C^*-ALGEBRAS

J. A. Erdos

Department of Mathematics
King's College
London WC2R 2LS
ENGLAND

Abstract

These notes form an introductory account of C^*-algebras. Sections 1, 2, and 4 provide a straightforward development of the subject up to the Gelfand-Naimark Theorem, namely the result that every C^*-algebra is isometrically isomorphic to a *-subalgebra of the algebra of operators on some Hilbert space. Some results on more general commutative Banach algebras, whose proofs require little extra effort, are included. In Section 3 there are accounts of two applications of the commutative theory: the C^*-algebra approach to the spectral theorem for bounded normal operators on Hilbert space and a brief introduction to the ideas of abstract harmonic analysis.

1 Definitions, the spectrum and other basics.

1.1 Banach algebras and C*-algebras

In these notes we shall use the term **algebra** to mean a linear associative algebra where, if no field is mentioned, the scalars will be the complex field $\mathbb{C}$. An algebra A is said to be a **normed algebra** if it has a norm that makes it into a normed linear space and the norm also satisfies

(i) $||ab|| \leq ||a||.||b||$

(ii) if A has an identity e then $||e|| = 1$.

If A is a normed algebra and A is a Banach space (that is, A, with its norm, is complete) then A is called a **Banach** algebra.

An **involution** on A is a map $^* : A \to A \quad (a \mapsto a^*)$ such that

(iii) $a^{**} = a$,

(iv) $(\lambda a + \mu b)^* = \bar{\lambda} a^* + \bar{\mu} b^*$,

(v) $(ab)^* = b^* a^*$.

If A is a Banach algebra with involution and also

(vi) $||aa^*|| = ||a||^2$,

then A is called a C*-algebra and (vi) is called the "C* condition".

The term "Banach algbra with involution" is normally reserved for the case when the involution is continuous; if for all $a \in A$ we have $||a^*|| = ||a||$ we say that the involution is **isometric**.

Some trivial consequences. The following results hold.

1. In a Banach algebra, multiplication $A \times A \to A, \quad ((a, b) \mapsto ab)$ is continuous.

This follows by applying (i) and the triangle inequality for the norm to the identity

$$ab - a_0 b_0 = (a - a_0)(b - b_0) + (a - a_0) b_0 + a_0 (b - b_0)$$

to obtain

$$||ab - a_0b_0|| \leq ||a - a_0||.||b - b_0|| + ||a - a_0||.||b_0|| + ||a_0||.||b - b_0||\,.$$

2. In a C*-algebra the involution is isometric.

Using the C* condition and (i), we have that for any $a \in A$, $||a||^2 = ||aa^*|| \leq ||a||.||a^*||$ so that $||a|| \leq ||a^*||$. Applying this to a^* gives the opposite inequality, so $||a|| = ||a^*||$.

3. In a C*-algebra

$$||a|| = \sup_{||x|| \leq 1} ||ax|| = \sup_{||x|| \leq 1} ||xa||\,.$$

Clearly $\sup_{||x||\leq 1} ||ax|| \leq \sup_{||x||\leq 1} ||a||.||x|| \leq ||a||$ and when $x = \frac{a^*}{||a^*||} = \frac{a^*}{||a||}$, this supremum is attained.

4. If a C*-algebra A has an identity e, then $e = e^*$.

In any algebra, the identity is unique. Now for all $a \in A$,

$$ae^* = (ea^*)^* = (a^*)^* = a$$

and similarly $a = e^*a$ so e^* is the identity, showing that $e = e^*$. Note also that the C* condition makes (ii) redundant since $||e|| = ||ee^*|| = ||e||^2$ and so $||e|| = 1$.

5. The C* condition (vi) may be replaced by $||aa^*|| \geq ||a||^2$.

This is because this is enough to deduce that $||a|| = ||a^*||$ in (2) above, and so the opposite inequality is a consequence of (i) : $||aa^*|| \leq ||a||.||a^*|| = ||a||^2$

In fact, (vi) may be replaced by $||aa^*|| = ||a||.||a^*||$ but this is not important (and the proof is very difficult).

Examples.

1. $C(X)$. Let X be a compact space and let $C(X)$ be the Banach space of all complex-valued functions on X with the usual norm, $||f|| = \sup_{x\in X} |f(x)|$. Multiplication in $C(X)$ is defined pointwise : $f.g(x) = f(x).g(x)$ and the involution by complex conjugation $f^*(x) = \overline{f(x)}$. It is easy to see that $C(X)$ is a commutative C*-algebra which has an identity, namely the function e where $e(x) = 1$ for all $x \in X$.

2. $C_0(X)$. Now X is a locally compact space and $C_0(X)$ is the Banach space of all complex-valued functions on X which vanish at infinity. The algebraic

operations and norm are defined as in Example 1. Once again, $C_0(X)$ is a commutative C*-algebra but this time it has no identity.

3. $\mathcal{B}(\mathcal{H})$. Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the algebra of all continuous linear operators on $\mathcal{H}$. For $A \in \mathcal{B}(\mathcal{H})$ let A^* be the usual adjoint. Then it is clear that $\mathcal{B}(\mathcal{H})$ is a C*-algebra. Furthermore, any subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ that is closed under adjoints (that is, $A^* \in \mathcal{A}$ whenever $A \in \mathcal{A}$) and is closed in the norm sense (hence complete) is an example of a C*-algebra.

If $\mathcal{X}$ is a Banach space, the algebra $\mathcal{B}(\mathcal{X})$ of all continuous operators on $\mathcal{X}$ is a Banach algebra but if $\mathcal{X}$ is not a Hilbert space then $\mathcal{B}(\mathcal{X})$ is not, in general, a C*-algebra.

4. $A(\Delta)$. Let Δ be the open unit disc and let $A(\Delta)$ be the set of functions that are analytic on Δ and continuous on its closure, $\bar{\Delta}$. $A(\Delta)$ is a subalgebra of $C(\bar{\Delta})$. Note that $A(\Delta)$ is not closed under the complex conjugation involution. However, there is another natural involution that can be defined on $A(\Delta)$, namely

$$f^*(z) = \overline{f(\bar{z})}\,.$$

It is easy to verify that this involution is isometric (that is, $||f|| = ||f^*||$). Also, if $fg = 0$ then one (or both) of f and g must vanish at an infinite number of points inside Δ and so, by a well-known theorem of complex function theory, must be zero. Thus $A(\Delta)$ is an integral domain. In particular, $f.f^* = 0 \Rightarrow f = 0$. However, $A(\Delta)$ is not a C*-algebra. To see this take, for example, $f(z) = e^{iz}$. It is trivial to verify that $||f.f^*|| = 1$ and $||f|| \geq e$ so that $||f.f^*|| \neq ||f||^2$.

5. $L^1(\mathbb{R})$, $L^1(\mathbb{T})$, $\ell^1(\mathbb{Z})$ and more generally $L^1(G)$ for a locally compact abelian groups. We first deal with the Banach space $L^1(\mathbb{R})$ of integrable functions (with usual Lebesgue measure) on the real line and the usual norm

$$||f||_1 = \int_{-\infty}^{\infty} f(t)\,dt\,.$$

Multiplication of two functions f and g is defined as their convolution :

$$(f * g)(t) = \int_{-\infty}^{\infty} f(s)g(t-s)\,ds\,.$$

The change of variable $u = t - s$ in the integration shows that, also,

$$(f * g)(t) = \int_{-\infty}^{\infty} f(t-u)g(u)\,du\,.$$

It is a simple exercise, using the theorems of Fubini and Tonelli to show that

$$||f * g||_1 \leq ||f||_1.||g||_1\,,$$

(see Lemma 3.2 for a proof of a slightly more general result). It follows that $L^1(\mathbb{R})$ is a commutative Banach algebra. It is not a C*-algebra (although one can define an isometric involution by $f^*(t) = \overline{f(-t)}$).

$L^1(\mathbb{T})$, the integrable functions on the unit circle is defined similarly. We can parametrise the unit circle $\mathbb{T}$ by $\{e^{it} : 0 \leq t < 2\pi\}$ and identify functions on $\mathbb{T}$ with periodic functions of period 2π on $\mathbb{R}$. The norm and multiplication are defined as above, except that the limits of integration are 0 and 2π.

$\ell^1(\mathbb{Z})$ is the space of sequences $\{\xi_n : -\infty < n < \infty\}$ such that $\sum_{-\infty}^{\infty} |\xi_n|$ converges. The norm of a sequence $x = (\xi_n)$ is given by

$$||x|| = \sum_{-\infty}^{\infty} |\xi_n|$$

and the product of x with $y = (\eta_n)$ is $z = (\zeta_n)$ where

$$\zeta_k = \sum_{n=-\infty}^{\infty} \xi_n \eta_{k-n}$$

All the above classical examples are special cases of the following situation. A topological group is a group with a topology on it that makes the group operations continuous. Let G be a topological group which is abelian as a group and locally compact as a topological space. It is shown in books on measure theory that on any such group one can find a measure μ that is invariant under the group operation. That is to say, for any measurable subset δ of G,

$$\mu(\delta x) = \mu(\delta)$$

for all $x \in G$. Here the group operation is written as multiplication and δx denotes the set $\{yx : y \in \delta\}$. This measure is unique (up to a multiplicative constant) and is called Haar measure.

The set $L^1(G)$ of complex-valued functions that are integrable with respect to Haar measure forms a Banach algebra under the norm

$$||f|| = \int_G |f(t)|\, dt\,.$$

and multiplication

$$(f * g)(t) = \int_G f(s)g(s^{-1}t)\, ds\,.$$

It is easy to verify that all the above examples fit into this framework. It is these examples that form the subject of abstract harmonic analysis which is treated in more detail in Section 3.

Looking ahead. The main aim of these notes is to prove that all C*-algebras are of the form of the examples above. Commutative C*-algebras are like Examples 1 or 2 (depending on whether there is an identity) and all C*-algebras are of the form of Example 3.

Much of the terminology of C*-algebras is inherited from operator theory on Hilbert space. Thus, an element a of a C*-algebra A is said to be **self-adjoint** if $a^* = a$, and **normal** if $aa^* = a^*a$. If A has an identity e then $u \in A$ is called **unitary** if $uu^* = u^*u = e$.

1.2 The spectrum

Let A be a Banach algebra with identity e (algebras without identity will be discussed later). The **spectrum** of an element a of A is defined to be the set of all complex numbers λ such that $\lambda e - a$ has no inverse in A. The spectrum of a is denoted by $\sigma(a)$. Thus

$$\sigma(a) = \{\lambda : \lambda e - a \text{ is singular } \}\,.$$

It may happen that an element a can be considered as a member of more than one algebra. If it is necessary to indicate the algebra, we write $\sigma_A(a)$ for the spectrum of a as a member of the algebra A.

The complement $\mathbb{C} \setminus \sigma(a)$ of $\sigma(a)$ is called the **resolvent set** of a and denoted by $\rho(a)$.

The above definitions are identical to those for operators. These in turn are generalisations of the eigenvalues of matrix theory.

To study the spectrum, it is necessary to discuss when elements have inverses. The simple but important Lemma 1.1 below uses a formal geometric series. For the proof, we need to recall the concept of absolute convergence for series in a Banach space. To say that $\sum x_n$ is absolutely convergent means that $\sum ||x_n||$ is convergent as a series of real numbers. For an absolutely convergent series $\sum x_n$, write $s_n = \sum_{r=1}^{n} x_r$. Then

$$||s_{n+p} - s_n|| = \left\| \sum_{r=n+1}^{n+p} x_r \right\| \leq \sum_{r=n+1}^{n+p} ||x_r||\,.$$

Since $\sum ||x_n||$ is convergent and hence a Cauchy series, we have that (s_n) is a Cauchy sequence thus converges. This shows that absolutely convergent series in a Banach space are convergent.

Lemma 1.1 *If* $||x|| < 1$ *then* $(e-x)^{-1}$ *exists.*

Proof. Since $||x^n|| \le ||x||^n$, and $||x|| < 1$, it is clear that $\sum_0^\infty x^n$ is absolutely convergent. Denote its sum by y. Now note the identity

$$e-x^{n+1} = (e-x)(e+x+x^2+x^3+\cdots+x^n) = (e+x+x^2+x^3+\cdots+x^n)(e-x)\,.$$

Let $n \to \infty$. Then $x^{n+1} \to 0$ and so

$$e = (e-x)y = y(e-x)$$

showing that y is the inverse of $e-x$. ■

The above lemma is often stated in the obviously equivalent form : if $||e - a|| < 1$ then a^{-1} exists.

Theorem 1.2 *Let* G *be the set of invertible elements of a Banach algebra* A*. Then*

(i) G *is open,*

(ii) *the map* $x \mapsto x^{-1}$ $\quad (G \to G)$ *is continuous.*

Proof. (i) We need to show that if $a \in G$ then, for x of sufficiently small norm, $a + x \in G$. Write $a + x = a(e + a^{-1}x)$. From 1.1, if $||a^{-1}x|| < 1$ then $(e + a^{-1}x)$ is invertible and hence so is $a + x$. Thus, if $||x|| < 1/||a^{-1}||$ then $a + x \in G$ and so the open ball centre a radius $||a^{-1}||^{-1}$ is a subset of G.

(ii) Given $a \in A$ and any $\epsilon > 0$, write

$$\begin{aligned} ||a^{-1} - b^{-1}|| &= ||b^{-1}(b-a)a^{-1}|| \\ &= ||(b^{-1} - a^{-1})(b-a)a^{-1} + a^{-1}(b-a)a^{-1}|| \\ &\le ||b^{-1} - a^{-1}||.||b-a||.||a^{-1}|| + ||b-a||.||a^{-1}||^2 \end{aligned}$$

So

$$\Big(||a^{-1} - b^{-1}||\Big)\Big(1 - ||b-a||.||a^{-1}||\Big) \le ||a^{-1}||^2.||b-a||\,.$$

First taking $||b - a||$ sufficiently small to make $1 - ||b-a||.||a^{-1}|| > \frac{1}{2}$, it is clear that $||a^{-1} - b^{-1}||$ may be made arbitrarily small by taking $||b-a||$ small enough. ■

Corollary 1.3 *The closure of a proper ideal is proper. Hence maximal ideals are closed.*

Proof. Continuity of the algebraic operations shows that the closure of an ideal is an ideal. But a proper ideal $\mathcal{I}$ contains no invertible elements and so is disjoint from some open ball centre the identity e. The closure $\bar{\mathcal{I}}$ is therefore also disjoint from this ball and so it is a proper ideal. ■

Theorem 1.4 *The spectrum $\sigma(a)$ of any element a of A is a non-empty compact subset of $\mathbb{C}$.*

Proof. We first show that $\sigma(a)$ is bounded. In fact, if $|\lambda| > ||a||$ then $||\lambda^{-1}a|| < 1$ and so, by 1.1 $\lambda e - a = \lambda(e - \lambda^{-1}a)$ has an inverse. Hence $\sigma(a)$ is contained in the ball centre 0, radius $||a||$.

