1. Math mode

- $\sqrt{a^2+b^2}$
- $f(x)=\sin^2 x$
- if $a \leq 0 \leq b$, then $0 \geq ab$
- $\lim_{x\to c} \frac{f(x)-f(c)}{x-c} \geq 0$
- $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$
- $\mathbf{e}_a = \mathbf{a}_0 = \frac{1}{|\mathbf{a}|}\mathbf{a}$
- $|\mathbf{a}| = \sqrt{\mathbf{a}^2}$
- $\overrightarrow{AB} \times \overrightarrow{BC} \perp \overrightarrow{AB} \in \mathbb{R}^2$
- $x \equiv y \pmod{a} \iff a \mid y-x$
- $\alpha \in \Delta \cap \Sigma \implies \Delta \cap \Sigma \neq \emptyset$
- $\sin^{(2)} x = -\sin x$
- $\sin'^2 x = \cos^2 x$
- $\sum_{n=1}^{\infty} \delta^n = \frac{\pi^2}{42}$
- $\sum_{i=1}^{n} \prod_{j=1}^{m_i} \delta_j^i$ if $m_1, m_2, \ldots, m_n < \Omega_n$
- $1^2+2^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$
- $\int \sin x \cos x \, dx = \frac{1}{2}\sin^2 x + c$
- $\int_0^1 \sin x \cos x \, dx = \frac{1}{2}\sin^2 x\Big|_0^1$

Theorem 1.1. *If a_n is a monotonically increasing sequence bounded from above, then it is convergent, and in fact $\lim_{n\to\infty} a_n = \sup\{a_n : n \in \mathbb{N}\}$.*

Theorem 1.2 (Bolzano–Weierstrass)**.** *Every bounded sequence has a convergent subsequence.*

Boundedness is a necessary condition, because for example for all subsequences a_{n_i} of the sequence $a_n = n$, we have $\lim_{i\to\infty} a_{n_i} = \infty$.

Proof. Let a_n be a bounded sequence. Because of Theorem 1.1, it's enough to show that a_n has a monotonic subsequence. [...] □

2. Multiline formulas

$$\int_0^1 \sin x \cos x \, dx = \frac{1}{2}\sin^2 x\Big|_0^1 \tag{$*$}$$

(use `\tag`!)

For the next formula, put this in the preamble

```
\DeclareMathOperator{\Dom}{Dom} %domain
\DeclareMathOperator{\Ran}{Ran} %range
\DeclareMathOperator{\Gr}{Gr}   %graph of a function
```

and use the new commands `\Gr` and `\Dom`.

$$\begin{aligned} \operatorname{Gr}(f^{-1}) &= \{(y, f^{-1}(y)) : y \in \operatorname{Dom} f^{-1} = f(\operatorname{Dom} f)\} \\ &= \{(f(x), f^{-1}(f(x))) : x \in \operatorname{Dom} f\} = \{(f(x), x) : x \in \operatorname{Dom} f\} \end{aligned}$$

Here the equality symbols are aligned:

$$\begin{aligned}\mathbf{v}_1 + \mathbf{v}_2 &= x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k} + x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k} \\ &= x_1\mathbf{i} + x_2\mathbf{i} + y_1\mathbf{j} + y_2\mathbf{j} + z_1\mathbf{k} + z_2\mathbf{k} \\ &= (x_1 + x_2)\mathbf{i} + (y_1 + y_2)\mathbf{j} + (z_1 + z_2)\mathbf{k} \\ &= (x_1 + x_2, y_1 + y_2, z_1 + z_2)\end{aligned}$$

For the next formula, put this

```
\DeclareMathOperator{\sgn}{sgn} %signum function
```

in the preamble and use the `\sgn` command.

$$\operatorname{sgn} x = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{otherwise.} \end{cases}$$

These formulas are not aligned, just gathered:

$$\begin{gathered}\int \cos^2 t\, dt = \int \cos t \cos t\, dt = \sin t \cos t + \int \sin^2 t\, dt \\ \int \cos^2 t\, dt = \int 1 - \sin^2 t\, dt = \int 1\, dt - \int \sin^2 t\, dt\end{gathered}$$

Unlike these:

$$\begin{aligned}&\int \cos^n x\, dx = \int \underbrace{\cos x}_{f'} \underbrace{\cos^{n-1} x}_{g}\, dx \\ &\qquad = \underbrace{\sin x}_{f} \underbrace{\cos^{n-1} x}_{g} - \int \underbrace{\sin x}_{f} \underbrace{(n-1)\cos^{n-2} x \cdot (-\sin x)}_{g'}\, dx \\ &\qquad = \sin x \cos^{n-1} x + (n-1) \int (1 - \cos^2 x) \cos^{n-2} x\, dx \\ &\qquad = \sin x \cos^{n-1} x + (n-1) \int \cos^{n-2} x\, dx - (n-1) \int \cos^n x\, dx \\ &\qquad \rightsquigarrow n \int \cos^n x\, dx = \sin x \cos^{n-1} x + (n-1) \int \cos^{n-2} x\, dx \\ &\qquad \rightsquigarrow \int \cos^n x\, dx = \frac{1}{n} \sin x \cos^{n-1} x + \frac{n-1}{n} \int \cos^{n-2} x\, dx\end{aligned}$$

