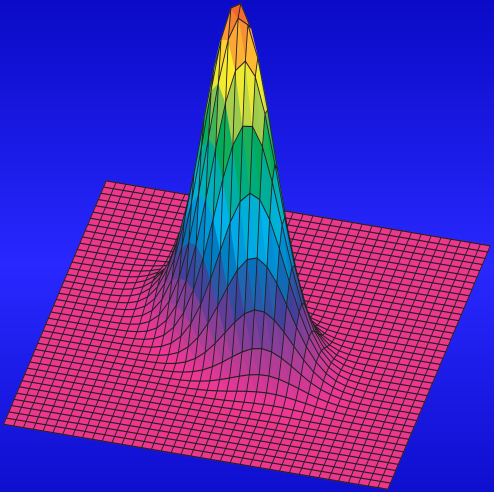

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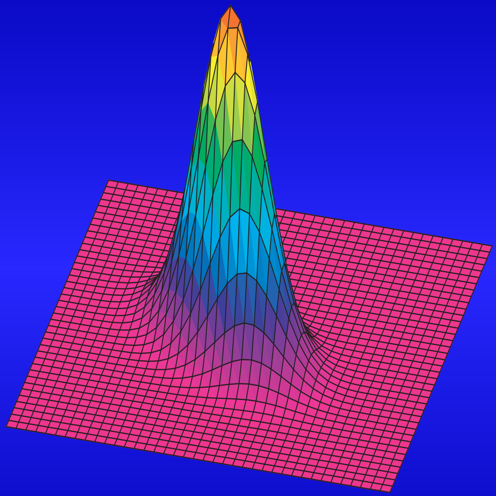

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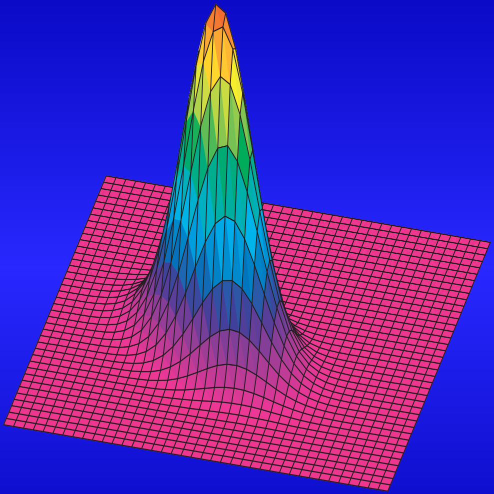

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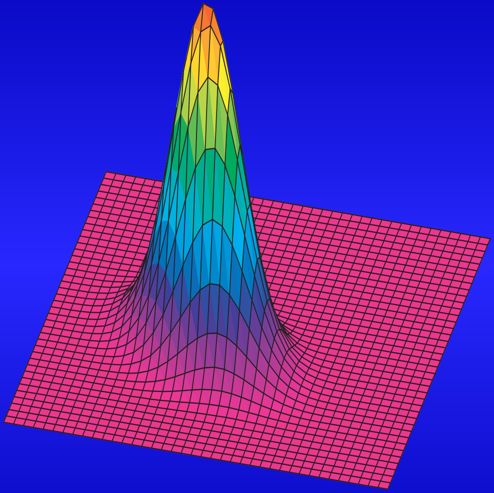

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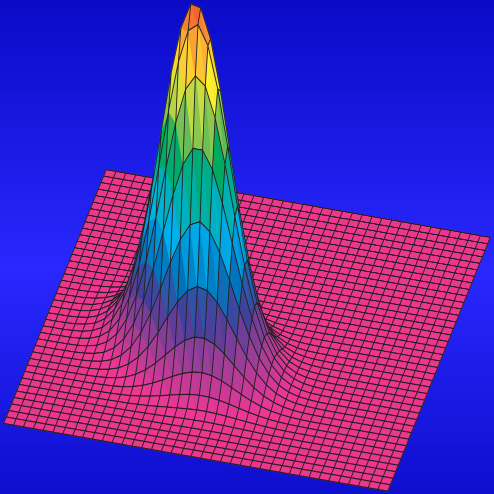

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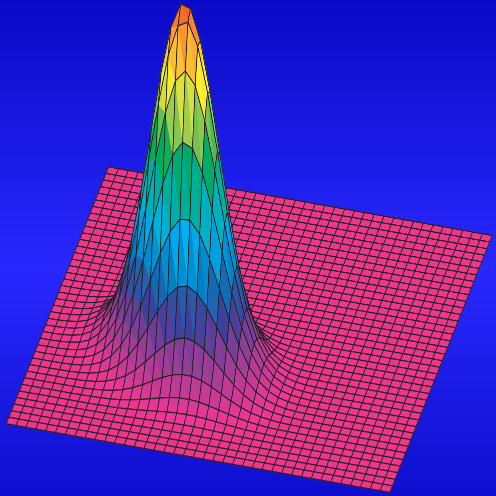