To prove that $\sigma(a)$ is closed, we show that $\rho(a)$ is open. If $\lambda \in \rho(a)$ then $\lambda e - a \in G$ where G, the set of invertible elements, is open (1.2). Hence, for some $\epsilon > 0$, any element b with $||b - (\lambda e - a)|| < \epsilon$ is in G. Thus if $|\mu - \lambda| < \epsilon$ then $\mu e - a \in G$ and so $\mu \in \rho(a)$, showing that $\rho(a)$ is open. (More briefly one can say that $\rho(a)$ is open since it is the pre-image of the open set G under the continuous map $\lambda \mapsto \lambda e - a$.)

For the proof that $\sigma(a) \neq \emptyset$, first note that, if $\lambda, \mu \in \rho(a)$ then

$$(\lambda e - a)^{-1} - (\mu e - a)^{-1} = [(\mu e - a)(\mu e - a)^{-1}](\lambda e - a)^{-1} - [(\lambda e - a)(\lambda e - a)^{-1}](\mu e - a)^{-1}$$

and so, since all the elements commute,

$$(\lambda e - a)^{-1} - (\mu e - a)^{-1} = (\mu - \lambda)(\lambda e - a)^{-1}(\mu e - a)^{-1}$$

(the above relation is called the **resolvent equation**).

Now take any $\phi \in A'$ (the Banach space dual of A) and let $f(\lambda) = \phi[(\lambda e - a)^{-1}]$. Then f is holomorphic on $\rho(a)$, since

$$\begin{aligned}
f'(\lambda) &= \lim_{\mu \to \lambda} \frac{f(\mu) - f(\lambda)}{\mu - \lambda} \\
&= \lim_{\mu \to \lambda} \phi \left\{ \frac{(\mu e - a)^{-1} - (\lambda e - a)^{-1}}{\mu - \lambda} \right\} \\
&= \lim_{\mu \to \lambda} \phi \left\{ -(\lambda e - a)^{-1}(\mu e - a)^{-1} \right\} \\
&= -\phi(\lambda e - a)^{-2} .
\end{aligned}$$

Note that above (and again below), we use the continuity of inversion, (Lemma 1.2 (ii)) several times and also the continuity of ϕ .

Now, if $\sigma(a)$ were empty, f would be entire. Also,

$$\lim_{|\lambda|\to\infty} |f(\lambda)| = \lim_{|\lambda|\to\infty} \frac{1}{|\lambda|}\phi\left[\left(e - \frac{a}{\lambda}\right)^{-1}\right] = 0\,.$$

The above shows that f is bounded and entire, so by Liouville's Theorem it is constant and its limit as $|\lambda| \to \infty$ shows $f = 0$. But, by the Hahn-Banach Theorem we can choose ϕ so that $\phi(-a^{-1}) = f(0) \neq 0$, giving us a contradiction. Thus f cannot be entire and so $\sigma(a)$ is not empty. $\blacksquare$

Note that it is essential for the above proof that the field of scalars of our Banach algebra is the complex field.

Theorem 1.5 (Gelfand-Mazur) *Every complex Banach division algebra is isometrically isomorphic to $\mathbb{C}$. This isomorphism is unique.*

Proof. In a division algebra A, the only singular element is 0. If $a \in A$, there exists an element λ of $\sigma(a)$ and $\lambda e - a$, being singular, equals 0. Therefore A consists of complex multiples of the identity. Clearly

$$||a|| = ||\lambda e|| = |\lambda|$$

and the map $a \mapsto \lambda$ where $a = \lambda e$ is an isometric isomorphism. ■

Theorem 1.6 (Spectral Mapping Theorem - elementary form) *Let a be an element of a Banach algebra with identity.*

(i) *If p is any polynomial, then*

$$\sigma\left(p(a)\right) = \{p(\lambda) : \lambda \in \sigma(a)\}$$

(ii) *if $0 \notin \sigma(a)$*

$$\sigma(a^{-1}) = \left\{\frac{1}{\lambda} : \lambda \in \sigma(a)\right\} .$$

Proof. (i) For any complex number λ,

$$p(\lambda)e - p(a) = (\lambda e - a)P(a) = P(a)(\lambda e - a)$$

where $P(a)$ is some polynomial in a. Hence, if $p(\lambda)e - p(a)$ has an inverse, so has $(\lambda e - a)$. Thus, $\lambda \in \sigma(a) \Rightarrow p(\lambda) \in p(\sigma(a))$.

Conversely, if $\mu \in \sigma(p(a))$ then $\mu e - p(a)$ has no inverse. Consider the polynomial $q(t) = \mu - p(t)$. This factorises as $q(t) = k(\lambda_1 - t)(\lambda_2 - t)\cdots(\lambda_n - t)$ where k is a constant and $\lambda_1, \lambda_2, \cdots \lambda_n$ are the zeros of q. Then

$$\mu e - p(a) = k(\lambda_1 e - a)(\lambda_2 e - a)\cdots(\lambda_n e - a)$$

and if all the factors $(\lambda_i e - a)$ had inverses, so would $\mu e - p(a)$ (since the factors commute). Therefore, at least one zero λ_i of q is in the spectrum of a and $p(\lambda_i) = \mu$; that is, $\mu \in \{p(\lambda) : \lambda \in \sigma(a)\}$.

(ii) This is obvious from the relation

$$\lambda e - a = -a\lambda\left(\frac{1}{\lambda}e - a^{-1}\right) .$$

Theorem 1.7 *In a C^*-algebra,*

(i) $\sigma(a) = \overline{\sigma(a^*)}$,

(ii) *if u is unitary then $|\lambda| = 1$ for all $\lambda \in \sigma(u)$,*

(iii) *if a is self-adjoint then $\sigma(a)$ is real.*

Proof. (i) Since $e^* = e$, we have $xy = yx = e$ if and only if $y^*x^* = x^*y^* = e$. Thus $(\lambda e - a)$ is invertible if and only if $(\lambda e - a)^* = (\bar{\lambda} e - a^*)$ is invertible showing that $\sigma(a) = \overline{\sigma(a^*)}$.

(ii) If u is unitary then $||u||^2 = ||uu^*|| = ||e|| = 1$. Now if $\lambda \in \sigma(u)$ then from 1.4, $|\lambda| \leq ||u|| = 1$. But u^* is also unitary and Theorem 1.6 (ii) shows that $\lambda^{-1} \in \sigma(u^*)$ so that $|\lambda^{-1}| \leq 1$. Thus $|\lambda| = 1$.

(iii) Suppose that $a = a^*$. We need to show that if $\lambda = \alpha + i\beta$ and the imaginary part β is non-zero, then $\lambda e - a$ has an inverse in A. First a reduction: note that

$$(\alpha + i\beta)e - a = \beta \left[\frac{\alpha e - a}{\beta} + ie \right]$$

and $x = \frac{\alpha e - a}{\beta}$ is self-adjoint. Hence it is sufficient to show that $x + ie$ is invertible for every self-adjoint x.

Suppose this is not the case so that $-i \in \sigma(x)$. Then for any ξ,

$$\xi + 1 \in \{\xi + it : t \in \sigma(x)\} = \sigma(\xi + ix)$$

and so, from 1.4, $|\xi + 1| \leq ||\xi + ix||$. Taking ξ to be real the C* condition shows that

$$(\xi + 1)^2 \leq ||\xi + ix||^2 = ||(\xi + ix)^*(\xi + ix)|| = ||\xi^2 + x^2|| \leq \xi^2 + ||x||^2 .$$

This implies that $1 + 2\xi \leq ||x||^2$ for all real ξ, which is impossible. Thus $-i \notin \sigma(x)$ and so the spectrum of any self-adjoint element is real. ∎

The following is a simple result on the spectrum of a product.

Lemma 1.8 *Let $a, b \in A$. Then for $\lambda \neq 0$, $\lambda \in \sigma(ab) \Leftrightarrow \lambda \in \sigma(ba)$.*

Proof. If $\lambda \neq 0$ and $\lambda \in \rho(ab)$, it can be verified by direct multiplication that

$$\lambda^{-1}e + \lambda^{-1}b(\lambda e - ab)^{-1}a.$$

is the inverse of $\lambda e - ba$. Thus $\lambda \in \rho(ba)$. ■

The **spectral radius** $\nu(a)$ of an element $a \in A$ is defined as

$$\nu(a) = \max\{|\lambda| : \lambda \in \sigma(a)\}\,.$$

It is clear from the proof of Theorem 1.4 that $\nu(a) \leq ||a||$. The next theorem establishes a formula for $\nu(a)$. It is reminiscent of the formula

$$\frac{1}{r} = \overline{\lim_{n\to\infty}}|a_n|^{\frac{1}{n}}$$

for the radius of convergence of the scalar power series $\sum_{n=0}^{\infty} a_n z^n$.

Theorem 1.9 (Spectral radius formula)

$$\nu(a) = \lim_{n\to\infty} ||a^n||^{\frac{1}{n}}$$

Proof. Consider the function $\lambda\phi[(\lambda e - a)^{-1}]$ where $\phi \in A'$. It follows as in the proof of Theorem 1.4 that f is holomorphic on $\rho(a)$ and so on $\{\lambda : |\lambda| > \nu(a)\}$. Make the substitution $z = 1/\lambda$ and define

$$g(z) = \frac{1}{z}\phi\left[(\frac{1}{z}e - a)^{-1}\right] = \phi\left[(e - za)^{-1}\right]\,.$$

Then g is holomorphic in $\{z : |z| < \frac{1}{\nu(a)}\}$.

We find the Taylor expansion of g. For $|z| < \frac{1}{||a||}$ we have from Lemma 1.1 that

$$(e - za)^{-1} = \sum_{n=0}^{\infty} z^n a^n\,.$$

Since ϕ is continuous,

$$g(z) = \phi\left((e - za)^{-1}\right) = \sum_{n=0}^{\infty} z^n \phi(a^n)\,.$$

Initially we only know that this power series expansion of g is valid for $|z| < \frac{1}{||a||}$. However, since g is holomorphic on the larger disc $\{z : |z| < \frac{1}{\nu(a)}\}$ and the Taylor expansion of a holomorphic function is unique, the above expansion is valid for $|z| < \frac{1}{\nu(a)}$. Thus, for $|z| < \frac{1}{\nu(a)}$ then $\lim_{n\to\infty} z^n\phi(a^n) = 0$. Actually, we only need to know that this sequence is bounded, that is, for each ϕ in the dual of A, there exists a constant K_ϕ such that

$$|\phi(z^n a^n)| < K_\phi \qquad \text{for all } n\,.$$

It now follows from the Uniform Boundedness Theorem that there exists a K independent of ϕ such that, for all n and all z with $|z| < \frac{1}{\nu(a)}$, $||z^n a^n|| < K$; that is,

$$||a^n||^{\frac{1}{n}} < \frac{K^{\frac{1}{n}}}{|z|}\,.$$

It follows that $||a^n||^{\frac{1}{n}} \leq K^{\frac{1}{n}}\nu(a)$ and so

$$\overline{\sup}||a^n||^{\frac{1}{n}} \leq \nu(a)\,.$$

In the other direction, if $\lambda \in \sigma(a)$ then, for all positive integers n, $\lambda^n \in \sigma(a^n)$ (1.6(i)) and so, (1.4) $|\lambda|^n \leq ||a^n||$. That is, $|\lambda| \leq ||a^n||^{\frac{1}{n}}$. Taking the maximum over $\lambda \in \sigma(a)$, we have that $\nu(a) \leq ||a^n||^{\frac{1}{n}}$ for all n. Therefore, combining all that we have shown,

$$\inf_n ||a^n||^{\frac{1}{n}} \geq \underline{\lim}||a^n||^{\frac{1}{n}} \geq \nu(a) \geq \overline{\sup}||a^n||^{\frac{1}{n}}$$

so the last three terms are equal. ■

Note that the existence of the limit is not part of the hypotheses of the theorem and is established by the above proof

Corollary 1.10 *If A is a C^*-algebra and a is normal then*

$$||a|| = \nu(a)\,.$$

Proof. It $x = x^*$ then by the C* condition, $||x^2|| = ||x||^2$ and by repeating this we have that $||x^{2^m}|| = ||x||^{2^m}$. Thus

$$||x|| = \lim_{m\to\infty} ||x^{2^m}||^{\frac{1}{2^m}} = \nu(x)\,.$$

For a normal a, using the above for aa^*,

$$\begin{aligned} ||a||^2 &= ||aa^*|| = \nu(aa^*) \\ &= \lim_{n\to\infty} ||(aa^*)^n||^{\frac{1}{n}} \\ &\leq \lim_{n\to\infty} ||a^n||^{\frac{1}{n}}.||(a^*)^n||^{\frac{1}{n}} \\ &= (\nu(a))^2 \leq ||a||^2 \end{aligned}$$

and so equality holds throughout. ■

We now consider the spectrum with respect to subalgebras. Let A be a Banach algebra. For a subalgebra B of A to be a Banach algebra in its own right, it must be complete and, since A is complete, this is equivalent to requiring it to be closed. The boundary of a set S will be denoted by ∂S, that is, $\partial S = \overline{S} \cap \overline{S^c}$.

Theorem 1.11 *Let B be a closed subalgebra of A containing the identity of A. Then*

$$\partial\sigma_B(b) \subseteq \sigma_A(b) \subseteq \sigma_B(b)\,.$$

Proof. If $b \in B$ and $\lambda e - b$ has no inverse in A then it can have no inverse in B. This shows that $\sigma_A(b) \subseteq \sigma_B(b)$.

