Volume of the space of qubit channels and the distribution of some scalar quantities on it

Attila Lovas and Attila Andai

Department of Mathematical Analysis, Budapest University of Technology and Economics

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Abstract

In quantum information theory, a qubit channel is the simplest quantum analogue of a probability transition matrix known from Kolmogorovian probability theory. The space of qubit channels can be identified with a convex submanifold of $\mathbb{R}^{12}$ via Choi representation. To each qubit channel a classical channel can be associated which is called the underlying classical channel. Our main goal is to investigate the distribution of scalar quantities, which are interesting in information geometrical point of view, on the space of qubit channels. Our approach is based on the positivity criterion for self-adjoint matrices by means of the left upper submatrices. This method was previously succesfully applied by A. Andai to compute the volume of density matrices [1]. The volume of the space of qubit channels with respect to the canonical Eucledian measure is computed, and explicit formulas are presented for the distribution of the volume over classical channels. We have constructed an efficient algorithm for generating uniformly distributed points in the space of qubit channels which enables us to investigate numerically the distribution of scalar quantities on the whole space or over a fixed classical channel. We computed the distribution of the Hilbert-Schmidt distance between the indentity matrix and its image under the action of a qubit channel. Distribution of trace-distance contraction coefficient (η^{Tr}) was investigated numerically by Monte-Carlo simulations. The range of possible the values of η^{Tr} over an arbitrary fixed classical channel was determined and the mode of η^{Tr} was calculated numerically. We have found that the distribution of trace-distance contraction coefficient over classical channels shows drammaticaly different behaviour for real and complex unital channels.

Analogies between classical and quantum probability and related geometrical objects

	Classical	Quantum
States	$\Delta_{n-1} = \left\{ p \in [0,1]^n : \sum_{i=1}^{n} p_i = 1 \right\}$	$\mathcal{M}_n^{\mathbb{C}} = \{ D \in \mathbb{C}^{n\times n} : D \geq 0, \operatorname{Tr} D = 1 \}$
Transitions	Stochastic matrices	Quantum channels

1. The volume of density matrices is $V(\mathcal{M}_n^{\mathbb{C}}) = \frac{\pi^{\binom{n}{k}}}{(n^2-1)!} \prod_{i=1}^{n-1} i!$ [1].
2. Classical analogues of identity-preserving $\mathcal{M}_n^{\mathbb{C}} \to \mathcal{M}_n^{\mathbb{C}}$ quantum channels are $n \times n$ doubly stochastic matrices. The class of $n \times n$ doubly stochastic matrices is a convex polytope known as the Birkhoff polytope B_n. Explicit value of $V(B_n)$ is known for $n \leq 10$ and the following asymptotic formula is presented for $V(B_n)$ by R. Canfield and B. McKay [2].

$$V(B_n) = \exp\left(-(n-1)^2 \log(n) + n^2 - \left(n - \frac{1}{2} \right) \log(2\pi) + \frac{1}{3} + o(1) \right)$$

Quantum channels

A linear map $Q : \mathbb{C}^{n\times n} \to \mathbb{C}^{m\times m}$ is *completely positive* (CP) if

$$\forall k \quad \mathrm{id}_k \otimes Q : \mathbb{C}^{k\times k} \otimes \mathbb{C}^{n\times n} \to \mathbb{C}^{k\times k} \otimes \mathbb{C}^{m\times m}$$

is positive. Completely positive and trace preserving linear maps are called *quantum channels* [4].

According to Choi's theorem [3], for a linear map $Q : \mathbb{C}^{n\times n} \to \mathbb{C}^{m\times m}$ the followings are equivalent:

i. $Q : \mathbb{C}^{n\times n} \to \mathbb{C}^{m\times m}$ is CP.
ii. $Q : \mathbb{C}^{n\times n} \to \mathbb{C}^{m\times m}$ is n-positive.
iii. The matrix $(\mathrm{id}_n \otimes Q)\left(\sum_{i,j=1}^{n} E_{ij} \otimes E_{ij} \right)$ is positive, where $E_{ij} \in \mathbb{C}^{n\times n}$ is the matrix with 1 in the ij-th entry and 0s elsewhere.

