Geometry Behind Robertson-type Uncertainty Relations

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Basic Concepts from Quantum Mechanics

State space: the set of $n \times n$ positive definite trace one matrices, denoted by $\mathcal{M}_n$.
Observables: $n \times n$ self adjoint matrices (M_{sa}).

For given state $D \in \mathcal{M}_n$ and observables $A, B \in M_{\mathrm{sa}}$ we have the following concepts.

Expectation value: $\mathbb{E}_D(A) = \operatorname{Tr}(DA)$.

Normalization of A: $A_0 = A - \mathbb{E}_D(A)I$. (One has $\mathbb{E}_D(A_0) = 0$.)

Variance: $\operatorname{Var}_D(A) = \mathbb{E}_D\left(A^2\right) - \left(\mathbb{E}_D(A)\right)^2$.

Covariance: $\operatorname{Cov}_D(A,B) = \mathbb{E}_D\left(\frac{1}{2}(AB+BA)\right) - \mathbb{E}_D(A)\mathbb{E}_D(B)$.

Brief History of Early Uncertainty Relations

The very first formalization of the uncertainty principle was

$$\textbf{1927, Heisenberg} \quad D_f D_{\mathcal{F}(f)} = \text{constant},$$

where f is a Gaussian distribution, D_f its width (its *uncertainty*) and $D_{\mathcal{F}(f)}$ is the width of the Fourier transformation of f.

Later for any state D and every observables A, B the following relations were proved.

1927, Kennard $\quad \operatorname{Var}_D(A)\operatorname{Var}_D(B) \geq \frac{1}{4} \quad$ if $[A,B] = -\mathrm{i}$,

1929, Robertson $\quad \operatorname{Var}_D(A)\operatorname{Var}_D(B) \geq \frac{1}{4}\left|\operatorname{Tr}(D[A,B])\right|^2$

1930, Schrödinger $\quad \operatorname{Var}_D(A)\operatorname{Var}_D(B) - \operatorname{Cov}_D(A,B)^2 \geq \frac{1}{4}\left|\operatorname{Tr}(D[A,B])\right|^2$

1934, Robertson: For every set of observables $(A_i)_{1,\dots,N}$

$$\det\left(\left[\operatorname{Cov}_D(A_h,A_j)\right]_{h,j=1,\dots,N}\right) \geq \det\left(\left[-\frac{\mathrm{i}}{2}\operatorname{Tr}(D[A_h,A_j])\right]_{h,j=1,\dots,N}\right).$$

Riemannian Metrics on the State Space

In classical probability setting the Fisher information matrix can be used to endow a statistical model with Riemannian metric.

Naturally arises the question: *What is the analogue of the Fisher information in the quantum mechanical framework?*

$\mathcal{F}_{\mathrm{op}}$: set of operator monotone functions $f : \mathbb{R}^+ \to \mathbb{R}$ with properties $f(x) = xf(x^{-1})$ and $f(1) = 1$.

Examples for such functions: $f(x) = \frac{2x}{1+x}, \quad \frac{1+x}{2}, \quad \left(\frac{1+\sqrt{x}}{2}\right)^2, \quad \frac{x-1}{\log x}$.

For every $f \in \mathcal{F}_{\mathrm{op}}$ introduce the notation $g_f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, $g_f(x,y) = (yf(x/y))^{-1}$.
(The function g_f is known as Chentsov–Morozova function.)

Theorem [Petz]. *In quantum setting there is a bijective correspondence between Fisher informations and functions in $f \in \mathcal{F}_{\mathrm{op}}$. For every $f \in \mathcal{F}_{\mathrm{op}}$ the Fisher information is given by*

$$\langle A,B\rangle_{D,f} = \operatorname{Tr}\left(A\left(g_f\left(L_D,R_D\right)(B)\right)\right),$$

where $L_D(X) = DX$, $R_D(X) = XD$. (The form $\langle\cdot,\cdot\rangle_{D,f}$ is known as Petz scalar product)

$\Longrightarrow$ For every $f \in \mathcal{F}_{\mathrm{op}}$ the pair $\left(\mathcal{M}_n, \langle\cdot,\cdot\rangle_{\cdot,f}\right)$ is a Riemannian manifold.