Now let $\lambda \in \partial\sigma_B(b) = \overline{\sigma_B(b)} \cap \overline{\rho_B(b)} = \sigma_B(b) \cap \overline{\rho_B(b)}$. Then there exists a sequence (λ_n) of elements of $\rho_B(b)$ with $(\lambda_n) \to \lambda$. If $\lambda \notin \sigma_A(b)$ then $(\lambda e - b)^{-1}$ exists in A and, using the continuity of inversion (1.2(ii)), the sequence $((\lambda_n e - b)^{-1})$ converges to $(\lambda e - b)^{-1}$ in A. But then $((\lambda_n e - b)^{-1})$ is a Cauchy sequence of elements of B and so converges in B to the same limit, $(\lambda e - b)^{-1}$, contradicting that $\lambda \in \sigma_B(b)$. ■

Corollary 1.12 *For $b \in B$,*

$$\nu_A(b) = \nu_B(b)$$

Proof. This is obvious either from the above result or from Theorem 1.9. ■

Corollary 1.13 *If $b \in B$, and $\sigma_B(b)$ has empty interior, then $\sigma_B(b) = \sigma_A(b)$*

Proof. If $\sigma_B(b)$ has empty interior, then $\partial\sigma_B(b) = \sigma_B(b)$. ■

If A and B are algebras, a **homomorphism** $\pi : A \to B$ is a map which preserves all the algebraic structure. If A and B are $*$-algebras and π preserves the involution then it is called a $*$-homomorphism. If A has an identity e then it is easy to see that $\pi(e)$ acts as the identity on the range of π (in fact, if B is a normed algebra, on the closure of the range of π). Thus there is essentially no loss of generality to assume that, where identities exist, all homomorphisms preserve identities - and we shall make this assumption.

Theorem 1.14 *If A and B are C*-algebras with identities and $\pi : A \mapsto B$ is a $*$-homomorphism then $||\pi(a)||_B \leq ||a||_A$. In particular, π is automatically continuous.*

Proof. If x has an inverse in A then $\pi(x^{-1})$ is the inverse of $\pi(x)$ in B. Applying this to $\lambda e - a$ shows that, $\rho_A(a) \subseteq \rho_B(\pi(a))$ for all $a \in A$. Hence

$$\nu_B(\pi(a)) \leq \nu_A(a) \leq ||a|| \,. \tag{1}$$

But in the C*-algebra B,

$$||\pi(a)||^2 = ||\pi(a)(\pi(a))^*|| = ||\pi(a)\pi(a^*)|| = ||\pi(aa^*)|| \,.$$

Since $\pi(aa^*)$ is selfadjoint in B, this, together with 1.10 shows that

$$||\pi(a)||^2 = \nu_B(\pi(aa^*)) \,.$$

Replacing a by aa^* in (1) and using the above we have

$$||\pi(a)||^2 = \nu_B(\pi(aa^*)) \leq ||aa^*|| = ||a||^2 \ . \blacksquare$$

An examination of the above proof shows that the C* condition was used only for the range algebra B and the theorem is valid with a weaker condition on A, namely that it be a Banach algebra with isometric involution. The following consequence is clear for algebras with identities.

Corollary 1.15 *If π is a $*$-isomorphism then it is isometric. In particular, there is at most one norm that makes a Banach algebra with isometric involution into a C*-algebra.*

The result above is also true for algebras without identities. It can be proved by using the procedure given in 1.3 below to embedd each algebra into an algebra with identity and apply Corollary 1.15. The detailsd are omitted.

Theorem 1.16 *If B is a closed $*$subalgebra of a C*-algebra A with the same identity then for all $b \in B$, $\sigma_A(b) = \sigma_B(b)$.*

Proof. Since the spectrum of a self-adjoint element of a C*-algebra is real (1.7), Corollary 1.13 shows that self-adjoint elements have the same spectra in A and B. Now, if $\lambda \in \rho_A(b)$, both $\lambda e - b$ and its adjoint have inverses in A. Thus 0 is not in $\sigma_A\left((\lambda e - b)(\lambda e - b)^*\right)$ and since $(\lambda e - b)(\lambda e - b)^*$ is self-adjoint, it therefore has an inverse in B. Writing x for this inverse, we have that

$$e = (\lambda e - b)(\lambda e - b)^* x$$

showing that $(\lambda e - b)$ has a right inverse in B. Similarly, $(\lambda e - b)$ has a left inverse in B and a simple algebra argument shows that it is then invertible. Thus $\sigma_B(b) \subseteq \sigma_A(b)$ for all $b \in B$ and since the opposite inclusion is obvious, equality holds. $\blacksquare$

1.3 Algebras without identity

It is a fact that, given an algebra A without identity, it is possible to construct an algebra A_1 with an identity e such that A is a subalgebra of A_1 and A_1 is the span of A and e. In fact the construction makes A into an *ideal* of A_1. If A is a C*-algebra then one can ensure that A_1 is also a C*-algebra. We now give a sketch of how this can be done.

From the algebraic point of view, the construction is very easy. A_1 is simply defined as $A \oplus \mathbb{C}$, the direct sum of A and $\mathbb{C}$ with the usual linear space structure. Multiplication is defined in A_1 by

$$(a, \lambda).(b, \mu) = (ab + \lambda b + \mu a, \lambda\mu)$$

(this definition is very natural if you think of (a, λ) as a plus λ times a formal identity). Of course, A is identified with the set $\{(a, 0) : a \in A\}$. Note that A is an ideal of A_1. The element $(0, 1) = e$ is the identity of the algebra A_1.

There are many ways to define a norm on A_1 to make it into a Banach algebra. For example

$$\|(a, \lambda)\| = \|a\| + |\lambda|$$

is complete and it is easy to prove that it satisfies the multiplicative norm condition (i) for a Banach algebra. However, this does not make A_1 into a C*-algebra.

To make A_1 into a C*-algebra, define a norm $|||.|||$ as follows :

$$|||(a, \lambda)||| = \sup_{\|x\| \leq 1} \|ax + \lambda x\|.$$

where x varies over A. Note that multiplication by (a, λ) acts as linear operator on the ideal A of A_1 and $|||(a, \lambda)|||$ is simply the norm of this operator. From this it follows easily that $|||.|||$ satisifies the multiplicative condition for the norm of a Banach algebra.

Recall (Section 1.1, consequence 3) that $\|a\| = \sup_{\|x\| \leq 1} \|ax\|$ showing that $|||(a, 0)||| = \|a\|$, so that the embedding of A into A_1 is isometric. To verify the C* condition,

$$\begin{aligned} |||(a, \lambda)|||^2 &= \sup_{\|x\| \leq 1} \|ax + \lambda x\|^2 \\ &= \sup_{\|x\| \leq 1} \|(x^* a^* + \bar{\lambda} x^*)(ax + \lambda x)\| \\ &= \sup_{\|x\| \leq 1} |||(x^*, 0)(a, \lambda)^*(a, \lambda)(x, 0)||| \end{aligned}$$

$$\begin{aligned}
&\leq \sup_{\|x\|\leq 1} |||(x^*,0)|||.|||(a,\lambda)^*(a,\lambda)|||.|||(x,0)||| \\
&= |||(a,\lambda)^*(a,\lambda)|||\,.
\end{aligned}$$

Hence (Section 1.1, consequence 5) $|||.|||$ satisfies the C* condition.

It remains to prove completeness. For this we use the fact that A has no identity.Let $((a_n, \lambda_n))$ be a Cauchy sequence in A_1. We first show that (λ_n) is a bounded sequence. Suppose this is not the case. Then by passing to a subsequence, we may assume that $|\lambda_n| \to \infty$. Then

$$\begin{aligned}
\left\|\frac{a_m}{\lambda_m} - \frac{a_n}{\lambda_n}\right\| &= \sup_{\|x\|\leq 1} \| \left(\frac{a_m}{\lambda_m} - \frac{a_n}{\lambda_n}\right) x\| \\
&= \sup_{\|x\|\leq 1} \left\|\left(\frac{a_m}{\lambda_m}x + x\right) - \left(\frac{a_n}{\lambda_n}x + x\right)\right\| \\
&\leq |\lambda_m|^{-1}|||(a_m,\lambda_m)||| + |\lambda_n|^{-1}|||(a_n,\lambda_n)|||\,.
\end{aligned}$$

Since (a_n, λ_n) is Cauchy in A_1 it is bounded and since $|\lambda_n| \to \infty$ it follows that $(\frac{a_n}{\lambda_n})$ is Cauchy in A and hence converges to some element f of A. But, for all $x \in A$,

$$\left\|\frac{a_n}{\lambda_n}x + x\right\| = \frac{1}{|\lambda_n|}|||(a_n,\lambda_n)||| \to 0$$

and so $fx + x = 0$. Thus A has an identity (namely $-f$), contrary to hypothesis. Therefore (λ_n) is a bounded sequence and, passing to a subsequence, we may assume that it is convergent to some number λ.

Now, using

$$\begin{aligned}
\|a_n - a_m\| &= |||(a_n,\lambda) - (a_m,\lambda)||| \\
&= |||(a_n,\lambda_n) + (0,\lambda-\lambda_n) - (a_m,\lambda_m) - (0,\lambda-\lambda_m)||| \\
&\leq |||(a_n,\lambda_n) - (a_m,\lambda_m)||| + |\lambda-\lambda_n| + |\lambda-\lambda_m|
\end{aligned}$$

it follows that (a_n) is a Cauchy sequence in A and so has a subsequence convergent to some element $a \in A$. It is now a subsequence of (a_n, λ_n) converges to $(a,\lambda) \in A_1$.

Most of the proofs in the remainder of these notes will be given for algebras with identities. The above construction will then be used to deduce results for algebras that need not have identities.

2 Commutative theory.

In this section we shall show that a commutative Banach algebra with identity can be mapped into the algebra of all continuous functions on some compact space. In the case of a C*-algebra the embedding is an isometric *-isomorphism.

Before embarking on the proofs, we need some remarks on quotients. Recall that if $\mathcal{X}$ is a linear space and $\mathcal{Y}$ is a subspace of $\mathcal{X}$ then we define an equivalence relation $\sim$ on $\mathcal{X}$ by

$$x_1 \sim x_2 \Longleftrightarrow x_1 - x_2 \in \mathcal{Y}\,.$$

Denote the equivalence class of x by $[x]$ (the notation $x + \mathcal{Y}$ is also used). The quotient space $\mathcal{X}/\mathcal{Y}$ is defined as the set of equivalence classes, with addition and multiplication by scalars defined in the natural way. If $\mathcal{X}$ is a Banach space and $\mathcal{Y}$ is a closed subspace then

$$\|[x]\| = \inf_{y \in Y} \|x + y\|$$

makes $\mathcal{X}/\mathcal{Y}$ into a Banach space. All the above is standard Banach space theory.

If A is an algebra and $\mathcal{I} \subseteq A$ then it can be verified that

$$[a].[b] = [a.b]$$

is a well-defined multiplication if and only if $\mathcal{I}$ is an ideal. In the case of a Banach algebra A and a closed ideal $\mathcal{I}$, this construction makes $A/\mathcal{I}$ a Banach algebra. Furthermore, if A is a C*-algebra then it canbe shown that $A/\mathcal{I}$ is also a C*-algebra (but this fact will not be needed in these notes).

From now on in this section, except where otherwise stated, A will denote a commutative Banach algebra with identity e. Obviously any ideal $\mathcal{I}$ is two-sided and any quotient algebra $A/\mathcal{I}$ is commutative.

Theorem 2.1 *For a maximal ideal M of A, then A/M is isometrically isomorphic to $\mathbb{C}$.*

Proof. From Corollary 1.3, M is closed. Also, A/M has no proper ideals since if $\mathcal{I}$ were such then $Z = \cup\{a : [a] \in \mathcal{I}\}$ would be a proper ideal of A containing M.

Now if $[a] \neq 0$ (i.e. $a \notin M$) then $(A/M)[a]$ the principal ideal generated by $[a]$ is non-zero and so equals A/M. Thus $[e] \in (A/M)[a]$ and so there exists $[x] \in A/M$ such that $[x].[a]$ is the identity of A/M. This shows that A/M is a division algebra and so, by the Gelfand-Mazur Theorem (1.5) it is isometrically isomorphic to $\mathbb{C}$. ■

Corollary 2.2 *There is a bijection between the set $\mathcal{M}$ of proper maximal ideals and the set Φ of all non-zero homomorphisms of A into $\mathbb{C}$.*

Proof. Given $M \in \mathcal{M}$, consider the composition of the natural map $A \to A/M$ and the homomorphism of the Gelfand-Mazur Theorem $A/M \to \mathbb{C}$. This is a non-zero homomorphism ϕ_M of A into $\mathbb{C}$ with kernel $\phi_M^{-1}(0) = M$.

In the other direction, given $\phi \in \Phi$ we show that the corresponding maximal ideal is its kernel $\phi^{-1}(0)$. Let $\phi, \psi \in \Phi$ with $\phi^{-1}(0) \subseteq \psi^{-1}(0)$ then for all $a \in A$,

$$a - \phi(a)e \in \phi^{-1}(0) \subseteq \psi^{-1}(0)$$

so that $0 = \psi(a - \phi(a)e) = \psi(a) - \phi(a)$, since $\psi(e) = 1$; that is, $\phi = \psi$. This shows firstly that for $\phi \in \Phi$, $\phi^{-1}(0)$ is maximal; for if $\phi^{-1}(0)$ were a proper subset of an $M \in \mathcal{M}$, the homomorphism $\psi = \phi_M$ would have a strictly larger kernel. Secondly, the above shows in particular that, if ϕ and ψ have the same kernel the they are equal. Hence the map $\phi \mapsto \phi^{-1}(0)$ is a bijection $\Phi \to \mathcal{M}$. ■

We call Φ (or $\mathcal{M}$) the **carrier space** of A. Other terms often used are : the **maximal ideal space** or the **spectrum** of A.

We now proceed to a construction similar to the embedding of a Banach space into its bidual. For each $\phi \in \Phi$ and $a \in A$, we have a complex number $\phi(a)$. This is really a map from $\Phi \times A$ into $\mathbb{C}$. So far ϕ has been thought of as a fixed function and a as the variable. However, this can be reversed. Define, for each $a \in A$ the function $\hat{a} : \Phi \to \mathbb{C}$ given by

$$\hat{a}(\phi) = \phi(a)\,.$$

The function $\hat{a}$ is called the **Gelfand transform** of a. We now establish its properties.

