Докл. Акад. Наук СССР
Том 288 (1986), № 6

Soviet Math. Dokl.
Vol. 33 (1986), No. 3

COINCIDENCE OF CRITICAL POINTS IN PERCOLATION PROBLEMS

UDC 519.217

M. V. MEN'SHIKOV

In this note we consider directed infinite connected graphs without loops and with a countable set of vertices $V = \{v\}$ and arcs $\mathcal{E} = \{\varepsilon\}$. We shall assume that the graph G satisfies Conditions 1 and 2, and in Theorems 2 and 3 also Condition 3.

CONDITION 1. The graph G is vertex-symmetric. This means that it has k types of symmetric vertices; that is, $V = V_1 \cup \cdots \cup V_k$, $V_i \cap V_j = \varnothing$ for $i \neq j$, and for every pair of vertices v, v' there is an automorphism that takes v to v' and preserves the subsets $V_1, \ldots, V_k$.

CONDITION 2. The degree of every vertex of the graph G is finite.

In G we fix a vertex v. We denote by $Y_m(v)$ the set of vertices attainable from v in at most m steps; we also put $S_m(v) = Y_m(v) \setminus Y_{m-1}(v)$, $m = 2, 3, \ldots$, and $S_1(v) = Y_1(v)$.

CONDITION 3. There are numbers $c_1, a_1 > 0$ and $0 < \gamma_1 < 1$ such that $|Y_n(v)| < c_1 \exp\{a_1 n^{\gamma_1}\}$ for all $v \in V$ and $n = 1, 2, \ldots$.

REMARK. The lattices in R^n considered in [1] satisfy all these conditions.

We shall be concerned with the so-called problem of vertices. Every vertex of a graph G independently of the others is occupied with probability p and the value $+1$ is assigned to it, or is free with probability $q = 1 - p$ and the value -1 is assigned to it. Thus, on the set V of vertices there is defined an independent random field. For an occupied vertex $v \in V$ we define the *cluster* $W(v)$ as the set of occupied vertices $v' \in V$ attainable from v through chains of occupied vertices $v_1, \ldots, v_l$, where $\{v, v_1\}, \{v_1, v_2\}, \ldots, \{v_{l-1}, v_l\}, \{v_l, v'\}$ are arcs of G. We denote by $|W(v)|$ the number of vertices in this cluster (it is possible that $|W(v)| = \infty$). We put

$$\theta_v(p) = \mathbf{P}_p\{|W(v)| = \infty\}; \tag{1}$$

that is, $\theta_v(p)$ is the probability that the vertex v belongs to an infinite cluster. Kesten [1], using the FKG-inequality and the fact that G is connected, proved that from the fact that $\theta_{v_1}(p)$ is positive for some vertex v_1 it follows that $\theta_v(p) > 0$ for any vertex v. A similar assertion is also true for $E_p\{|W(v)|\}$. Therefore, the critical percolation points p_H and p_T are well defined as follows:

$$p_H = p_H(G) = \sup\{p: p \in [0,1],\ \theta(p) = 0\}, \tag{2}$$

$$p_T = p_T(G) = \sup\{p: p \in [0,1],\ \mathbf{E}_p\{|W(v)|\} < \infty\}, \tag{3}$$

where $\mathbf{E}_p(\cdot)$ is the expectation.

Let $D_n(v) = D_n$ be the event consisting in the existence of a path of occupied vertices that joins v to one of the vertices belonging to S_n (we shall say that the flow has reached the sphere S_n).

Our main results are the following.

THEOREM 1. *Suppose that the graph G satisfies Conditions 1 and 2. Then for any $p < p_H$ and γ $(0 < \gamma < 1)$ we can find a number N_1 such that for all $n > N_1$ and $v \in V$*

$$\mathbf{P}_p\{D_n(v)\} < \exp\{-n^{\gamma}\}. \tag{4}$$

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 60K35; Secondary 05C20.

THEOREM 2. *Suppose that the graph G satisfies Conditions* 1–3. *Then for any* $p < p_H$ *there are numbers* $c_1, a_2 > 0$ *and* N_2 *such that for all* $n > N_2$ *and* $v \in V$

$$\mathbf{P}_p\{D_n(v)\} < c_2 \exp\{-a_2 n\}. \tag{5}$$

THEOREM 3. *For a graph G satisfying Conditions* 1–3 *the critical percolation points coincide*: $p_H = p_T$.