The *underlying classical channel* corresponding to the quantum channel $Q : \mathcal{M}_n^{\mathbb{C}} \to \mathcal{M}_m^{\mathbb{C}}$ is defined by $P_Q = \pi_m \circ Q \circ \iota_n$, where $\iota_n : \Delta_{n-1} \to \mathcal{M}_n^{\mathbb{C}}$ is the diagonal embedding and $\pi_m : \mathcal{M}_m^{\mathbb{C}} \to \Delta_{m-1}$ is the projection.

The manifold of qubit channels

The block matrix $Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}$ is the Choi matrix of a qubit channel iff $Q_{11}, Q_{22} \in \mathcal{M}_2$, $Q_{21} = Q_{12}^*$, $\operatorname{Tr} Q_{12} = 0$ and $Q > 0$. The space of such a matrices with real (complex) entries is denoted by $\mathcal{Q}_{\mathbb{R}}$ ($\mathcal{Q}_{\mathbb{C}}$) and it is a convex submanifold of $\mathbb{R}^7$ ($\mathbb{R}^{12}$).

Identity preserving requires that $Q_{11} + Q_{22} = \mathrm{id}_2$ which means that the space of unital qubit channels with real (complex) entries are convex submanifolds of $\mathbb{R}^5$ ($\mathbb{R}^9$). These sets are denoted by $\mathcal{Q}_{\mathbb{R}}^1$ and $\mathcal{Q}_{\mathbb{C}}^1$, respectively.

Distribution of volume over classical channels

Theorem 1. The volume of the space of qubit channels with respect to the canonical Eucledian measure is presented in the following table.

	$\mathbb{K} = \mathbb{R}$	$\mathbb{K} = \mathbb{C}$
$V(\mathcal{Q}_{\mathbb{K}})$	$\frac{4\pi^3}{105}$	$\frac{2\pi^5}{4725}$
$V(\mathcal{Q}_{\mathbb{K}}^1)$	$\frac{4\pi^2}{15}$	$\frac{2\pi^4}{315}$

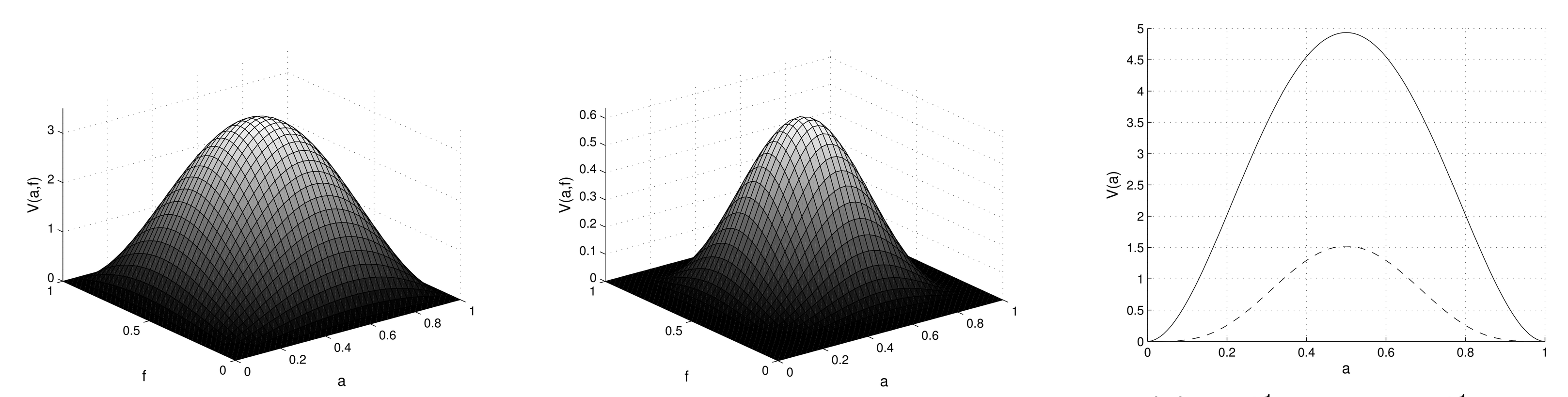
The distribution of volume over classical channels can be expressed as follows:

	Real qubit channels	Complex qubit channels
$V(a,f)$	$\frac{128\pi^2}{45}(af)^{3/2}(5(1-a)(1-f)-af)$	$\frac{16\pi^5}{45}a^3f^3\left[\frac{3}{8}(5(1-a)(1-f)-af)^2 + \frac{5}{8}((1-a)(1-f)-af)^2\right]$,
$V(a)$	$8\pi^2a^2(1-a)^2$	$4\pi^4a^4(1-a)^4$

where the underlying classical channel corresponding to the parametervalues is

$$(a,f) \mapsto \begin{bmatrix} a & 1-a \\ 1-f & f \end{bmatrix} \text{ and } a \mapsto \begin{bmatrix} a & 1-a \\ 1-a & a \end{bmatrix} \tag{1}$$

in the unital case. The formula presented for $V(a,f)$ is valid only for $a + f < 1$. Function values for $a + f \geq 1$ can be computed by using the substitution $(a,f) \mapsto (1-a, 1-f)$.



(a) $V(a,f)$ for $\mathcal{Q}_{\mathbb{R}}$. (b) $V(a,f)$ for $\mathcal{Q}_{\mathbb{C}}$. (c) $V(a)$ for $\mathcal{Q}_{\mathbb{R}}^1$ (solid) and $\mathcal{Q}_{\mathbb{C}}^1$ (dashed).

Figure : Distribution of volume over classical qubit channels. Real channels (a), complex channels (b) and unital qubit channels (c).

Uniform sampling algorithm (presented for $\mathcal{Q}_{\mathbb{R}}$)

Step 1: Generate $a, f \sim \mathcal{U}([0,1])$ independently.

Step 2: Generate $x_1 \sim \mathcal{U}(-\sqrt{af}, \sqrt{af})$ and set $A_2 = \begin{bmatrix} a & x_1 \\ x_1^* & f \end{bmatrix}$.

Step 3: Generate $y_2 \sim \mathcal{U}(\{r \in \mathbb{R}^2 : \|r\| \leq \sqrt{f}\})$ and set $A_3 = \begin{bmatrix} A_2 & x_2 \\ x_2^* & 1-a \end{bmatrix}$, where $x_2 = \sqrt{A_2}y_2$.

Step 4: Compute the projection P onto the subspace Span($\{\sqrt{A_3}e_3\}$) and set $z = -x_1(1-f)^{-1/2}P\sqrt{A_3^{-1}}e_3$.

Step 5: **If** $\|z\| > 1$, **then goto** Step 2.

Step 6: Generate $y_3 \sim \mathcal{U}(\{r \in \mathbb{R}^2 : \|r\| \leq \sqrt{1 - \|z\|^2}\})$ and set $A = \begin{bmatrix} A_3 & x_3 \\ x_3^* & 1-f \end{bmatrix}$, where $x_3 = \sqrt{1-f}\sqrt{A_3}([e_1, e_2]y_3 + z)$.

Step 7: Apply the transform $Q = UAU^*$.

Distribution of the HS distance between identity matrix and its image

For a quantum channel $Q : \mathcal{M}_n^{\mathbb{C}} \to \mathcal{M}_m^{\mathbb{C}}$ the Hilbert-Schmidt distance between id_m and $Q(\mathrm{id}_n)$ measures the "non-unitality" of the channel. For a qubit channel represented by the Choi matrix