Theorem 2.3 *The map $a \mapsto \hat{a}$ is an algebraic homomorphism of A into the set of complex-valued functions on Φ.*

Proof. This is a routine verification using the fact that each ϕ is a homomorphism. For example, to show that $\widehat{ab} = \hat{a}.\hat{b}$, simply note that

$$(\hat{a}.\hat{b})(\phi) = \phi(a).\phi(b) = \phi(ab) = \widehat{ab}(\phi)\,. \;\blacksquare$$

Theorem 2.4

(i) *$\hat{e}$ is the unit function (i.e. $\hat{e}(\phi) = 1$ for all ϕ).*

(ii) *$\hat{a} = 0$ if and only if $a \in \bigcap\{\phi^{-1}(0) : \phi \in \Phi\} = \bigcap\{M : M \in \mathcal{M}\}$.*

(iii) *$\sigma(a) = \{\hat{a}(\phi) : \phi \in \Phi\}$*

(iv) *$\nu(a) = \max\{|\hat{a}(\phi)| : \phi \in \Phi\}$*

(v) *If $\phi_1 \neq \phi_2$ then for some $a \in A$, $\hat{a}(\phi_1) \neq \hat{a}(\phi_2)$.*

Proof. (i) For each $\phi \in \Phi$, $\phi(e) = 1$, that is $\hat{e}(\phi) = 1$.

(ii) For $a \in A$,

$\hat{a} = 0 \Leftrightarrow \hat{a}(\phi) = 0, \quad \forall \phi \in \Phi \Leftrightarrow \phi(a) = 0, \quad \forall \phi \in \Phi \Leftrightarrow a \in \phi^{-1}(0), \quad \forall \phi \in \Phi$.

(iii) For any $\phi \in \Phi$, $\phi(\phi(a)e - a) = 0$. Hence $\phi(a)e - a$ is in the kernel of ϕ which is a proper ideal. Therefore it has no inverse, that is $\phi(a) \in \sigma(a)$.

Conversely, if $\lambda \in \sigma(a)$ then $\lambda e - a$ has no inverse. Therefore the principal ideal $A(\lambda e - a)$ is proper and so is a subset of some maximal ideal M, that is, it is a subset of the kernel of some $\phi \in \Phi$. In particular $\phi(\lambda e - a) = 0$ and so $\lambda = \phi(a)$.

(iv) This is obvious from (iii)

(v) If $\phi_1 \neq \phi_2$ then the functions must differ at some element $a \in A$. Then $\phi_1(a) \neq \phi_2(a)$, that is, $\hat{a}(\phi_1) \neq \hat{a}(\phi_2)$. $\blacksquare$

Note that the function $\hat{a}$ (where $a \in A$) in the algebra $B(\Phi)$ of bounded functions on Φ, has norm

$$\|\hat{a}\| = \sup_{\phi \in \Phi} |\hat{a}(\phi)|\,.$$

Hence the above Theorem shows that

$$\|\hat{a}\| = \nu(a) \leq \|a\|$$

so the Gelfand transform is continuous $A \to B(\Phi)$. The Gelfand transform is one to one (i.e. a monomorphism) if A has the property that $\nu(a) = 0 \Rightarrow a = 0$; it is isometric if $\|a\| = \nu(a)$ for all $a \in A$, and this is so in the case when A is a commutative C*-algebra (Theorem 1.10).

An element of a Banach algebra is called **quasinilpotent** if the only element in its spectrum is 0. Thus (iii) above shows that the Gelfand transform of $a \in A$ is the zero function if and only if a is quasinilpotent.

We mention the **Jacobson radical** for general interest, although we shall not be making any use of the concept in the rest of these notes. The Jacobson radical $\mathcal{R}$ of any Banach algebra A with identity is defined by

$$\mathcal{R} = \cap\{\text{all maximal left ideals of A }\} = \cap\{\text{all maximal right ideals of A }\}$$

(and it is a fact that the two intersections are equal). When A is commutative then,

$$\mathcal{R} = \cap\{\text{all maximal two-sided ideals of A }\}.$$

Now,

$$\hat{a} = 0 \Leftrightarrow \hat{a}(\phi) = 0 \quad \forall \phi \in \Phi \Leftrightarrow a \in M \quad \forall M \in \mathcal{M} \Leftrightarrow a \in \mathcal{R}.$$

Thus the kernel of the Gelfand map is the Jacobson radical.

We now proceed to define a topology on the carrier space. If Φ is the carrier space of the commutative Banach algebra A, the Gelfand topology on Φ is the weakest topology (that is, the one with fewest open sets) to make each function $\hat{a} : \Phi \to \mathbb{C}$ continuous.

A base for the neighbourhoods of ϕ in the Gelfand topology is given by all sets of the form

$$\{\psi \in \Phi : |\phi(a_i) - \psi(a_i)| < \epsilon, i = 1, 2, \cdots, n\}$$

that is

$$\{\psi \in \Phi : |\hat{a}_i(\phi) - \hat{a}_i(\psi)| < \epsilon, i = 1, 2, \cdots, n\}$$

as the quantities $a_1, a_2, \cdots, a_n$ in A and $\epsilon > 0$ range over all choices.

Convergence of sequences in this topology is given by

$$\begin{aligned} \phi_n \to \phi \quad &\Leftrightarrow \quad \hat{a}(\phi_n) \to \hat{a}(\phi) \text{ in } \mathbb{C} \quad \forall a \in A \\ &\Leftrightarrow \quad \phi_n(a) \to \phi(a) \quad \forall a \in A \end{aligned}$$

that is, convergence of sequences in the Gelfand topology is pointwise convergence.

From here on, when we refer to the "carrier space" of a commutative banach algebra, we shall understand the space Φ of homomorphisms with the Gelfand topology . Note that the Gelfand topology is Hausdorff, since if $\phi_1 \neq \phi_2$ then there exists $a \in A$ such that $\phi_1(a) \neq \phi_2(a)$ so, if $\delta = \frac{1}{2}|\phi_1(a) - \phi_2(a)|$ then

$$N_1 = \{\phi : |\phi_1(a) - \phi(a)| < \delta\} \text{ and } N_2 = \{\phi : |\phi_2(a) - \phi(a)| < \delta\}$$

are neighbourhoods of ϕ_1 and ϕ_2 respectively and are disjoint.

The lemma below uses Tychonoff's theorem which states that the product of compact spaces with the product topology is compact. Recall that if $\{X_\gamma : \gamma \in \Gamma\}$ is a family of non-empty sets then the **cartesian product**, $\prod_{\gamma \in \Gamma} X_\gamma$ has, as a typical member, the function $\gamma \mapsto x_\gamma$ where $x_\gamma \in X_\gamma$. The proof below exploits this by identifying each member of the carrier space of the algebra A with an element of a suitable product space. It is the same technique as that usually used to prove the Banach-Alaoglu Theorem on the weak* compactness of the unit ball of the dual of a Banach space. An alternative (shorter) proof could be given deducing the result from the Banach-Alaoglu Theorem.

Lemma 2.5 *The carrier space of a commutative Banach algebra with identity is compact.*

Proof. Let Φ be the carrier space of A and let $\phi \in \Phi$. From Theorem 2.4 (iii), $\phi(a) \in \sigma(a)$ and so $|\phi(a)| \leq \nu(a) \leq \|a\|$. Thus the closed disc Z_a in $\mathbb{C}$, centre 0 radius $\|a\|$ is compact and contains $\phi(a)$ for all $a \in A$. Let

$$Z = \prod_{a \in A} Z_a \, .$$

Then Z with the product topology is compact by Tychonoff's Theorem. Recall that the product topology on Z is the weakest topology to make the projection maps p_a corresponding to each element of the index set A continuous. Note that the elements of Z are all functions $f : A \to \mathbb{C}$ such that $|f(a)| \leq \|a\|$. Then $p_a(f) = f(a)$. Now the elements of Φ are particular cases of these functions (i.e. elements of Z) and $p_a(\phi) = \phi(a) = \hat{a}(\phi)$. Thus the topology on Φ inherited from the product topology of Z is the same as the Gelfand topology. It is therefore sufficent to show that Φ is a closed subset of Z.

The elements of Φ are those elements of Z that also satisfy the algebraic properties of homomorphisms. Thus if $\phi \in \Phi$ then for all $a, b \in A$,

$$\phi(a+b) = \phi(a) + \phi(b)\,.$$

Define the function $L_{a,b} : Z \to \mathbb{C}$ by

$$L_{a,b}(f) = f(a+b) - f(a) - f(b) = p_{a,b}(f) - p_a(f) - p_b(f)\,.$$

Then, directly from the definition of the product topology it is clear that $L_{a,b}$ is continuous and hence its kernel $L_{a,b}^{-1}(0)$ is closed. The algebraic condition on ϕ given above is that ϕ is in the kernel of each $L_{a,b}$. Similarly we define, for $a, b \in A$ and $\lambda \in \mathbb{C}$,

$$\begin{aligned} M_{a,b}(f) &= f(ab) - f(a).f(b) \\ S_{a,\lambda}(f) &= f(\lambda a) - \lambda f(a) \\ I(f) &= f(e) - 1 \end{aligned}$$

and all these functions are continuous. It is easy to see that an element of Z is in Φ exactly when it is in the intersection of the kernels of all the functions $L_{a,b}, M_{a,b}, S_{a,\lambda}, I$ as a, b, λ take all possible values. Therefore it is a closed subset of Z and so it is compact. ■

The theory developed in the results above, when put together, constitute a proof of the following important theorem.

Theorem 2.6 (Gelfand map) *Given any commutative Banach algebra A with identity, there is a homomorphism of A into $C(\Phi)$, the algebra of all continuous complex-valued functions on a compact Hausdorff space Φ. The kernel of this map is the set of all quasinilpotent elements of A.*

The carrier space appears to be a rather abstract topological space. However, in some important cases it can be realised as a very concrete space. The following lemma gives a simple and useful example of this situation.

Lemma 2.7 *If A is generated by the identity e and one element a then the carrier space Φ of A is homeomorphic to $\sigma(a)$.*

Proof. We show that the required homeomorphism is the function $\hat{a} : \Phi \to \sigma(a)$. Indeed, by the definition of the Gelfand topology, $\hat{a}$ is continuous and from Theorem 2.4 (iii) it is onto $\sigma(a)$. To show that $\hat{a}$ is injective we use our hypothesis that the elements $\{p(a)\}$, as p ranges over all polynomials, form a dense subset of A. Then

$$\begin{aligned}
\hat{a}(\phi_1) = \hat{a}(\phi_2) \quad &\Rightarrow \quad \phi_1(a) = \phi_2(a) \\
&\Rightarrow \quad \phi_1(p(a)) = \phi_2(p(a)) \;\; \forall \text{ polynomials, since } \phi_1, \phi_2 \text{ are homomorphisms} \\
&\Rightarrow \quad \phi_1(x) = \phi_2(x) \qquad\quad \forall x \in A, \text{ since } \phi_1, \phi_2 \text{ are continuous} \\
&\Rightarrow \quad \phi_1 = \phi_2
\end{aligned}$$

showing that $\hat{a}$ is injective. So $\hat{a}$ is a continuous bijection. Now apply the general theorem that a continuous bijection between compact Hausdorff spaces is a homeomorphism. ■

If X and Y are topological spaces and $h : X \leftrightarrow Y$ is a homeomorphism between them, the mapping $f \leftrightarrow \tilde{f}$ given by

$$\tilde{f}(h(x)) = f(x) \quad \text{or, equivalently} \quad \tilde{f}(y) = f(h^{-1}(y))$$

is clearly an isomorphism between the function spaces $C(X)$ and $C(Y)$. This observation, together with Lemma 2.7 shows that, for the case of a singly generated algebra, we can compose this isomorphism with the Gelfand map to get a map of A into the space $C(\sigma)$ where $\sigma \subseteq \mathbb{C}$ is the spectrum of the generator. (Loosely speaking, we make $C(\sigma)$ the range of the Gelfand map.) Moreover (with the notation of Lemma 2.7) using the function $\hat{a}$ as the homeomorphism, the function f_x on $\mathbb{C}$ corresponding to an element $x \in A$ is given by

$$f_x(s) = \hat{x}(\hat{a}^{-1}(s))\,.$$

For the case of $x = a$ this reduces to $f_a(s) = s$. This shows that when we use Lemma 2.7 and the Gelfand transform to map the singly generated algebra A into $C(\sigma)$, we obtain that the image of the generator a is the identity function $f(s) = s$.

So far in this section our algebra A was a commutative Banach algebra. For the special case of a C*-algebra A, a much stronger result holds. Here the algebra is not merely mapped onto some subalgebra of $C(\Phi)$ but essentially it "is" the whole of $C(\Phi)$.

Theorem 2.8 (The commutative Gelfand-Naimark Theorem) *Given any commutative C*-algebra A with identity, there is an isometric $*$-isomorphism of A onto $C(\Phi)$, the algebra of all continuous complex-valued functions on a compact Hausdorff space Φ.*

Proof. Let Φ be the carrier space of A and let $a \mapsto \hat{a}$ be the Gelfand map as above. Clearly every element of a commutative C*-algebra is normal and so from Corollary 1.10 the norm of each element of A is equal to its spectral radius. Thus $\|a\| = \|\hat{a}\|$.

Now A is a Banach space and thus complete. Therefore, since the Gelfand map is isometric, $\{\hat{a} : a \in A\}$ is a complete subset of $C(\Phi)$ and so closed. We shall use the Stone-Weierstrass Theorem to show that it is the whole of $C(\Phi)$.

We show that the Gelfand map preserves the involution (where the involution in $C(\Phi)$ is defined in the natural way : $f^*(\phi) = \overline{f(\phi)}$. For $a \in A$ we write $a = x + iy$ where $x = \frac{1}{2}(a + a^*), y = \frac{1}{2i}(a - a^*)$ are selfadjoint. From Theorem 1.7 (iii), x and y have real spectra and so, since $\hat{(}\phi) \in \sigma(x)$, the functions $\hat{x}$ and $\hat{y}$ are real-valued. Therefore

$$(\hat{a})^* = \overline{\hat{x} + i\hat{y}} = \hat{x} - i\hat{y} = \widehat{x - iy} = \widehat{a^*}\,.$$

This shows that the image of $\hat{A}$ is a *-subalgebra and also that the Gelfand map preserves the adjoint operation. The remaining conditions needed to apply the Stone-Weierstrass Theorem have already been proved, namely that $\hat{A}$ separates the points of Φ and contains the unit function (Theorem 2.4 (i), (v)). Therefore $\hat{A} = C(\Phi)$. ■

Lemma 2.9 *If A is a commutative C^*-algebra generated by the identity e and the elements a and a^* of A, then the carrier space Φ of A is homeomorphic to $\sigma(a)$.*

Proof. We again show that the required homeomorphism is the function $\hat{a} : \Phi \to \sigma(a)$. The only difference between this lemma and Lemma 2.7 is that the dense subset of A is $\{p(a, a^*)\}$ as p ranges over polynomials in 2 variables. So, to prove that $\hat{a}$ is an injection we note that

$$\hat{a}(\phi_1) = \hat{a}(\phi_2) \Rightarrow \widehat{a^*}(\phi_1) = (\hat{a}(\phi_1))^* = \overline{\hat{a}(\phi_1)} = \overline{\hat{a}(\phi_2)} = \widehat{a^*}(\phi_2)$$

and from this it follows that

$$\phi_1(p(a, a^*)) = \phi_2(p(a, a^*))$$

for all polynomials in two variables. Thus, by the hypothesis, $\phi_1(x) = \phi_2(x)$ for all $x \in A$ so that $\phi_1 = \phi_2$. The remainder of the proof is exactly the same as that of Lemma 2.7. ■

The observations following Lemma 2.7 also apply to Lemma 2.9 above. In view of Theorem 2.8 the conclusion here is that A is isometrically isomorphic to $C(\sigma)$. Once again, the image of the generator a is the identity function $f(s) = s$

Theorem 2.10 (The Gelfand-Naimark Theorem - algebras with no identity) *Given any commutative C^*-algebra A, there is an isometric $*$-isomorphism of A onto $C_0(\Phi)$, the algebra of all continuous complex-valued functions vanishing at infinity on a locally compact Hausdorff space Φ.*

Proof. As shown in Section 1, A can be embedded in an algebra A_1 which is A with an identity adjoined. The above theory can then be applied to A_1.