Covariances

For observables $A, B \in M_{\mathrm{sa}}$, state $D \in \mathcal{M}_n$ and function $f \in \mathcal{F}_{\mathrm{op}}$ let us define the following covariances.

Covariance: $\operatorname{Cov}_D(A,B) = \frac{1}{2}\operatorname{Tr}(DAB + DBA) - \operatorname{Tr}(DA)\operatorname{Tr}(DB)$.

Quantum f-covariance: $\operatorname{Cov}_D^f(A,B) = \operatorname{Tr}\left(Af(L_D R_D^{-1})R_D(B)\right)$.

Antisymmetric f-covariance: $\operatorname{qCov}_{D,f}^{as}(A,B) = \frac{f(0)}{2}\langle \mathrm{i}\,[D,A], \mathrm{i}\,[D,B]\rangle_{D,f}$.

Symmetric f-covariance: $\operatorname{qCov}_{D,f}^{s}(A,B) = \frac{f(0)}{2}\langle \{D,A\}, \{D,B\}\rangle_{D,f}$.

Here $[.,.]$ is the commutator of matrices and $\{.,.\}$ denotes the anticommutator respectively.

For a fixed density matrix $D \in \mathcal{M}_n$, a function $f \in \mathcal{F}_{\mathrm{op}}$ and an N-tuple of matrices $(A^{(k)})_{k=1,\dots,N} \in M_{\mathrm{sa}}$ we define the following $N \times N$ matrices Cov_D, Cov_D^f, $\operatorname{qCov}_{D,f}^{as}$ and $\operatorname{qCov}_{D,f}^{s}$ with entries

$$[\operatorname{Cov}_D]_{ij} = \operatorname{Cov}_D(A_0^{(i)},A_0^{(j)}) \qquad \left[\operatorname{Cov}_D^f\right]_{ij} = \operatorname{Cov}_D^f(A_0^{(i)},A_0^{(j)})$$

$$\left[\operatorname{qCov}_{D,f}^{as}\right]_{ij} = \operatorname{qCov}_{D,f}^{as}(A_0^{(i)},A_0^{(j)}) \qquad \left[\operatorname{qCov}_{D,f}^{s}\right]_{ij} = \operatorname{qCov}_{D,f}^{s}(A_0^{(i)},A_0^{(j)}).$$

Main Mathematical Ingredients

Generalization of Petz's scalar product:

Define

$$\mathcal{C}_{\mathcal{M}} = \left\{ g : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ \;\middle|\; \begin{array}{l} g \text{ is a symmetric smooth function, with analytical} \\ \text{extension defined on a neighborhood of } \mathbb{R}^+ \times \mathbb{R}^+ \end{array} \right\}.$$

Fix a function $g \in \mathcal{C}_{\mathcal{M}}$. Define for every $D \in \mathcal{M}_n$ and for every $A, B \in M_{\mathrm{sa}}$

$$(A,B)_{D,g} = \operatorname{Tr}\left(Ag(L_D,R_D)(B)\right).$$

Theorem [Andai, Lovas]. *For every $g \in \mathcal{C}_{\mathcal{M}}$ the pair $\left(\mathcal{M}_n, (\cdot)\cdot_{,g}\right)$ is a Riemannian manifold.*

Relation between covariance matrices:

Theorem [Andai, Lovas]. *Consider a density matrix $D \in \mathcal{M}_n$, an N-tuple of observables $(A^{(k)})_{k=1,\dots,N}$ and functions $g_1, g_2 \in \mathcal{C}_{\mathcal{M}}$ such that*

$$g_1(x,y) \geq g_2(x,y) \qquad \forall x,y \in \mathbb{R}^+.$$

For the $N \times N$ matrices $\mathfrak{Cov}_{D,g_1}$ and $\mathfrak{Cov}_{D,g_2}$ with entries $\left[\mathfrak{Cov}_{D,g_k}\right]_{ij} = (A_0^{(i)},A_0^{(j)})_{D,g_k}$ $(k=1,2)$ one has

$$\mathfrak{Cov}_{D,g_1} \geq \mathfrak{Cov}_{D,g_2}, \qquad \textit{(as positive matrices).}$$

Generalized Minkowski inequality for positive matrices:

For an $n \times n$ matrix A define the matrix invariants $\alpha_k(A)$ (for $k \in \{1,\dots,n\}$) as

$$\det(A + tI) = t^n + \sum_{k=0}^{n-1}\alpha_{n-k}(A)t^k.$$

($\alpha_1(A) = \operatorname{Tr} A$, $\alpha_n(A) = \det A$.)