3. Matrices

(1)

$$\begin{pmatrix} 2 & 4 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & 8 \end{pmatrix}^{-1} = \frac{1}{7} \begin{pmatrix} 9/2 & 16 & -2 \\ -1/2 & -8 & 1 \\ -1/2 & -1 & 1 \end{pmatrix}$$

(2)

$$\begin{bmatrix} 2 & 4 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & 8 \end{bmatrix}^{-1} = \frac{1}{7} \begin{bmatrix} 9/2 & 16 & -2 \\ -1/2 & -8 & 1 \\ -1/2 & -1 & 1 \end{bmatrix}$$

(3)

$$\begin{pmatrix} 1 & 1 & 4 \\ 3 & 3 & 6 \end{pmatrix} \quad \overset{L_2 - 3L_1}{\sim} \quad \begin{pmatrix} 1 & 1 & 4 \\ 0 & 0 & -6 \end{pmatrix} \quad \text{this was easy}$$

(4)

$$\begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & \dots & a_{2n} \\ \vdots & & \vdots \\ a_{n2} & \dots & a_{nn} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n3} & \dots & a_{nn} \end{vmatrix}$$

$$+ \dots + (-1)^{n+1} a_{1n} \begin{vmatrix} a_{11} & \dots & a_{1,n-1} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{n,n-1} \end{vmatrix}$$

4. Fields

Definition 4.1. $\langle F, +, \cdot, 0, 1 \rangle$ is a field if

(1) $+$ (addition) and $\cdot$ (multiplication) are commutative and associative
(2) 0 is a neutral element for addition and 1 is a neutral element for multiplication: $x + 0 = x$ and $1x = x$
(3) Every element of F has an additive, and every non-0 element of F has a multiplicative inverse, that is,

$$(\forall x)(\exists y) x + y = 0$$

(notation for y: $-x$), and

$$(\forall x \neq 0)(\exists y) xy = 1$$

(notation for y: $1/x$).
(4) multiplication distributes over addition: $x(y + z) = xy + xz$

Abbreviations: $a - b = a + (-b)$, $a/b = a \cdot 1/b$, $n = \underbrace{1 + 1 + \dots + 1}_{n \text{ times}}$.

Proposition 4.2. (1) 0 *is the only neutral element for addition.*
(2) *The additive inverse is unique. (So* $-$ *in* (3) *of the definition is a function.)*
(3) 1 *is the only neutral element for multiplication.*
(4) *The multiplicative inverse is unique. (So the reciprocal* $(1/)$ *in* (3) *of the definition is a function.)*
(5) $x + y = x + z \implies y = z$
(6) $0x = 0$ *(in particular, if* $\exists x \neq 0$*, then* $0 \neq 1$*)*
(7) $(-1)x = -x$

Proof. 1. If $0'$ is also neutral for addition, then we have $0 = 0 + 0' = 0'$.
2. if y and z are both additive inverses of x, then

$$\begin{aligned} y &= y + 0 && \text{0 is neutral} \\ &= y + (x + z) && z \text{ is an additive inverse of } x \\ &= (y + x) + z && + \text{ is associative} \\ &= 0 + z && y \text{ is an additive inverse of } x \\ &= z && \text{0 is neutral} \end{aligned}$$

3,4. Similar to the previous two.

5.

$$\begin{aligned} y &= 0 + y && \text{0 is neutral} \\ &= (-x + x) + y && -x \text{ is the additive inverse of } x \\ &= -x + (x + y) && + \text{ is associative} \\ &= -x + (x + z) && \text{assumption} \\ &= (-x + x) + z && + \text{ is associative} \\ &= 0 + z && -x \text{ is the additive inverse of } x \\ &= z && \text{0 is neutral} \end{aligned}$$

6. $xy = x(y + 0) = xy + x0$ due to distributivity, from which adding $-(xy)$ to both sides (cf. (5)!) gives the result.
7. $x + (-1)x = 1x + (-1)x = (1 + (-1))x = 0x = 0$ and this is enough because of the uniquness of $-$. In the first equality, we used (3), in the second, (4) of the definition, in the third, the fact that -1 is the additive inverse of 1, and in the last, (6).

□