From the construction of adjoining an identity it is easy to see that A (strictly speaking $\{(a, 0) : a \in A\}$) is a maximal ideal of A_1. Therefore, if Φ_1 is the carrier space of A_1, by Corollary 2.2 there is a unique element of Φ_1 with kernel A. Call this element ϕ_∞ and put

$$\Phi = \Phi_1 \setminus \{\phi_\infty\}$$

Since the identity of A_1 is $(0, 1)$, it follows that $\phi_\infty((a, \lambda)) = \lambda$. Also, every element of Φ is (by restriction) a non-zero homomorphism of A into $\mathbb{C}$. Conversely, if ψ is a non-zero homomorphism of A into $\mathbb{C}$, then $\phi : A_1 \to \mathbb{C}$ defined by $\phi((a, \lambda)) = \psi(a) + \lambda$ is easily shown to be an element of Φ_1. Thus Φ is, as before, the space of all non-zero homomorphisms of A into $\mathbb{C}$ and is called the carrier space of A. Since Φ_1 is compact, the topology induced on Φ is locally compact (i.e. every point has a compact neighbourhood).

From Theorem 2.8, the Gelfand map $(a, \lambda) \mapsto \widehat{(a, \lambda)}$ is an isometric isomorphism of A_1 onto $C(\Phi_1)$. The composition of this with the embedding $A \to (A, 0)$ takes A onto the algebra of all functions of $C(\Phi_1)$ that vanish at ϕ_∞. Restricting the functions to Φ shows that A is mapped onto an algebra $\hat{A}$ of continuous functions on Φ.

A function $f \in C(\Phi_1)$ satisfies $f(\phi_\infty) = 0$ if and only if, given any $\epsilon > 0$, there exists an open neighbourhood N of ϕ_∞ such that $|f(\phi)| < \epsilon$ for $\phi \in N$. Since the complement of N is compact, this can be rephrased in terms of elements of Φ alone as follows : given any $\epsilon > 0$, there exists a compact set K such that $|f(\phi)| < \epsilon$ for all $\phi \notin K$. This is exactly the definition of a "function vanishing at infinity" on a locally compact space. ∎

3 Applications of the Commutative Theory

3.1 The Spectral Theorem for Normal Operators

Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the C*-algebra of all bounded linear operators on $\mathcal{H}$. It $T \in \mathcal{B}(\mathcal{H})$ is a normal operator then $TT^* = T^*T$ and so the algebra $\mathcal{A}$ generated by T, T^* and I (the identity operator) is a commutative C*-subalgebra of $\mathcal{B}(\mathcal{H})$.

Let σ be the spectrum of T in $\mathcal{B}(\mathcal{H})$. Then Theorem 1.16 shows that

$$\sigma_{\mathcal{A}}(T) = \sigma\,.$$

It follows from Lemma 2.9 that the carrier space of $\mathcal{A}$ is homeomorphic to σ. Using the remarks following Lemma 2.7 and Lemma 2.9 we see that the Gelfand map composed with this homeomorphism gives an isometric *-isomorphism of $\mathcal{A}$ onto $C(\sigma)$. Moreover, the homeomorphism can be chosen so that T is mapped onto the identity function; i.e. onto the function f_T where $f_T(t) = t$. If $A \in \mathcal{A}$, we denote the image of A under this map by f_A and if $f \in C(\sigma)$, we denote the operator mapped onto f by A_f.

The spectral theorem for T is now easily derived using two standard theorems. Each of these is usually referred to as the "Riesz representation theorem". The first is the description of the dual of $C(\sigma)$ as the set of Borel measures on σ and the second is the correspondence between sesqui-linear functionals on $\mathcal{H} \times \mathcal{H}$ and bounded operators on $\mathcal{H}$.

First we note that, for any $x, y \in \mathcal{H}$, the map $\phi : C(\sigma) \to \mathbb{C}$ defined by

$$\phi(f) = \langle A_f x, y\rangle$$

is a continuous linear functional. Thus there exists a Borel measure $\mu_{x,y}$ on σ such that,

$$\langle A_f x, y\rangle = \int_\sigma f(t)\, d\mu_{x,y}\,.$$

We shall frequently use the fact that if two measures give the same integral of continuous functions then they are the same. If $f \in C(\sigma)$, and $|f(t)| \leq 1$ then $\|A_f\| = \|f\| \leq 1$ and so

$$\left|\int_\sigma f(t)\, d\mu_{x,y}\right| = |\langle A_f x, y\rangle| \leq \|x\|.\|y\|$$

which, by a standard argument using regularity, shows that for all Borel sets δ,

$$|\mu_{x,y}(\delta)| \leq \|x\|.\|y\|.$$

Also, it is easy to verify that for a fixed Borel set δ, $\mu_{x,y}(\delta)$ is linear in x and conjugate linear in y. For example, if $f \in C(\sigma)$

$$\begin{aligned}\int f\,d\mu_{x+x',y} &= \langle A_f(x+x'),y\rangle = \langle A_f x,y\rangle + \langle A_f x',y\rangle \\ &= \int f\,d(\mu_{x,y}+\mu_{x',y})\end{aligned}$$

showing that $\mu_{x+x',y}(\delta) = \mu_{x,y}(\delta) + \mu_{x',y}(\delta)$ for any Borel set δ.

The next step is to extend the definition of A_f to all functions f in the set $B(\sigma)$ of bounded Borel measurable functions on σ. To do this, note that for $f \in B(\sigma)$,

$$\phi(x,y) = \int_\sigma f\,d\mu_{x,y}$$

is a bounded sesquilinear form on $\mathcal{H}\times\mathcal{H}$. The second "Riesz representation theorem" states that every such form equals $\langle Ax,y\rangle$ for some bounded linear operator A. Thus we extend the definition of A_f by

$$\langle A_f x,y\rangle = \phi(x,y) = \int_\sigma f\,d\mu_{x,y}\,.$$

We now show that the extended map $f \mapsto A_f$ preserves the algebraic structure, that is, it is a *-homomorphism.

Since for $f \in C(\sigma)$ we have that $A_f^* = A_{\bar f}$, it follows easily that

$$\int_\sigma f\,d\mu_{x,y} = \langle A_f x,y\rangle = \overline{\langle A_f^* y,x\rangle} = \overline{\int_\sigma \bar f\,d\mu_{y,x}} = \int_\sigma f\,d\overline{\mu_{y,x}}\,.$$

This shows that integrating with respect to $\mu_{x,y}$ is the same as integrating with respect to $\overline{\mu_{y,x}}$. The relation therefore holds for all $f \in B(\sigma)$. Thus $\mu_{x,y} = \overline{\mu_{y,x}}$ and consequently $A_f^* = A_{\bar f}$ for all $f \in B(\sigma)$.

The proof that $f \mapsto A_f$ preserves multiplication requires two stages. If $f,g \in C(\sigma)$, $A_{fg} = A_f.A_g$ and so, for all x,y

$$\int_\sigma fg\,d\mu_{x,y} = \int_\sigma f\,d\mu_{A_g x,y}\,.$$

So "$g\,d\mu_{x,y} = d\mu_{A_g x,y}$" and the above relation holds for all $f \in B(\sigma)$. Thus for all $f \in B(\sigma)$ and $g \in C(\sigma)$,

$$A_{fg} = A_f.A_g\,.$$

Also

$$A_{fg} = (A_{\bar{f}\bar{g}})^* = (A_{\bar{f}}A_{\bar{g}})^*(A_{\bar{g}})^*(A_{\bar{f}})^* = A_g A_f \,.$$

But then

$$\int_\sigma gf \, d\mu_{x,y} = \langle A_{gf}x, y\rangle = \langle A_g A_f x, y\rangle = \int_\sigma g \, d\mu_{A_f x, y}$$

and the equality of the integrals for all $g \in B(\sigma)$ follows as before. Thus for all $f, g \in B(\sigma)$ we have $A_f A_g = A_{fg} = A_g A_f$.

The isometry of the Gelfand map is not extended to $f \in B(\sigma)$, since f may be large on sets of measure 0. However, it is easy to verify that $\|A_f\| \le \|f\|$ for $f \in B(\sigma)$.

We now look at the operators A_f for the cases when f is the characteristic function χ_δ of some Borel set δ. We write $E(\delta) = A_{\chi_\delta}$. Clearly each $E(\delta)$ commutes with T and $E(\sigma) = I$. We show that $E(\cdot)$ is an instance of what is called a "spectral measure". Since $(\chi_\delta)^2 = \chi_\delta$ and χ_δ is real, it is immediate that $E(\delta)^2 = E(\delta)$ and $E(\delta)^* = E(\delta)$ so that each $E(\delta)$ is an orthogonal projection on $\mathcal{H}$. It is easy to see that the following properties hold :

1. $E(\alpha \cap \beta) = E(\alpha).E(\beta)$
2. $E(\alpha \cup \beta) = E(\alpha) + E(\beta) - E(\alpha).E(\beta)$

To demonstrate the appropriate countable additivity of $E(\cdot)$, let $\{\delta_i\}$ be a countable family of disjoint Borel sets whose union is δ. Then, since for all $x \in \mathcal{H}$, $\mu_{x,x}(.)$ is a measure, $\mu_{x,x}(\delta) = \sum_1^\infty \mu_{x,x}(\delta_i)$ and so, since $E(\delta_i).E(\delta_j) = 0$ when $i \ne j$, a simple calculation shows that

$$\|E(\delta)x - \sum_{i=1}^n E(\delta_i)x\|^2 = \|E(\delta)x\|^2 - \|\sum_{i=1}^n E(\delta_i)x\|^2 = \mu_{x,x}(\delta) - \sum_{i=1}^n \mu_{x,x}(\delta_i)\,.$$

Thus, $\sum_{i=1}^\infty E(\delta_i)x$ converges in norm to $E(\delta)x$. Thus $\sum_{i=1}^\infty E(\delta_i)$ converges to $E(\delta)$ in the strong operator topology and we express this property by saying that

3. $E(\cdot)$ is strongly countably additive.

We extend the definition of $E(\delta)$ to all Borel subsets of $\mathbb{C}$ by $E(\delta) = E(\delta \cap \sigma)$. Clearly all the properties are preserved. A map from the Borel subsets of $\mathbb{C}$ with values in the orthogonal projections on Hilbert space $\mathcal{H}$ satisfying conditions 1 – 3 above is called a **spectral measure**. In the case of the spectral measure we have derived from the bounded normal operator T, we have that $E(\sigma) = I$ and $E(\cdot)$ has support σ. (The support of a Borel measure is the complement of the union of all open sets of measure zero. Thus it is always closed – but in this case it is also compact.)

The fact that $T = A_f$ where f is the identity function shows that, with the new notation,

$$\langle Tx, y\rangle = \int_\sigma \lambda \, \langle E(d\lambda)x, y\rangle$$

for all $x, y \in \mathcal{H}$. This is a weak form of the Spectral Theorem.

To obtain a stronger version, we need to discuss what is meant by $\int_\sigma f(\lambda)E(d\lambda)$. For a simple function $f = \sum_{i=1}^n \alpha_i \chi_{\delta_i}$, one defines

$$\int_\sigma f(\lambda)E(d\lambda) = \sum_{i=1}^n \alpha_i E(\delta_i)\,.$$

Clearly for a simple function f, $\| \int_\sigma f(\lambda)E(d\lambda)\| \le \|f\|$. Now for every $f \in B(\sigma)$, there is a sequence (f_k) of simple functions converging uniformly to f on σ. It follows easily that the sequence $(\int_\sigma f_k(\lambda)E(d\lambda))$ converges in the norm of $\mathcal{B}(\mathcal{H})$ to an operator which is dependent only on f and not the choice of the sequence (f_k). Define

$$\int_\sigma f(\lambda)E(d\lambda) = \lim_{k\to\infty} \int_\sigma f_k(\lambda)E(d\lambda)\,.$$

We clearly would like to have that $\int_\sigma f(\lambda)E(d\lambda) = A_f$ and this is indeed the case. To prove it we first note that the equality is true for simple functions. Now if (f_k) is a sequence of simple functions converging uniformly to f, since $\|A_f - A_{f_k}\| \le \|f - f_k\|$ it is clear that A_f equals the integral. Looking at the special case when f is the identity function, we obtain the spectral theorem.

Spectral Theorem for Normal Operators. For a normal operator on a Hilbert space $\mathcal{H}$, there exists a spectral measure $E(\cdot)$ with support the spectrum of T such that

$$T = \int \lambda \, E(d\lambda)\,.$$

We shall not go further with this topic in any detail in these notes. However we should mention a few developments.

1. It follows that defining $f(T) = \int_\sigma f(\lambda)E(d\lambda)$ gives a functional calculus for bounded Borel functions of a normal operator. In fact one could

deal with functions that are essentially bounded (in the appropriate sense) but we will not go into these technicalities.

2. It is easy to show that, when T is selfadjoint, each $E(\delta)$ commutes with any operator that commutes with T. This fact is the foundation stone of the theory of von Neumann algebras, since it shows that every strongly closed *-subalgebra of $\mathcal{B}(\mathcal{H})$ is generated by the selfadjoint projections it contains.