Theorem [Andai, Lovas]. *Consider the $n \times n$ positive matrices A, B. Then for every matrix invariant $(\alpha_k,\ k \in \{1,\dots,n\})$ one has*

$$\alpha_k(A+B) \geq \alpha_k(A) + \alpha_k(B).$$

Recent Uncertainty Relations

Around 2000 it turned out that the lower bound in the Robertson uncertainty relation can be sharpened using (antisymmetric) quantum covariances. First partial results considered very specific functions from the set $\mathcal{F}_{\mathrm{op}}$, few (generally 2) observables and the inequalities were expressed in a more complicated form.

Unification of these modern attempts:

The key point was to realize that these early results can be combined together.

Conjecture [Gibilisco and Isola in 2006]. $\quad \det(\operatorname{Cov}_D) \geq \det(\operatorname{qCov}_{D,f}^{as})$

Theorem [*Andai and Gibilisco, Imparato and Isola in 2008*]. *For any operator monotone function $f \in \mathcal{F}_{\mathrm{op}}$ at every state $D \in \mathcal{M}_n$ for every N-tuple of observables $(A^{(k)})_{k=1,\dots,N}$ we have for the covariance matrices*

$$\det(\operatorname{Cov}_D) \geq \det(\operatorname{qCov}_{D,f}^{as}).$$

A more accurate inequality:

Theorem [*Lovas, Andai, 2016*]. *For any operator monotone function $f \in \mathcal{F}_{\mathrm{op}}$ at every state $D \in \mathcal{M}_n$ for every N-tuple of observables $(A^{(k)})_{k=1,\dots,N}$, the following holds.*

$$\det(\operatorname{Cov}_D) \geq \det(\operatorname{qCov}_{D,f}^{s}) \geq \det(\operatorname{qCov}_{D,f}^{as})$$

We have an estimation for the gap between the symmetric and antisymmetric covariance:

Theorem [*Lovas, Andai, 2016*]. *Using the same notation as in the previous Theorem we have*

$$\det(\operatorname{qCov}_{D,f}^{s}) - \det(\operatorname{qCov}_{D,f}^{as}) \geq (2f(0))^N \det(\operatorname{Cov}_D^{f_0}),$$

where $f_0(x) = \frac{2x}{1+x}$.

Moreover, we have shown that the symmetric covariance generated by the operator monoton function $f_{opt}(x) = \frac{1+x}{4} + \frac{x}{1+x}$ is universal in the following sense.

Theorem [*Lovas, Andai, 2016*]. *For every function $g \in \mathcal{F}_{\mathrm{op}}$ the inequality*

$$\det(\operatorname{qCov}_{D,f_{opt}}^{s}) \geq \det(\operatorname{qCov}_{D,g}^{as})$$

holds and f_{opt} gives the best upper bound in $\mathcal{F}_{\mathrm{op}}$.

Inequality for other invariants of the covariance matrix:

Theorem [*Andai, Lovas, 2019*]. *For any operator monotone function $f \in \mathcal{F}_{\mathrm{op}}$ at every state $D \in \mathcal{M}_n$ for every N-tuple of observables $(A^{(k)})_{k=1,\dots,N}$ and every matrix invariant $(\alpha_k,\ k \in \{1,\dots,N\})$, the following holds.*

$$\alpha_k(\operatorname{Cov}_D) \geq \alpha_k(\operatorname{qCov}_{D,f}^{s}) \geq \alpha_k(\operatorname{qCov}_{D,f}^{as})$$