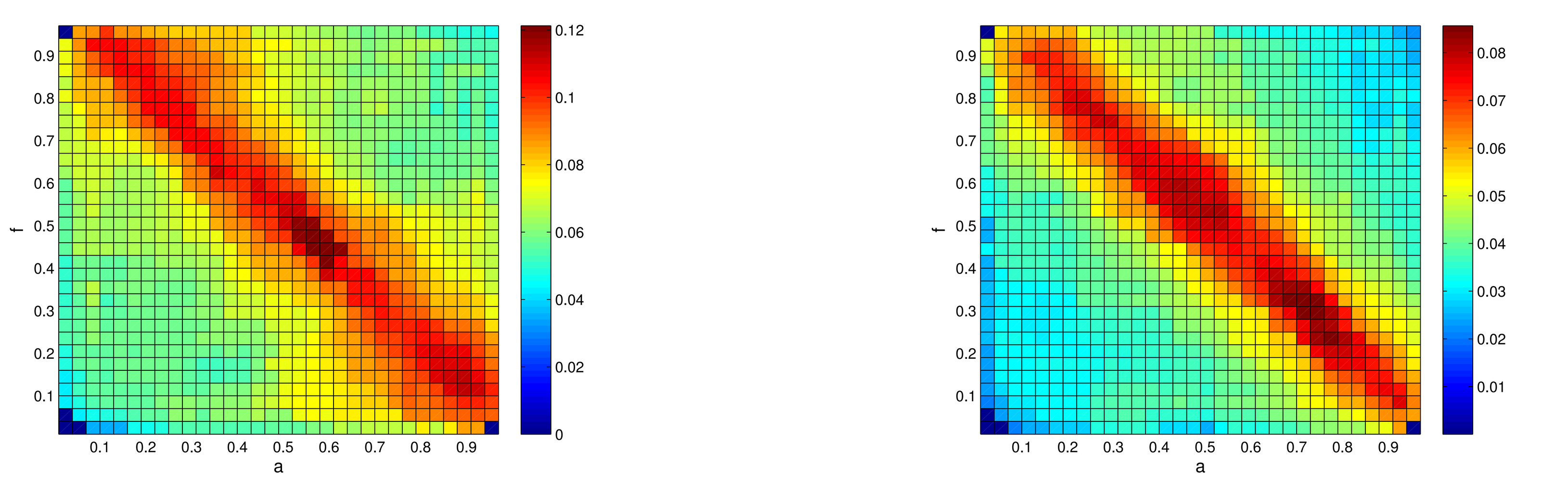
$$Q = \begin{bmatrix} a & b & c & d \\ \bar{b} & 1-a & e & -c \\ \bar{c} & \bar{e} & f & g \\ \bar{d} & -\bar{c} & \bar{g} & 1-f \end{bmatrix}, \quad \|Q(I) - I\|_{HS} = \sqrt{2}\left((a+f-1)^2 + |b+g|^2 \right)^{1/2}.$$

Let us denote the set of qubit channels over the classical channel (1) by $\mathcal{Q}_{\mathbb{R}}(a,f)$ and $\mathcal{Q}_{\mathbb{C}}(a,f)$. The next theorem gives the range of $\|Q(I) - I\|_{HS}$ on these sets.

Theorem 2. For the Hilbert-Schmidt distance between id_2 and $Q(\mathrm{id}_2)$, the following identity holds

$$\{\|Q(I) - I\|_{HS} : Q \in \mathcal{Q}_{\mathbb{C}}(a,f)\} = \{\|Q(I) - I\|_{HS} : Q \in \mathcal{Q}_{\mathbb{R}}(a,f)\} = [m, M],$$

where $m = \sqrt{2}\left|af - (1-a)(1-f)\right|$, $M = \sqrt{2}\left|\sqrt{af} + \sqrt{(1-a)(1-f)}\right|$ and $a, f \in [0,1]$.



(a) $\mathcal{Q}_{\mathbb{R}}$ (b) $\mathcal{Q}_{\mathbb{C}}$

Figure : Monte Carlo simulation of the the expectation of the rescaled Hilbert-Schmidt distance: $\frac{\|Q(I)-I\|_{HS}-m}{M-m}$. In each cell of the 33×33 grid 20 independent qubit channel were generated.