The proof of the main Theorem 1 will be developed in a number of lemmas. A vertex v_1 of a graph G is said to be *essential* for the event $D_n(v)$ and a given configuration ω if $\omega(v_1) = +1$, the event $D_n(v)$ has happened, but when $\omega(v_1) = -1$, the event $D_n(v)$ does not happen. The set of essential points (vertices) of a given configuration and event $D_n(v)$ will be denoted by $N(D_n(v)) = N(D_n)$.

It is easy to verify that the event $D_n(v)$ is increasing (see [1]); that is, if for some configuration ω the event $D_n(v)$ has happened, then it will also happen if any -1 is replaced by $+1$. If the event $D_n(v)$ has not happened, then it will not happen if $+1$ is replaced by -1. Consequently, Russo's theorem is true for the family of events $D_n(v)$ (see [1] and [2]).

We take arbitrary $\overline{p}_0$ and $\overline{p}_1$, $0 < \overline{p}_1 < \overline{p}_0 \leq 1$, and put

$$p(t) = t\overline{p}_0 + (1-t)\overline{p}_1, \quad 0 \leq t \leq 1, \qquad p(0) = \overline{p}_1, \qquad p(1) = \overline{p}_0.$$

Then the probability $\mathbf{P}_p(D_n(v))$ of the event $D_n(v)$ for fixed n and vertex v is a function of t, $0 \leq t \leq 1$, and this probability satisfies the following inequalities.

THEOREM (RUSSO).

$$\frac{d}{dt}\mathbf{P}_p(D_n(v)) \geq \alpha \mathbf{E}_p\{N(D_n(v))\}, \qquad t \in [0,1]; \tag{6}$$

$$\frac{d}{dt}\mathbf{P}_p(D_n(v)) \geq \alpha \mathbf{E}_p\{N(D_n(v)) | D_n(v)\} \cdot \mathbf{P}_p(D_n(v)); \tag{7}$$

$$\mathbf{P}_{\overline{p}_1}(D_n(v)) \leq \mathbf{P}_{\overline{p}_0}(D_n(v)) \exp\left\{-\alpha \int_0^1 \mathbf{E}_{p(t)}\{N(D_n(v))|D_n(v)\}\, dt\right\}, \tag{8}$$

where $\alpha > 0$ *is a constant depending on the points* $\overline{p}_0$ *and* $\overline{p}_1$.

Inequality (8) is obtained from (7) by integration. The main difficulties in the proof of Theorem 1 consist in estimating $\mathbf{E}_p\{N(D_n(v))|D_n(v)\}$.

LEMMA 1. *For any* $p < p_H$ *and* $v \in V$, $\lim_{n\to\infty} \mathbf{P}_p(D_n(v)) = 0$ *and there is a constant* $\alpha_1 > 0$ *depending on* p *such that* $\mathbf{P}_p(D_n(v_i)) \leq \alpha_1 \mathbf{P}_p(D_n(v_j))$, *where* v_i *and* v_j *are any vertices of* V.

The proof follows from the FKG-inequality, the fact that the number of types of vertices is finite, and the fact that G is connected.

Suppose that for some configuration the event $D_n(v)$ has happened, and that $a_1, \ldots, a_m$ are all the essential points of this configuration. The next lemma gives the geometrical picture of this situation.

LEMMA 2. *The essential points can be indexed so that all paths of occupied vertices from* v *to* $S_n(v)$ *intersect these points in the order of indexing* (*once each*) *and for two points with adjacent numbers there are at least two paths disjoint on the interval between them.*

LEMMA 3. *For any* k, $1 \leq k \leq n$, *and* $0 < p < p_H$

$$\mathbf{P}_p\{N(D_n) \geq k | D_n\} \geq (1 - \alpha_1 \mathbf{P}_p(D_{[n/k]}))^k.$$

