

Matematika 1 Vegyész- és környezetmérnököknek Gyakorlat

6. feladatsor – Megoldás

1. Határozzuk meg az alábbi határértékeket!

$$\begin{array}{lll} \text{a)} \lim_{x \rightarrow -\infty} (x^3 - 2x + 3), & \text{c)} \lim_{x \rightarrow -\infty} \frac{2x^5 - 3x^2 + 1}{x^7 + 4x^3 + 5}, & \text{e)} \lim_{x \rightarrow +\infty} \frac{x^7 - 5x^4 - x^2}{10x - 3x^5 + 11x^2}, \\ \text{b)} \lim_{x \rightarrow +\infty} \frac{-x^4 - 2x^2}{3x^3 + 1}, & \text{d)} \lim_{x \rightarrow +\infty} \frac{3x^3 - 3}{5x - 4x^3 + 1}, & \text{f)} \lim_{x \rightarrow -\infty} x \left(\sqrt{x^2 + 1} - \sqrt{x^2 - 3} \right). \end{array}$$

Megoldás:

a)

$$\lim_{x \rightarrow -\infty} (x^3 - 2x + 3) = \lim_{x \rightarrow -\infty} x^3 \left(1 - \frac{2}{x^2} + \frac{3}{x^3} \right) = -\infty.$$

b)

$$\lim_{x \rightarrow +\infty} \frac{-x^4 - 2x^2}{3x^3 + 1} = \lim_{x \rightarrow +\infty} \frac{x^4 \left(-1 - \frac{2}{x^2} \right)}{x^3 \left(3 + \frac{1}{x^3} \right)} = \lim_{x \rightarrow +\infty} x \frac{-1 - \frac{2}{x^2}}{3 + \frac{1}{x^3}} = -\infty.$$

c)

$$\lim_{x \rightarrow -\infty} \frac{2x^5 - 3x^2 + 1}{x^7 + 4x^3 + 5} = \lim_{x \rightarrow -\infty} \frac{x^5 \left(2 - \frac{3}{x^3} + \frac{1}{x^5} \right)}{x^7 \left(1 + \frac{4}{x^4} + \frac{5}{x^7} \right)} = \lim_{x \rightarrow -\infty} \frac{1}{x^2} \frac{2 - \frac{3}{x^3} + \frac{1}{x^5}}{1 + \frac{4}{x^4} + \frac{5}{x^7}} = 0.$$

d)

$$\lim_{x \rightarrow +\infty} \frac{3x^3 - 3}{5x - 4x^3 + 1} = \lim_{x \rightarrow +\infty} \frac{x^3 \left(3 - \frac{3}{x^3} \right)}{x^3 \left(\frac{5}{x^2} - 4 + \frac{1}{x^3} \right)} = \lim_{x \rightarrow +\infty} \frac{3 - \frac{3}{x^3}}{\frac{5}{x^2} - 4 + \frac{1}{x^3}} = -\frac{3}{4}.$$

e)

$$\lim_{x \rightarrow +\infty} \frac{x^7 - 5x^4 - x^2}{10x - 3x^5 + 11x^2} = \lim_{x \rightarrow +\infty} \frac{x^7 \left(1 - \frac{5}{x^3} - \frac{1}{x^5} \right)}{x^5 \left(\frac{10}{x^4} - 3 + \frac{11}{x^3} \right)} = \lim_{x \rightarrow +\infty} x^2 \frac{1 - \frac{5}{x^3} - \frac{1}{x^5}}{\frac{10}{x^4} - 3 + \frac{11}{x^3}} = -\infty.$$

f)

$$\begin{aligned} \lim_{x \rightarrow -\infty} x \left(\sqrt{x^2 + 1} - \sqrt{x^2 - 3} \right) &= \lim_{x \rightarrow -\infty} x \frac{(\sqrt{x^2 + 1} - \sqrt{x^2 - 3})(\sqrt{x^2 + 1} + \sqrt{x^2 - 3})}{\sqrt{x^2 + 1} + \sqrt{x^2 - 3}} = \\ &= \lim_{x \rightarrow -\infty} \frac{4x}{\sqrt{x^2 + 1} + \sqrt{x^2 - 3}} = \lim_{x \rightarrow -\infty} \frac{4x}{\sqrt{x^2 \left(1 + \frac{1}{x^2} \right)} + \sqrt{x^2 \left(1 - \frac{3}{x^2} \right)}} = \\ &= \lim_{x \rightarrow -\infty} \frac{4x}{|x| \left(\sqrt{1 + \frac{1}{x^2}} + \sqrt{1 - \frac{3}{x^2}} \right)} = \lim_{x \rightarrow -\infty} \frac{-4}{\sqrt{1 + \frac{1}{x^2}} + \sqrt{1 - \frac{3}{x^2}}} = -2. \end{aligned}$$