Distribution of trace-distance contraction coefficient

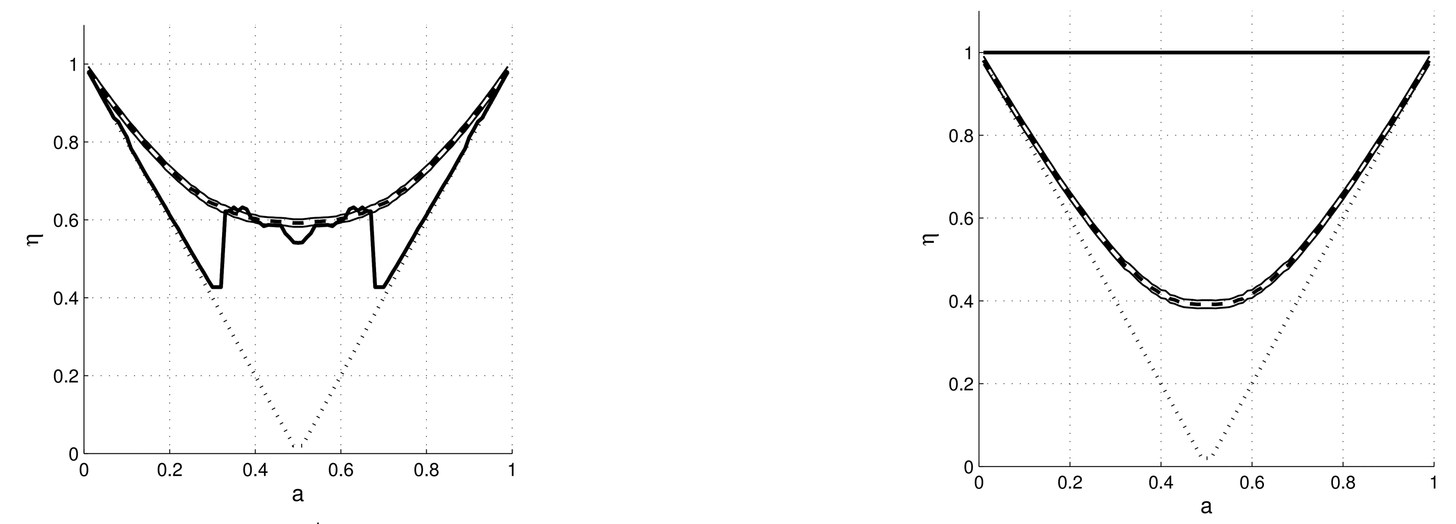
The trace-distance contraction coefficient of a quantum channel $Q : \mathcal{M}_n^{\mathbb{C}} \to \mathcal{M}_m^{\mathbb{C}}$ decribes the maximal contraction under Q it is defined by

$$\eta^{\mathrm{Tr}}(Q) = \sup\left\{ \frac{\|Q(\rho) - Q(\sigma)\|_1}{\|\rho - \sigma\|_1} : \rho, \sigma \in \mathcal{M}_n^{\mathbb{C}} \right\}.$$

The next theorem gives the range of $\eta^{\mathrm{Tr}}(Q)$ on $\mathcal{Q}_{\mathbb{R}}(a,f)$, $\mathcal{Q}_{\mathbb{C}}(a,f)$, $\mathcal{Q}_{\mathbb{R}}^1(a)$ and $\mathcal{Q}_{\mathbb{C}}^1(a)$.

Theorem 3. For arbitrary $a, f \in [0,1]$ and $x \in [|a-f|, \sqrt{(1-a)f} + \sqrt{a(1-f)})$ there exists a qubit channel $Q \in \mathcal{Q}_{\mathbb{R}}(a,f) \subset \mathcal{Q}_{\mathbb{C}}(a,f)$ for which $\eta^{\mathrm{Tr}}(Q) = x$. Furthermore, $\inf\{\eta(Q) : Q \in \mathcal{Q}_{\mathbb{C}}(a,f)\} = |a-f|$ which is equal to the trace-distance contraction coefficient of the underlying classical channel.

The mode of η^{Tr} on $\mathcal{Q}_{\mathbb{R}}^1(a)$ and $\mathcal{Q}_{\mathbb{C}}^1(a)$ $a \in [0,1]$ was investigated by Monte Carlo simulation. The interval $[0,1]$ was divided into 100 equidistant parts. Expectation and mode of η^{Tr} was estimated from the sample of size $n = 1000$ in each point.



(a) $\mathcal{Q}_{\mathbb{R}}^1$ (b) $\mathcal{Q}_{\mathbb{C}}^1$

Figure : Minimal value (dotted) of η^{Tr}, mode of η^{Tr} (thick), expectation of η^{Tr} (dashed) and confidence band (solid) corresponding to the expectation ($n = 1000$, $\alpha = 5 \times 10^{-5}$).

The mode shows irregular behaviour in case of real unital channels. Small deviations of mode from infimum can be observed near $a \approx 0.1$ and $a \approx 0.9$ The distribution of η^{Tr} changes dramatically near $a \approx 0.33$ & $a \approx 0.67$. Qubit channels over the complex field are condensed near the extremal $\eta^{\mathrm{Tr}} = 1$ isosurface of η^{Tr}.

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 lovas@math.bme.hu, andaia@math.bme.hu