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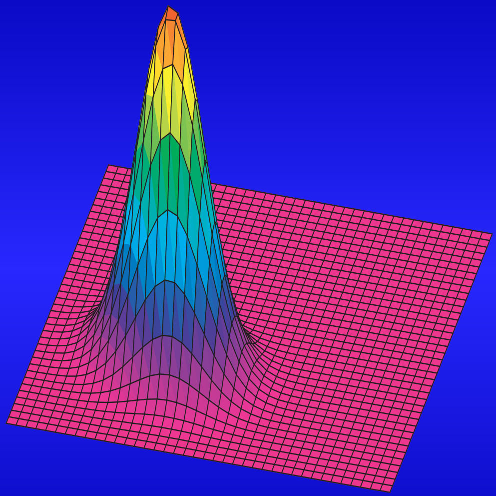

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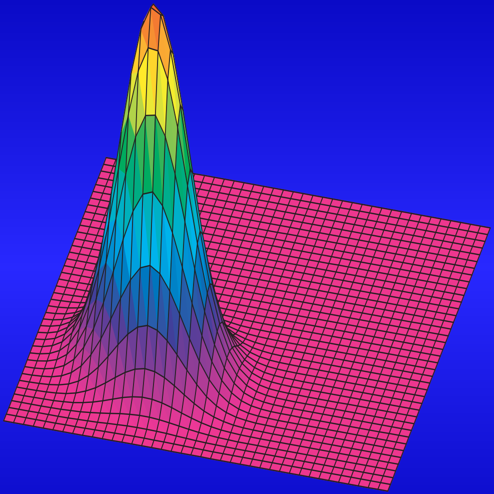

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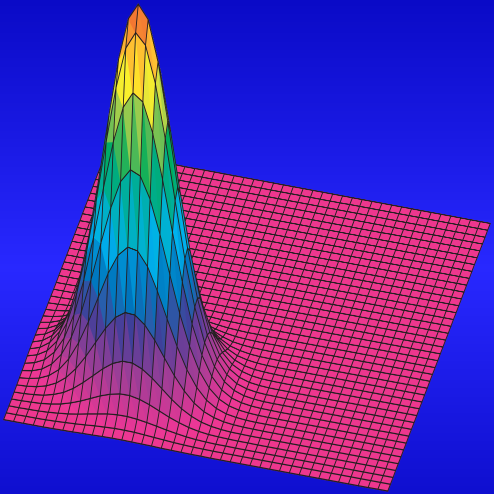

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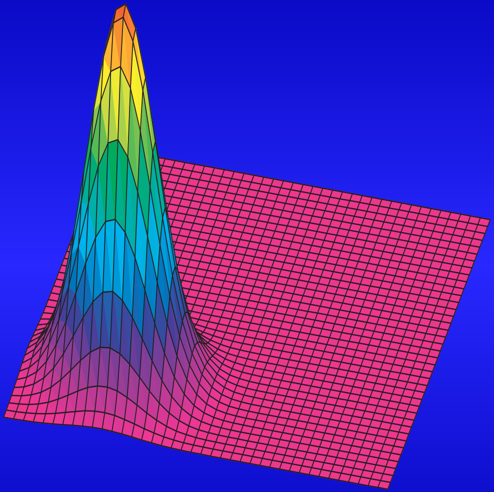

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$$\ddot{\gamma}(t) - 2\dot{\gamma}(t)\gamma(t)^{-1}\dot{\gamma}(t) = 0$$

which satisfies $\gamma(0) = X$, $\gamma(1) = Y$

$$\gamma(t) = (X^{-1} + (Y^{-1} - X^{-1})t)^{-1}$$

in this case

$$\tilde{M}(X,Y) = 2(X^{-1} + Y^{-1})^{-1}\ .$$

Example:

5. $\mathcal{M} := \mathcal{M}_n$, and

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The equation of the geodesic line $\gamma(t) : \mathbb{R} \to \mathcal{M}_n$

$$\ddot{\gamma}(t) - 2\dot{\gamma}(t)\gamma(t)^{-1}\dot{\gamma}(t) = 0$$

and its solution $\gamma(t) = (C_1 + C_2 t)^{-1}$

which satisfies $\gamma(0) = X$, $\gamma(1) = Y$

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$$\tilde{M}(X,Y) = 2(X^{-1} + Y^{-1})^{-1}\ .$$

Questions

1. Which Riemannian metrics guarantee the scaling property:

$$t\tilde{M}(X,Y) = \tilde{M}(tX,tY) \text{ ?}$$

2. How one can find a Riemannian metric for a given mean?

3. Can the geometrical background help to prove that the iteration

$$x_{n+1} := M(y_n, z_n), \; y_{n+1} := M(z_n, x_n), \; z_{n+1} := M(x_n, y_n)$$

is convergent in the space of matrices?

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