2. Határozzuk meg az alábbi határértékeket!

$$\begin{array}{lll} \text{a)} \lim_{x \rightarrow 2} \frac{-x^4 - 2x^2}{2x^3 + 2}, & \text{c)} \lim_{x \rightarrow 1} \frac{x^2 - 3x + 2}{x - 1}, & \text{e)} \lim_{x \rightarrow 1} \frac{x^2 - 3x + 2}{2x - 2x^2}, \\ \text{b)} \lim_{x \rightarrow \pi} (\sin x + \tan(2x)), & \text{d)} \lim_{x \rightarrow 0} \frac{x^2 + 5x}{2x - x^2}, & \text{f)} \lim_{x \rightarrow -3} \frac{x^2 + 6x + 9}{x^2 - 9}. \end{array}$$

Megoldás:

a)

$$\lim_{x \rightarrow 2} \frac{-x^4 - 2x^2}{2x^3 + 2} = \frac{-2^4 - 2 \cdot 2^2}{2 \cdot 2^3 + 2} = \frac{-16 - 8}{16 + 2} = -\frac{4}{3}.$$

b)

$$\lim_{x \rightarrow \pi} (\sin x + \tan(2x)) = \lim_{x \rightarrow \pi} (\sin \pi + \tan(2\pi)) = 0 + 0 = 0.$$

c)

$$\lim_{x \rightarrow 1} \frac{x^2 - 3x + 2}{x - 1} = \lim_{x \rightarrow 1} \frac{(x-1)(x-2)}{x-1} = \lim_{x \rightarrow 1} x - 2 = 1 - 2 = -1.$$

d)

$$\lim_{x \rightarrow 0} \frac{x^2 + 5x}{2x - x^2} = \lim_{x \rightarrow 0} \frac{x(x+5)}{x(2-x)} = \lim_{x \rightarrow 0} \frac{x+5}{2-x} = \frac{0+5}{2-0} = \frac{5}{2}.$$

e)

$$\lim_{x \rightarrow 1} \frac{x^2 - 3x + 2}{2x - 2x^2} = \lim_{x \rightarrow 1} \frac{(x-1)(x-2)}{-2x(x-1)} = \lim_{x \rightarrow 1} \frac{x-2}{-2x} = \frac{1-2}{-2 \cdot 1} = \frac{1}{2}.$$

f)

$$\lim_{x \rightarrow -3} \frac{x^2 + 6x + 9}{x^2 - 9} = \lim_{x \rightarrow -3} \frac{(x+3)^2}{(x+3)(x-3)} = \lim_{x \rightarrow -3} \frac{x+3}{x-3} = \frac{-3+3}{-3-3} = 0.$$

3. Határozzuk meg az alábbi határértékeket, amennyiben léteznek!

a) $\lim_{x \rightarrow 5} \frac{70}{2x - 10},$

c) $\lim_{x \rightarrow 1} \frac{x^2 - 6}{x^2 - 3x + 2},$

b) $\lim_{x \rightarrow 5} \frac{1}{(2x - 10)^2},$

d) $\lim_{x \rightarrow -2} \frac{x^2 + 3x - 10}{(x^2 - 4)^2}.$

Megoldás: A jobb- és bal oldali határértékek egyezését vizsgáljuk.

a)

$$\lim_{x \rightarrow 5^-} \frac{70}{2x - 10} = \frac{70}{2 \cdot 5^- - 10} = \frac{70}{0^-} = -\infty, \quad \lim_{x \rightarrow 5^+} \frac{70}{2x - 10} = \frac{70}{2 \cdot 5^+ - 10} = \frac{70}{0^+} = +\infty.$$

A határérték nem létezik.

b)

$$\lim_{x \rightarrow 5^-} \frac{1}{(2x - 10)^2} = \frac{1}{(2 \cdot 5^- - 10)^2} = \frac{1}{(0^-)^2} = +\infty,$$

$$\lim_{x \rightarrow 5^+} \frac{1}{(2x - 10)^2} = \frac{1}{(2 \cdot 5^+ - 10)^2} = \frac{1}{(0^+)^2} = +\infty.$$

$\lim_{x \rightarrow 5} \frac{1}{(2x - 10)^2} = +\infty.$

c)

$$x^2 - 3x + 2 = (x-1)(x-2).$$

$$\lim_{x \rightarrow 1^-} \frac{x^2 - 6}{(x-1)(x-2)} = \frac{-5}{0^- \cdot (-1)} = -\infty, \quad \lim_{x \rightarrow 1^+} \frac{x^2 - 6}{(x-1)(x-2)} = \frac{-5}{0^+ \cdot (-1)} = +\infty.$$