3. One can prove that, for any Borel set δ, the spectrum of $TE(\delta)$, regarded as an operator on the range of $E(\delta)$, is $\overline{\delta}$. The converse to this is true and shows the uniqueness of the spectral measure of an operator: namely if $E(\cdot)$ is a spectral measure commuting with T such that $\sigma(TE(\delta)) \subseteq \overline{\delta}$ for all Borel subsets δ of $\mathbb{C}$, the $E(\cdot)$ is the spectral measure of T.

3.2 An Interlude on Harmonic Analysis

The purpose of this section is to connect the Gelfand theory and Fourier analysis on groups. The most familiar and the most important groups for Fourier analysis are the real line $\mathbb{R}$, euclidean n-space $\mathbb{R}^n$, the integers $\mathbb{Z}$ (all with addition as the operation) and the circle group $\mathbb{T}$ of complex numbers of modulus 1 under multiplication (equivalently $(0, 2\pi]$ with addition modulo 2π). We shall refer to these groups as “our specific groups”.

The appropriate setting for elementary abstract harmonic analysis is to consider functions on a locally compact abelian group G (the more advanced portions of the subject deal with more general groups). We have seen that such a group G has a Haar measure and that $L^1(G)$, with convolution as multiplication, is a Banach algebra. We shall introduce the statements of the results in this generality. However, when the proofs are much easier (or even trivial) in the case of our specific groups mentioned above, we shall omit the general proofs.

For the whole of this section, G will denote a locally compact abelian group and all integrations will be with respect to Haar measure over the whole group. Recall (Section 1, Example 5) that $L^1(G)$ is a Banach algebra with multiplication

$$(f * g)(t) = \int_G f(s)g(s^{-1}t)\, ds\,.$$

Some preliminary results

Theorem 3.1 *$L^1(G)$ is a commutative algebra.*

Proof. For the general case one needs the fact that given any subset δ of G, then δ and $\delta^{-1} = \{x^{-1} : x \in \delta\}$ have the same Haar measure. For our specific groups this fact is clear and the rest of the proof is a trivial verification using a change of variable in the integration. ■

Lemma 3.2 *If $f \in L^1(G)$ and $g \in L^p(G), (1 \le p < \infty)$ then $(f * g)(y)$ exist for almost all y and $f * g \in L^p(G)$ with*

$$\|f * g\|_p \le \|f\|_1 \|g\|_p\,.$$

Proof. The proof uses the theorems of Fubini and Tonelli on double integration. Let $q = \frac{p}{p-1}$ be conjugate to p and let h be an arbitrary element of $L^q(G)$. Then $f(x)g(x^{-1}y)\overline{h(y)}$ is measurable on $G \times G$. So, using the invariance of Haar measure, we have that,

$$\begin{aligned} \int\int |f(x)g(x^{-1}y)\overline{h(y)}|\,dy\,dx &\le \int |f(x)| \left\{\int |g(x^{-1}y)|^p\,dy\right\}^{\frac{1}{p}} \left\{\int |h(y)|^q\,dy\right\}^{\frac{1}{q}} dx \\ &= \|f\|_1 \|g\|_p \|h\|_q \end{aligned}$$

Tonelli's theorem now implies that $f(x)g(x^{-1}y)\overline{h(y)}$ is integrable over $G \times G$ and Fubini's theorem shows that its integral is

$$\int \overline{h(y)} \int f(x)g(x^{-1}y)\,dx\,dy = \int (f * g)(y)\overline{h(y)}\,dy\,.$$

Essentially the same calculation as above now shows that $h \mapsto \int h(y).\overline{(f * g)(y)}\,dy$ is in the dual of L^q with norm at most $\|f\|_1\|g\|_p$ and so $f * g \in L^p$ with

$$\|f * g\|_p \le \|f\|_1 \|g\|_p \quad ■.$$

The above result shows, in particular, that if $T_f : L^2(G) \to L^2(G)$ is defined by $T_f g = f * g$ the map $f \mapsto T_f$ is continuous from $L^1(G)$ to $\mathcal{B}(L^2(G))$. Easy calculations show that this map is an algebraic homomorphism and that, if f^* is defined by $f^*(x) = \overline{f(x^{-1})}$, then $T_f^* = T_{f^*}$.

We shall need the fact that $T_f \ne 0$ whenever $f \ne 0$. For general groups, the standard proof of this involves the concept of an "approximate identity" but we shall not formally introduce this. We indicate proofs for our specific groups.

In $\ell^1(\mathbb{Z})$, the sequence $e = (\epsilon_n)$ where $\epsilon_0 = 1$ and $\epsilon_n = 0$ for $n \neq 0$ acts as an identity (and also $e \in \ell^2(\mathbb{Z})$). Thus if $x = (\xi_n) \neq 0$, $T_x e = x \neq 0$.

The proof for $L^1(\mathbb{R})$ is similar to the general case and involves some technicalities. We first deal with a continuous function f of compact support. If $f \neq 0$, choose $\epsilon > 0$ such that $\epsilon < \|f\|_1$. Clearly it is enough to find a function $g \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ such that $\|f * g - f\|_1 < \epsilon$. Since f vanishes outside a compact set, there exists $k > 0$ such that $f(x) = 0$ for $|x| > k$. Also an elementary result shows that f is uniformly continous so there is a $\delta > 0$ with $\delta < 1$ such that for $|s| < \delta$ and all t, $|f(t-s) - f(t)| < \frac{\epsilon}{2(k+1)}$. The condition $\delta < 1$ is imposed so that if $|s| < \delta$, $|f(t-s) - f(t)|$ vanishes for $|t| > k+1$. Therefore, for $|s| < \delta$,

$$\int |f(t-s) - f(t)|\, dt < \epsilon\,.$$

Define g by

$$g(t) = \begin{cases} \frac{1}{2\delta} & \text{if } |s| < \delta \\ 0 & \text{otherwise.} \end{cases}$$

Then $\int g(s)\, ds = 1$ so $f(t) = \int f(t) g(s)\, ds$ and

$$\begin{aligned} \|f * g - f\|_1 &= \int \left| \int [f(t-s) - f(t)] g(s)\, ds \right| dt \\ &\leq \int \int |f(t-s) - f(t)|\, dt\, g(s)\, ds \\ &< \epsilon\,. \end{aligned}$$

For an arbitrary function $f \in L^1(\mathbb{R})$, we use the fact that the continuous functions of compact support are dense in $L^1(\mathbb{R})$. It then follows by a routine approximation argument that, for $f \neq 0$, the operator T_f is non-zero.

The proof for $L^1(\mathbb{T})$ (and indeed for a general locally compact abelian group G) follows along the same lines. Instead of the interval $[-\delta, \delta]$ one needs a suitable compact neighbourhood of 1. The details are omitted.

Abstract harmonic analysis

Lemma 3.3 *The Gelfand map on $L^1(G)$ is injective.*

Proof. In view of Theorem 2.4.(iv), it is sufficient to show that if f is a non-zero element of $L^1(G)$, then $\nu(f) \neq 0$

But if $f \neq 0$, then the operator $T_f : L^2(G) \to L^2(G)$ given by $T_f g = f * g$, is non-zero and $\|T_f\| \leq \|f\|$. Since $T_f^* = T_{f^*}$, we have that T_f is normal and so, using Lemma 2.7,

$$\begin{aligned} 0 \neq \|T_f\| &= \lim_n \|T_f^n\|^{\frac{1}{n}} \\ &\leq \lim_n \|\overbrace{f * f * \cdots * f}^{n \text{ times}}\|^{\frac{1}{n}} = \nu(f). \end{aligned}$$

This proves the lemma. ■

Note that another way of stating the above lemma is : $L^1(G)$ *is semi-simple*.

A continuous homomorphism of G into the group $\mathbb{T}$ of complex numbers of modulus 1 is called a **character** of G. Denote the set of characters of G by $\hat{G}$. If $\xi, \eta \in G$, it is easy to verify that the function $\xi\eta : G \mapsto \mathbb{C}$ defined by,

$$(\xi\eta)(x) = \xi(x).\eta(x)$$

is also a character and that with this operation, $\hat{G}$ is an abelian group. We shall establish a correspondence between $\hat{G}$ and the carrier space Φ of $L^1(G)$.

For any function f on G, we define the function f_y (where $y \in G$) by $f_y(x) = f(yx)$. It is an easy exercise to show that if $f \in L^1(G)$ the map $y \mapsto f_y$ is continuous ($G \to L^1(G)$). (First consider the case when f is a continuous function of compact support and then use the fact that such functions are dense in $L^1(G)$.)

Theorem 3.4 *Given any character ξ of G, the function $\phi_\xi : L^1(G) \to \mathbb{C}$, given by*

$$\phi_\xi(f) = \int_G f(x)\overline{\xi(x)}\, dx$$

is a non-zero homomorphism. The map $\xi \mapsto \phi_\xi$ is a bijection between the group $\hat{G}$ of characters and the carrier space Φ of $L^1(G)$.

Proof. Since characters are continuous and bounded, there is no problem concerning integrability in the definition of ϕ_ξ. It is clear that ϕ_ξ is linear on $L^1(G)$. If $f, g \in L^1(G)$, using the invariance of Haar measure, Fubini's theorem and the fact that $\xi(xy) = \xi(x)\xi(y)$,

$$\begin{aligned} \phi_\xi(f * g) &= \int\int f(y)g(y^{-1}x)\overline{\xi(x)}\, dy\, dx \\ &= \int\int f(y)g(x)\overline{\xi(yx)}\, dy\, dx \\ &= \int f(y)\overline{\xi(y)}\, dy \int g(x)\overline{\xi(x)}\, dx \\ &= \phi_\xi(f)\phi_\xi(g) \end{aligned}$$

and so ϕ_ξ is a homomorphism.

Since ξ is a non-zero function, clearly $\phi_\xi \neq 0$. Similarly, it is clear that the map $\xi \mapsto \phi_\xi$ is an injection.

To show that $\xi \mapsto \phi_\xi$ is a surjection, we use the fact that the dual of L^1 is L^∞. Suppose that $\phi \in \Phi$. Then ϕ is a continuous linear functional on $L^1(G)$ and so, for some $\alpha \in L^\infty(G)$,

$$\phi(f) = \int f(x)\overline{\alpha(x)}\,dx\,.$$

Choose $g \in L^1(G)$ such that $\phi(g) \neq 0$. Then,

$$\begin{aligned}
\phi(f)\phi(g) &= \phi(f * g) \\
&= \int (f * g)(x)\overline{\alpha(x)}\,dx \\
&= \int\int f(y)g(y^{-1}x)\overline{\alpha(x)}\,dy\,dx \\
&= \int f(y)\phi(g_{y^{-1}})\,dy\,.
\end{aligned}$$

Thus

$$\phi(f) = \int f(y)\frac{\phi(g_{y^{-1}})}{\phi(g)}\,dy.$$

It therefore remains to prove that if ξ is defined by

$$\overline{\xi(y)} = \frac{\phi(g_{y^{-1}})}{\phi(g)}\,,$$

then ξ is a character. Continuity of ξ is clear. Also, an easy calculation shows that for $x, y \in G$,

$$g_{x^{-1}} * g_{y^{-1}} = g_{(xy)^{-1}} * g$$

and so $\phi(g_{x^{-1}}).\phi(g_{y^{-1}}) = \phi(g_{(xy)^{-1}}).\phi(g)$. Therefore,

$$\xi(x).\xi(y) = \xi(xy)\,. \tag{2}$$

Since $\phi(f) = \int f(y)\overline{\xi(y)}\,dy$ and $|\phi(f)| \leq \nu(f) \leq \|f\|$, we have that $|\xi(x)| \leq 1$ except on a set of measure zero. Continuity of ξ shows that $|\xi(x)| \leq 1$ for all $x \in G$. But also, from (1), $|\xi(x^{-1})| = |\xi(x)|^{-1} \leq 1$ and so $|\xi(x)| = 1$ all $x \in G$. Thus ξ is a character of G. ∎

Note that it is implicit in the above proof that $\frac{\phi(g_{y^{-1}})}{\phi(g)}$ is independent of of the choice of g (provided $\phi(g) \neq 0$). This can also be seen from the relation $f * g_y = f_y * g$. Note also that $\xi(y) = \overline{\xi(y^{-1})} = \frac{\phi(g_y)}{\phi(g)}$.

The group $\hat{G}$ inherits a topology from the Gelfand topology on Φ via the bijection of the above theorem. That is, a set δ is open in $\hat{G}$ when $\{\phi_\xi : \xi \in \delta\}$ is open in Φ. It is a fact that this makes $\hat{G}$ into a topoplogical group (i.e. with this topology, the group operations are continuous). We shall verify this for our specific groups but the general proof will not be given in these notes. We call the group $\hat{G}$ with this topology the **dual group** of the group G.

For $f \in L^1(G)$, the Gelfand transform of f is a function with domain Φ, the carrier space of $L^1(G)$. Since Φ and $\hat{G}$ are homeomorphic, we may therefore regard the transform of f as a function on $\hat{G}$. Thus

$$\hat{f}(\xi) = \phi_\xi(f)$$

where the map $\xi \mapsto \phi_\xi$ ($\hat{G} \to \Phi$) is as given in the Theorem above. Therefore, $\hat{f}$ may be written as

$$\hat{f}(\xi) = \int_G f(x)\overline{\xi(x)}\, dx\,.$$

The function $\hat{f}$ (regarded as a function on $\hat{G}$) is called the **Fourier transform** of f. Since the Fourier transform is essentially the same as the Gelfand transform, the following properties follow immediately from the general theory.

For $f, g \in L^1(G)$,

1. $\widehat{f * g} = \hat{f}.\hat{g}$
2. $\hat{f}$ is continuous and vanishes at infinity,
3. $\|\hat{f}\|_\infty \leq \|f\|_1\,.$

We now examine our specific groups and will show that the above indeed gives the classical Fourier transform. The results quoted should be familiar in the case of the classical theory.

The additive group $\mathbb{R}$ of real numbers.

We first identify the characters. If ξ is a character, $|\xi(x)| = 1$ for all $x \in \mathbb{R}$, $\xi(0) = 1$ and ξ is continuous. Therefore there exists a δ such that $|\arg \xi(x)| < \pi/2$ for $|x| < \delta$.