A határérték nem létezik.

d)

$$x^2 + 3x - 10 = (x + 5)(x - 2), \quad (x^2 - 4)^2 = (x - 2)^2(x + 2)^2.$$

$$\frac{x^2 + 3x - 10}{(x^2 - 4)^2} = \frac{(x + 5)(x - 2)}{(x - 2)^2(x + 2)^2} = \frac{x + 5}{(x - 2)(x + 2)^2}.$$

$$\lim_{x \rightarrow -2^-} \frac{x + 5}{(x - 2)(x + 2)^2} = \frac{3}{(-4) \cdot 0^+} = -\infty, \quad \lim_{x \rightarrow -2^+} \frac{x + 5}{(x - 2)(x + 2)^2} = \frac{3}{(-4) \cdot 0^+} = -\infty.$$

$$\lim_{x \rightarrow -2} \frac{x^2 + 3x - 10}{(x^2 - 4)^2} = -\infty.$$

4. Határozzuk meg a következő határértékeket! (*Kónya 3.16 3.17 3.18*) emlékeztessétek őket a híres $\frac{\sin x}{x}$ határértékre!

a) $\lim_{x \rightarrow 0} \frac{2x}{\sin(x)},$

c) $\lim_{x \rightarrow 0} \frac{\sin(3x)}{\sin(6x)},$

e) $\lim_{x \rightarrow 0} \frac{1 - \cos(x)}{x^2},$

b) $\lim_{x \rightarrow 0} \frac{\sin(5x)}{\sin(x)},$

d) $\lim_{x \rightarrow 0} \frac{\sin(5x^2)}{\tan(3x^2)},$

f) $\lim_{x \rightarrow 0} \frac{1 - \cos(9x^2)}{x^2}.$

Megoldás:

a)

$$\lim_{x \rightarrow 0} \frac{2x}{\sin x} = \lim_{x \rightarrow 0} \frac{2}{\frac{\sin x}{x}} = \frac{2}{1} = 2.$$

b)

$$\lim_{x \rightarrow 0} \frac{\sin(5x)}{\sin x} = \lim_{x \rightarrow 0} \frac{\frac{\sin(5x)}{5x}}{\frac{\sin x}{x}} \cdot 5 = \frac{1}{1} \cdot 5 = 5.$$

c)

$$\lim_{x \rightarrow 0} \frac{\sin(3x)}{\sin(6x)} = \lim_{x \rightarrow 0} \frac{\frac{\sin(3x)}{3x}}{\frac{\sin(6x)}{6x}} \cdot \frac{3}{6} = \frac{1}{1} \cdot \frac{3}{6} = \frac{1}{2}.$$

d)

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin(5x^2)}{\tan(3x^2)} &= \lim_{x \rightarrow 0} \frac{\sin(5x^2)}{\frac{\sin(3x^2)}{\cos(3x^2)}} = \lim_{x \rightarrow 0} \frac{\sin(5x^2) \cos(3x^2)}{\sin(3x^2)} \\ &= \lim_{x \rightarrow 0} \frac{\frac{\sin(5x^2)}{5x^2}}{\frac{\sin(3x^2)}{3x^2}} \cdot \frac{5}{3} \cos(3x^2) = \frac{1}{1} \cdot \frac{5}{3} \cdot 1 = \frac{5}{3}. \end{aligned}$$

e)

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} = \lim_{x \rightarrow 0} \frac{2 \sin^2(x/2)}{x^2} = \frac{1}{2} \lim_{x \rightarrow 0} \left(\frac{\sin(x/2)}{x/2} \right)^2 = \frac{1}{2}.$$

f)

$$\lim_{x \rightarrow 0} \frac{1 - \cos(9x^2)}{x^2} = \lim_{x \rightarrow 0} \frac{2 \sin^2(9x^2/2)}{x^2} = \frac{81}{2} \lim_{x \rightarrow 0} \left(\frac{\sin(9x^2/2)}{9x^2/2} \right)^2 x^2 = 0.$$

5.* Számítsuk ki a következő határértékeket!

a)

$$\lim_{x \rightarrow 2} \frac{\sin(x^2 - 4)}{x^2 - 3x + 2} = 4,$$

b)

$$\lim_{x \rightarrow 0} \frac{\tan^2(3x) - \sin^2(3x)}{x^3 \sin(2x)} = \frac{81}{2}.$$