For any x, $\xi(x) = \xi(\frac{x}{2} + \frac{x}{2}) = (\xi(\frac{x}{2}))^2$ and so,

$$\arg \xi(\tfrac{x}{2}) = \tfrac{1}{2} \arg \xi(x) \quad \text{or} \quad \arg \xi(\tfrac{x}{2}) = \tfrac{1}{2} \arg \xi(x) + \pi \,.$$

But $|x| < \delta$ implies that $|\frac{x}{2}| < \delta$ and so $|\arg \xi(\frac{x}{2})| < \frac{\pi}{2}$ and the second possibility is excluded for such x.

Choose one particular number $t \neq 0$ with $|t| < \delta$ and suppose that $\xi(t) = e^{i\theta}$. Then from above, we see that

$$\xi(\tfrac{t}{2^n}) = e^{i\frac{\theta}{2^n}}$$

for every positive integer n. Also, for every integer m

$$\xi(\tfrac{mt}{2^n}) = \Big(\xi(\tfrac{t}{2^n})\Big)^m = e^{i\frac{m\theta}{2^n}}$$

and so for all dyadic rationals α, (i.e. finite sums of the form $\sum \dfrac{m_k}{2^{n_k}}$)

$$\xi(\alpha t) = e^{i\alpha\theta} \,.$$

Since dyadic rationals are dense in $\mathbb{R}$, using the continuity of ξ we have that the above hold for all real numbers α. Now let $y = \frac{\theta}{t}$. The for any $x \in \mathbb{R}$

$$\xi(x) = \xi(\tfrac{x}{t}.t) = e^{ixy} \,.$$

Hence every character is of the form

$$\xi(x) = e^{ixy}$$

for some $y \in \mathbb{R}$.

Conversely it is easy to see that distinct $y \in \mathbb{R}$ give distinct characters. Also it is clear that multiplication of characters corresponds to addition of the numbers that give rise to them. Thus, as far as algebraic properties are concerned, the dual group of $\mathbb{R}$ is $\mathbb{R}$.

We now verify that the topology induced on $\mathbb{R}$ by the Gelfand topology coincides with the usual topology. Note that a base for the neighbourhoods of 0, (that is, of the character ξ_0 corresponding to the map $x \mapsto e^{ixy}$ when $y = 0$, hence $\xi_0(x) \equiv 1$) in the Gelfand-induced topology is given by

$$N(\xi_0, f_1, f_2, \cdots, f_n; \epsilon) = \left\{ y : \left| \int f_r(x)(1 - e^{ixy})\, dx \right| < \epsilon,\ 1 \leq r \leq n \right\}$$

for all choices of $f_1, f_2, \cdots, f_n$ and ϵ. Since the map $y \mapsto \int f(x)(1 - e^{ixy})\, dx$ is continuous, it is clear that every neighbourhood of this type contains an open interval about 0. Conversely, choosing $f_1 = \chi_{[-1,1]}$, an explicit calculation shows that $N(\xi_0, f_1; \epsilon) = \{y : |\frac{\sin y}{y}| > 1 - \frac{\epsilon}{2}\}$ which, for suitable ϵ is contained in any open interval containing 0. Thus the neighbourhoods of 0 are the same in both topologies. An easy translation argument shows that all neighbourhoods are the same in the two topologies. Thus the dual of $\mathbb{R}$ is $\mathbb{R}$ (strictly one should say that the dual of $\mathbb{R}$ is algebraically isomorphic and homeomorphic to $\mathbb{R}$).

When dealing with Fourier transforms, Haar measure on $\mathbb{R}$ is usually taken as $\frac{1}{\sqrt{2\pi}}$ times Lebesgue measure. Then, from above, we have that the Fourier transform $\hat{f}$ of a function $f \in L^1(\mathbb{R})$ is

$$\hat{f}(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ixy}\, dy\,.$$

The properties 1 – 3 above are then familiar results of Fourier theory, Property 2 being the Riemann-Lebesgue lemma.

The additive group of $\mathbb{R}^n$.

Note that each subspace of $\mathbb{R}^n$ is a subgroup. If ξ is a character of $\mathbb{R}^n$, it induces a character on each one-dimensional subspace. Let $\{e_r : 1 \le r \le n\}$ be the usual basis of $\mathbb{R}^n$. Then $x \mapsto \xi(xe_r)$ is a character of $\mathbb{R}$ and so from above we have that for some y_r,

$$\xi(xe_r) = e^{ixy_r}\,.$$

Thus, if $\mathbf{x} = (x_1, x_2, \cdots, x_n)$,

$$\xi(\mathbf{x}) = \xi\left(\sum x_r e_r\right) = e^{i\mathbf{x}.\mathbf{y}}$$

where $\mathbf{y} \in \mathbb{R}^n$ and $\mathbf{x}.\mathbf{y}$ is the usual inner product. It follows easily that the dual of $\mathbb{R}^n$ is $\mathbb{R}^n$ and the Gelfand transform of $f \in L^1(\mathbb{R}^n)$ is therefore

$$\hat{f}(\mathbf{y}) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \int_{\mathbb{R}^n} f(\mathbf{x}) e^{-i\mathbf{x}.\mathbf{y}}\, d\mathbf{x}\,,$$

that is, the Fourier transform.

The circle group $\mathbb{T}$.

It follows just as for $\mathbb{R}$, that every character of ξ of $\mathbb{T}$ is of the form

$$\xi(x) = e^{ixy}\,.$$

However, since x and $x + 2\pi$ represent the same member of $\mathbb{T}$, we must have $e^{ixy} = e^{i(x+2\pi)y}$. Therefore y is an integer. Thus the dual of $\mathbb{T}$ is the additive group $\mathbb{Z}$ of integers. The Gelfand transform of $f \in L^1(\mathbb{T})$ is

$$\hat{f}(n) = \int_0^{2\pi} f(x)\, e^{-inx}\, dx\,.$$

The doubly infinite sequence $\{\hat{f}(n)\}$ is usually called the sequence of Fourier coefficients of f.

The group $\mathbb{Z}$ of integers.

Once again, the characters are of the form $\xi(n) = e^{iny}$ but now any two numbers differing by 2π give the same character. Thus the dual of $\mathbb{Z}$ is $\mathbb{T}$. (The proof that the topology of $\mathbb{T}$ arising from the Gelfand topology is the usual topology is an easy verification).

Haar measure on $\mathbb{Z}$ gives unit mass to every point and so $L^1(\mathbb{Z})$ is the space of absolutely convergent doubly infinite sequences. The Fourier transform of a member $(a_n)_{-\infty<n<\infty}$ of $L^1(\mathbb{Z})$ is the function $f \in \mathrm{C}(\mathbb{T})$ given by,

$$f(y) = \sum_{-\infty}^{\infty} a_n e^{iny}$$

and this is just a Fourier series.

4 Representations - the Gelfand-Naimark Theorem

4.1 The GNS construction

Until further notice, we shall be considering a C*-algebra A with identity e. We begin with a lemma which is the first of many occasions when the commutative Gelfand-Naimark theorem is used to prove a fact about general C*-algebras.

Lemma 4.1 *Let x be a selfadjoint element of A such that $\sigma(x) \subseteq \mathbb{R}^+$. Then there exists a unique $y \in A$ such that y is selfadjoint, $\sigma(y) \subseteq \mathbb{R}^+$ and $y^2 = x$.*

Proof. Let C be the commutative C*-subalgebra of A generated by e and x. Then C is isometrically isomorphic, via the Gelfand map, to $C(S)$, the continuous functions on a compact space S. Since the range of $\hat{x}$ is $\sigma(x) \subseteq \mathbb{R}^+$, the function $\hat{x}$ is non-negative and so there exists a unique non-negative function $f \in C(S)$ such that $f^2 = \hat{x}$. As the Gelfand map is onto, $f = \hat{y}$ for some $y \in C$. Since $\hat{y}$ is a non-negative function, it is clear that $y = y^*$ and $\sigma(y) \subseteq \mathbb{R}^+$.

To prove uniqueness, let z be any element satisfying the conclusion of the lemma. Then z commutes with x and so the C*-algebra generated by x, z and e is commutative and contains C. Under the Gelfand map applied to this algebra, both y and z are represented by positive functions whose square is the same (namely the function corresponding to x). Therefore $y = z$. ■

We now introduce the main concept of this section. A **representation**[1] ρ of A is a *-homomorphism of A into the C*-algebra $\mathcal{B}(\mathcal{H})$ of all bounded linear operators on some Hilbert space $\mathcal{H}$ such that $\rho(e) = I$. That is, a representation is a map $\rho : A \mapsto \mathcal{B}(\mathcal{H})$ such that

$$\begin{aligned} \rho(\lambda a + \mu b) &= \lambda\rho(a) + \mu\rho(b) \\ \rho(ab) &= \rho(a).\rho(b) \\ \rho(a^*) &= \rho(a)^* \\ \rho(e) &= I\,. \end{aligned}$$

[1]In a more general context this would be called a *-representation of A on a Hilbert space.

Note that in many other treatments $\rho(e) = I$ is not included in the definition of the term representation. However, the condition is convenient and, as will be seen later, there is no essential loss in generality.

A representation is said to be **faithful** if it is an injection. Note that Theorem 1.14 shows that (with the above definition) any representation is automatically contiuous with $\|\rho(a)\| \leq \|a\|$. The following lemma is included to motivate the development.

Lemma 4.2 *Let ρ be a representation of A on $\mathcal{H}$ and let ξ be a unit vector of $\mathcal{H}$. Then the function $f : A \mapsto \mathbb{C}$ defined by*

$$f(a) = \langle \rho(a)\xi, \xi \rangle$$

satisfies

(i) *f is a linear functional,*

(ii) *$f(a^*a) \geq 0$ for all $a \in A$,*

(iii) *$f(e) = 1$,*

(iv) *f is continuous*

(v) *$\|f\| = 1$.*

Proof. The only part that is not completely trivial is (ii) whose proof is as follows :

$$f(a^*a) = \langle \rho(a^*a)\xi, \xi \rangle = \langle \rho(a^*)\rho(a)\xi, \xi \rangle = \|\rho(a)\xi\|^2 \geq 0.$$

Parts (iv) and (v) follow from the fact (Theorem 1.14) that $\|\rho(a)\| \leq \|a\|$. ■

A linear functional $f : A \mapsto \mathbb{C}$ is said to be a **positive linear functional** if $f(a^*a) \geq 0$ for all $a \in A$. If f satisfies properties (i) – (v) of Lemma 4.2 then f is said to be a **state** of A. In Lemma 4.3 below we shall prove that every positive linear functional is continuous and is a positive scalar multiple of a state.

From the above, we have that every representation gives rise to a state. The object of the GNS construction is to reverse the process; that is, to construct a representation from any given state.

Lemma 4.3 *Let f be a positive linear functional on A. Then for all $a, b \in A$,*

(i) $f(b^*a) = \overline{f(a^*b)}\,,$

(ii) $|f(b^*a)|^2 \leq f(a^*a).f(b^*b)$

Proof. These results arise from the fact that $\phi(x,y) = f(y^*x)$ is a non-negative sesquilinear form on A. The details follow.

(i) This is proved using polarization. We have

$$4b^*a = (a+b)^*(a+b) - (a-b)^*(a-b) + i(a+ib)^*(a+ib) - i(a-ib)^*(a-ib)$$

$$4a^*b = (a+b)^*(a+b) - (a-b)^*(a-b) - i(a+ib)^*(a+ib) + i(a-ib)^*(a-ib)$$

and, since f is positive and linear,

$$\begin{aligned} 4\overline{f(a^*b)} &= f[(a+b)^*(a+b)] - f[(a-b)^*(a-b)] \\ &\quad + if[(a+ib)^*(a+ib)] - if[(a-ib)^*(a-ib)] = 4f(b^*a). \end{aligned}$$

(ii) This is the Cauchy-Schwartz inequality. For all $\lambda \in \mathbb{C}$,

$$f[(\lambda a+b)^*(\lambda a+b)] = |\lambda|^2 f(a^*a) + \lambda f(b^*a) + \overline{\lambda} f(a^*b) + f(b^*b) \geq 0\,.$$

Choosing $\lambda = ke^{-i\theta}$ where $\theta = \arg f(b^*a)$ with k real gives, using (i), that

$$k^2 f(a^*a) + 2k|f(b^*a)| + f(b^*b) \geq 0$$

for all $k \in \mathbb{R}$, and the result merely states the fact that the discriminant of the above quadratic is negative. ■

Corollary 4.4

(i) $f(a^*) = \overline{f(a)}\,,$

(ii) $|f(a)|^2 \leq f(e)f(a^*a)\,.$

Proof. Put $b = e$ in the lemma. ■

Lemma 4.5 *If f is a positive linear functional on A then f is continuous and $\|f\| = f(e)$.*

Proof. Suppose $a = a^*$ and $\|a\| \le 1$. Then $\sigma(a) \subseteq [-1, 1]$ and the spectral mapping theorem shows that $\sigma(e-a) \subseteq [0, 2]$. Thus by Lemma 4.1 A contains a selfadjoint element b such that $b^2 = e - a$. Therefore $f(e-a) = f(b^*b) \ge 0$ and so

$$f(a) \le f(e)\,.$$

Now if $x \in A$, applying the above to $\dfrac{x^*x}{\|x^*x\|}$ and using Corollary 4.4 (ii) we have that

$$|f(x)|^2 \le f(e).f(x^*x) \le [f(e)]^2\|x^*x\| \le [f(e)]^2\|x\|^2\,.$$

Thus f is continuous with $\|f\| \le f(e)$. But also since $\|e\| = 1$,

$$f(e) = |f(e)| \le \|f\|$$

and so $\|f\| = f(e)$. ■

Note that the only time that the C* condition, via Lemma 4.1, is used in this section is in the above proof. By justifying the formal binomial expasion of $(e-x)^{\frac{1}{2}}$ for $\|x\| \le 1$, the above lemma can be proved for Banach *-algebras with isometric involution. This shows that the GNS construction is valid for these more general algebras.

The following lemma gives an inner product space on which the GNS construction will provide a representation.

Lemma 4.6 *Let f be a positive linear functional on A and let $N = \{x : f(x^*x) = 0\}$. Then N is a left ideal and if $[a]$ denotes the equivalence class of a in the quotient space A/N then*

$$\langle [a], [b] \rangle = f(b^*a)$$

defines an inner product on A/N.

Proof. If $x \in N$ then from Lemma 4.3 (ii), for $a \in A$,

$$|f(ax)|^2 \le f(x^*x).f(a^*a) = 0\,.$$

Therefore we have the following characterisation of N :

$$N = \{x : f(ax) = 0 \text{ for all } a \in A\}\,.$$

It is clear from this that N is a left ideal.

We now show that $f(b^*a)$ depends only on the equivalence classes of a and b. For all $x, y \in N$, using Lemma 4.3 (i) and the above,

$$\begin{aligned} f[(b+y)^*(a+x)] &= f(b^*a) + f(b^*x) + f[y^*(a+x)] \\ &= f(b^*a) + f(b^*x) + f\overline{[(a+x)^*y]} \\ &= f(b^*a)\,. \end{aligned}$$

Hence the relation

$$\langle [a], [b] \rangle = f(b^*a)$$

gives well-defined form on A/N and is clearly a non-negative sesquilinear form. Also $\langle [a], [a] \rangle = 0$ if and only if $[a] = [0]$ (the zero of A/N). Therefore$\langle \cdot, \cdot \rangle$ is an inner product on A/N. ■

Lemma 4.7 *If f is a positive linear functional on A, then for all $x, y \in A$*

$$|f(y^*xy)| \le \|x\| f(y^*y)\,.$$

Proof. let $g(a) = f(y^*ay)$. Then $g(a^*a) = f(y^*a^*ay) \ge 0$ and so g is positive. From Lemma 4.5 both f and g are continuous and $\|g\| = g(e) = f(y^*y)$. Hence

$$|f(y^*xy)| = |g(x)| \le \|g\|.\|x\| = \|x\| f(y^*y)\,,$$

and the lemma is proved. ■

Recall that if K is any inner product space then its completion H is a Hilbert space and K is (strictly one should say "can be identified with") a dense subspace of H.

The following theorem is called after Gelfand, Naimark and Segal.

Theorem 4.8 (GNS construction) *Given any state on a C^*-algebra with identity, there exists a representation ρ_f on some Hilbert space $H_f$such that*

$$f(a) = \langle \rho_f(a)\xi, \xi \rangle$$

for some unit vector ξ of H_f.

Proof. Let $N = \{x : f(x^*x) = 0\}$. Define H_f as the completion of the inner product space A/N constructed as in Lemma 4.6 . Then H_f is a Hilbert space.

For each $[x] \in A/N$ and $a \in A$ define

$$\rho_f(a)[x] = [ax]\,.$$

Since N is a left ideal it is clear that $\rho_f(a)$ is a well- defined map form A/N to itself. Clearly $\rho_f(a)$ is linear and we now show that $\|\rho_f(a)\| \leq \|a\|$. (Note that this could not be deduced from Theorem 1.14 since A/N is not complete and so $\mathcal{B}\,(A/N)$ is not a C*-algebra.) Using Lemma 4.7 we have that,

$$\begin{aligned} \|\rho_f(a)x\|^2 &= \|[ax]\|^2 = f(x^*a^*ax) \\ &\leq \|a^*a\|f(x^*x) \\ &\leq \|a\|^2\langle[x],[x]\rangle \\ &= \|a\|_A^2.\|[x]\|_{A/N}^2\,. \end{aligned}$$

Therefore $\rho_f(a)$ is a continuous linear operator on a dense subspace of H_f and so it has a unique continuous extension to H_f. It should not cause any confusion if the extension is also denoted by $\rho_f(a)$.

Now a trivial verification shows that the map $a \mapsto \rho_f(a)$ is a homomorphism and $\rho_f(e)$ is easily seen to be the identity operator on A/N. In addition

$$\begin{aligned} \langle\rho_f(a)[x],[y]\rangle &= \ = \langle[ax],[y]\rangle = f(y^*ax) = f\left((a^*y)^*x\right) \\ &= \langle[x],[a^*y]\rangle \\ &= \langle[x],\rho_f(a^*)[y]\rangle \end{aligned}$$

which proves that $\rho_f(a)^* = \rho_f(a^*)$ on the dense subspace A/N of H_f and hence, since both $\rho_f(a)^*$ and $\rho_f(a^*)$ are bounded, they are equal.

We have thus proved that ρ_f is a representation of A on H_f. Also

$$\langle\rho_f(a)[e],[e]\rangle = \langle[a],[e]\rangle = f(a)$$

and, since f is a state, $\|[e]\|_{H_f}^2 = f(e^*e) = f(e) = 1$ so $[e]$ is a unit vector. This completes the proof. ■

4.2 Positive elements and the Gelfand-Naimark theorem

In this section, A continues to denote a C*-algebra with identity e. The question of showing that A has a sufficiently rich supply of states hinges on the study of elements of the form a^*a. The major tool in the investigation of this is the commutative Gelfand-Naimark theorem.

An element a of A is said to be **positive** if it is selfadjoint and its spectrum consists of non-negative real numbers. The main technical problem will be to show that every element of the form x^*x is positive. Note that, from Theorem 1.16 , if B is a C*-subalgebra of A, with the same identity as A, then $\sigma_B(b) = \sigma_A(b)$ for all $b \in B$. Thus an element a is positive in A if and only if it is positive in any one *-subalgebra containing it and so it is sufficient to look at its spectrum in the C*-algebra generated by a and e.

The set of all positive elements of A will be denoted by A^+. Note that the spectral mapping theorem (Theorem 1.6) shows that if $a \in A^+$ then $a^n \in A^+$ for all positive integers n.

Lemma 4.9 *If a is a selfadjoint element of A then*

(i) $\|e - a\| \leq 1 \Rightarrow a \in A^+$,

(ii) $a \in A^+, \|a\| \leq 1 \Rightarrow \|e - a\| \leq 1$,

(iii) $a \in A^+ \iff \Big\| \|a\|e - a \Big\| \leq \|a\|$.

Proof. Let C be the C*-algebra generated by a and e and suppose that Φ is the carrier space of C. Since the range of the Gelfand transform $\hat{a}$ of a is the spectrum of a, it follows that $a \in A^+$ if and only if $\hat{a}$ is a positive function on Φ.

To prove (i), if $\|e - a\| \leq 1$ then $\|\hat{e} - \hat{a}\| \leq 1$ and so $|1 - \hat{a}(\phi)| \leq 1$ for all $\phi \in \Phi$. Since a is selfadjoint, $\hat{a}$ is real-valued and so it follows that $\hat{a}(\phi) \geq 0$ for all $\phi \in \Phi$. Therefore $a \in A^+$.

For (ii), if $a \in A^+$ and $\|a\| \leq 1$ then $\|\hat{a}\| \leq 1$ and so $0 \leq \hat{a}(\phi) \leq 1$ for all $\phi \in \Phi$. Therefore $\|e - a\| = \|\hat{e} - \hat{a}\| \leq 1$.

Finally (iii) results from applying (i) and (ii) to $\dfrac{a}{\|a\|}$. ■

Theorem 4.10 A^+ *is a closed convex cone with* $A^+ \cap (-A^+) = (0)$.

Proof. If $a \in A^+$ then clearly $\lambda a \in A^+$ for all $\lambda \geq 0$. To prove that A^+ is convex suppose $a, b \in A^+$ and $\lambda, \mu \geq 0$ with $\lambda + \mu = 1$. If $a = b = 0$ then $\lambda a + \mu b = 0 \in A^+$. Otherwise, $\dfrac{a}{\|a\| + \|b\|}$ and $\dfrac{b}{\|a\| + \|b\|}$ are both positive and have norm at most 1. Hence, using Lemma 4.9 ,

$$\begin{aligned} \left\| e - \frac{\lambda a + \mu b}{\|a\| + \|b\|} \right\| &\leq \left\| \lambda \left(e - \frac{a}{\|a\| + \|b\|} \right) \right\| + \left\| \mu \left(e - \frac{b}{\|a\| + \|b\|} \right) \right\| \\ &\leq \lambda + \mu = 1 \end{aligned}$$

and so by part (i) of Lemma 4.9 , $\dfrac{\lambda a + \mu b}{\|a\| + \|b\|} \in A^+$. Consequently, $\lambda a + \mu b \in A^+$ and so A^+ is a convex cone.

If $x \in A^+ \cap (-A^+)$ then x is selfadjoint and $\sigma(x) = \{0\}$. Hence $\nu(x) = 0$ and so from Corollary 1.10, $x = 0$.

To show that A^+ is closed, note that the set S of all selfadjoint elements of A is closed. Lemma implies that A^+ is a closed subset of S and so A^+ is closed. ■

Theorem 4.11 *The following conditions on an element a of A are equivalent.*

(i) $a \in A^+$,

(ii) $a = x^2$ *for some selfadjoint element x of A ,*

(iii) $a = y^* y$ *for some element y of A .*

Proof. Lemma 4.1 establishes (i)$\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) is trivial. The substance of the Theorem is to prove that (iii) $\Rightarrow$ (i).

If $a = y^* y$ then a is self-adjoint. Let Φ be the carrier space of the commutative C*-algebra C generated by a and e. Then $\hat{a}$ is a real function on Φ and, writing $f(\phi) = \max[\hat{a}(\phi), 0]$ and $g(\phi) = -\min[\hat{a}(\phi), 0]$, we have that f and g are non-negative functions on Φ with

$$\hat{a} = f - g \text{ and } fg = gf = 0\,.$$

Since $f, g \in C(\Phi)$, from the commutative Gelfand-Naimark theorem (Theorem 2.8) we have that for some $b, c \in C$, $\hat{b} = f$ and $\hat{c} = g$. Thus

$$a = y^* y = b - c$$

where $bc = cb = 0$. Since an element of C is in A^+ if and only if its Gelfand transform is non-negative, we have that $b, c \in A^+$. To prove that $a \in A^+$ we shall show that $c = 0$.

Let u and v be, respectively, the real and imaginary parts of cy^*. Since u and v are selfadjoint the spectrum of each is real and the spectral mapping theorem shows that u^2 and v^2 are positive. Since A^+ is a convex cone,

$$(cy^*)^* cy^* + cy^*(cy^*)^* = 2(u^2 + v^2) \in A^+\,. \tag{1}$$

Also

$$cy^*(cy^*)^* = cy^* yc = cbc - c^3 = -c^3 \in -A^+\,. \tag{2}$$

Therefore, using (1), (2) and Theorem 4.10 ,

$$(cy^*)^* cy^* = c^3 + 2(u^2 + v^2) \in A^+\,. \tag{3}$$

However, since $(cy^*)^* cy^*$ and $cy^*(cy^*)^*$ have the same non-zero numbers in their spectrum (Lemma 1.8), (3) implies that

$$cy^*(cy^*)^* = -c^3 \in A^+\,.$$

Thus $c^3 \in A^+ \cap (-A^+)$ and so, using Theorem 4.10 , $c^3 = 0$. This implies that $c = 0$ and the Theorem is proved. ■

The above theorem shows that the various notions of positivity are identical. As a result we now have that a linear functional is positive if and only if it is positive on positive elements. This is needed for the following result.

Lemma 4.12 *Let f be a linear functional on A such that $\|f\| = f(e) = 1$. Then f is a state of A.*

Proof. We show that f is positive and therefore a state. First we show that if a is selfadjoint then $f(a)$ is real. We may suppose that $\|a\| \leq 1$. Then, for any real k, using the C*- condition,

$$\|a + ike\|^2 = \|(a - ike)(a + ike)\| = \|a^2 + k^2 e\| \leq 1 + k^2 \,.$$

Now, if $f(a) = \alpha + i\beta$, using $\|f\| = f(e) = 1$ it follows that

$$|f(a + ike)|^2 = |\alpha + i(k + \beta)|^2 = \alpha^2 + k^2 + \beta^2 + 2k\beta \leq \|a + ike\|^2 \leq 1 + k^2 \,.$$

If $\beta \neq 0$, this produces a contradiction for positive or negative k of sufficiently large modulus. Hence $\beta = 0$ and $f(a)$ is real. Now suppose that a is positive with $\|a\| \leq 1$. Then $\|e - a\| \leq 1$ so, using $\|f\| = 1$, $-1 \leq f(e - a) \leq 1$. Since $f(e - a) = 1 - f(a)$ this implies that $f(a) \geq 0$. ■

Lemma 4.13 *For any non-zero element x of A there exists a state f of A such that $f(x^*x) \neq 0$.*

Proof. Let C be the C*-algebra generated by e and x^*x. The Gelfand mapping theorem shows that for any λ in the spectrum of x^*x, there is a homomorphism ϕ of C into $\mathbb{C}$ such that $\phi(x^*x) = \lambda$. In particular, ϕ is a linear functional on C such that $\|\phi\| = \phi(e) = 1$ and clearly, by the choice of λ we can ensure that $\phi(x^*x) \neq 0$. A simple application of the Hahn-Banach theorem gives an extension f of ϕ to A such that $\|f\| = \|\phi\|$. It follows from Lemma 4.12 that f is a state. ■

Theorem 4.14 (Gelfand-Naimark) *Any C*-algebra is isometrically *-isomorphic to a closed *-subalgebra of operators on some Hilbert space.*

Proof. We first prove the theorem for the case of a C*-algebra A with identity e. For each non-zero $x \in A$ let f_x be a state (as found in the preceding lemma) such that $f_x(x^*x) \neq 0$. Let ρ_x be the representation of A on a Hilbert space H_x constructed from f_x by the GNS construction Theorem 4.8 . Then

$$\|\rho_x(x)[e]\|^2 = f_x(ex^*xe) = f_x(x^*x) \neq 0$$

and so $\rho_x(x) \neq 0$.

Let $H = \oplus\{H_x : x \in A\}$ and define $\rho : A \to \mathcal{B}(H)$ by

$$\rho(a) = \oplus\{\rho_x(a) : x \in A\}\,.$$

Then ρ is clearly a representation and, since $\rho_a(a) \neq 0$, ρ is injective. Hence, from Corollary 1.15 , it is isometric. Thus the theorem is proved for algebras with identity.

If A has no identity and A_1 is the C*-algebra formed by adjoining an identity to A, then A is a closed *-subalgebra of A_1. Hence, applying the result proved above to A_1 we find an isometric *-representation ρ_1 of A_1 into $\mathcal{B}(H)$ for some Hilbert space H. The result now follows just by restricting ρ_1 to A. ■

The above theorem may be rephrased as follows : every C*-algebra has a faithful representation.

Suggestions for further reading

1. R.V. Kadison and J.R. Ringrose, *Fundamentals of the theory of operator algebras,* Academic Press, New York, 1983.

2. L.H. Loomis, *Abstract harmonic analysis,* Van Nostrand, Princeton, 1953.

3. G.J. Murphy, C^**-algebras and operator theory,* Academic Press, San Diego, 1990.

4. G.K. Pedersen C^**-algebras and their automorphism groups,* Academic Press, London, 1979.

5. W. Rudin, *Fourier analysis on groups,* Interscience, New York, 1962.