PLAN OF THE PROOF. Suppose that $a_1 \in S_{i_1}(v), \ldots, a_m \in S_{i_m}(v)$. We put $\xi_1 = i_1$, $\xi_2 = \max\{i_2 - i_1, 0\}, \ldots, \xi_m = \max\{i_m - i_{m-1} 0\}$, and

$$\text{(9)} \quad \mathbf{P}_p\{N(D_n) \geq k | D_n\} \geq \mathbf{P}_p\{\xi_1 + \xi_2 + \cdots + \xi_k \leq n | D_n\} \geq \mathbf{P}_p\left\{\bigcap_{i=1}^{k} \xi_i \leq n/k | D_n\right\}$$
$$= \mathbf{P}_p\{\xi_1 \leq n/k | D_n\} \cdot \mathbf{P}_p\{\xi_2 \leq n/k | D_n, \xi_1 \leq n/k\} \cdots \mathbf{P}_p\{\xi_k \leq n/k | D_n,\ \xi_1 \leq n/k, \ldots, \xi_{k-1} \leq n/k\}.$$

We can also show that each of the factors on the right-hand side of (9) is not less than $(1 - \alpha_1 \mathbf{P}_p(D_{[n/k]}))$.

The next lemma follows from Lemma 3.

LEMMA 4. *For any* $k \leq n$

$$\text{(10)} \qquad \mathbf{E}_p\{N(D_n) | D_n\} \geq k(1 - \alpha_1 \mathbf{P}_p(D_{[n/k]}))^k.$$

The probability $\mathbf{P}_p(D_n)$ does not increase as n increases. Therefore,

$$\varphi_n(k) = k\mathbf{P}_p(D_{[n/k]})$$

for fixed n is an increasing function of $k = 1, \ldots, n$. Let k_n be determined by the relations

$$\text{(11)} \qquad k_n \mathbf{P}_p(D_{[n/k_n]}) \leq 1, \qquad (k_n + 1)\mathbf{P}_p(D_{[n/k_n+1]}) > 1.$$

LEMMA 5. *There is a number* $d > 0$ *such that for all* n

$$\text{(12)} \qquad \mathbf{E}_p\{N(D_n) | D_n\} \geq d k_n.$$

We put $f_p(n) = f_p(D_n(v)) = 1/\mathbf{P}_p(D_n(v))$.

LEMMA 6. *For any* $c_2 > 0$ *there exist a number* $a > 0$ *and a sequence of numbers* $\{n_i\}$, $n_i \to \infty$, *such that*

$$\text{(13)} \qquad f_{\overline{p}_1}(n_i) \geq a(n_i)^{c_2}.$$

PLAN OF THE PROOF. We rewrite (8) in a new notation:

$$\text{(14)} \qquad f_{\overline{p}_1}(n) \geq f_{\overline{p}_0}(n) \exp\left\{\alpha \int_0^1 \mathbf{E}_{p(t)}\{N(D_n) | D_n\}\right\} dt.$$

For any p' and p'', $\overline{p}_1 \leq \overline{p}'' < p' \leq \overline{p}_0$, we have

$$\text{(15)} \qquad f_{p''}(n) \geq f_{p'}(n) \exp\left\{\alpha \int_{t''}^{t'} \mathbf{E}_{p(t)}\{N(D_n) | D_n\}\right\} dt,$$

where $p(t') = p'$, $p(t'') = p''$, $t' - t'' = (p' - p'')/(\overline{p}_0 - \overline{p}_1)$.

We define recursively three sequences $\{p_i\}$, $\{n_i\}$, and $\{f_i\}$, $i = 0, 1, 2, \ldots$;

$$\text{(16)} \qquad f_i = f_{p_i}(n_i), \quad n_{i+1} = n_i[f_i], \quad p_0 = \overline{p}_0, \quad \Delta_i = p_{i+1} - p_i = c \ln f_i / f_i.$$

Taking account of Lemma 5 and the fact that $[f_i]$ is a root of the equation $x = [f_{p_i}(n_{i+1}/x)]$, we obtain

$$\text{(17)} \qquad \mathbf{E}_{p_i}\{N(D_{n_{i+1}}) | D_{n_{i+1}}\} \geq d k_{n_{i+1}} = d[f_i];$$
$$\text{(18)} \qquad f_{i+1} \geq f_{p_i}(n_{i+1}) \exp\{\alpha d [f_i] \Delta_i\} \geq f_i^{c_1},$$

where by the choice of constant c in (16) we can arrange that c_1 is arbitrarily large, which leads to the convergence of the series $\sum_1^\infty \Delta_i$, and for sufficiently large n_0 and f_0 we have $\sum_1^\infty \Delta_i < \overline{p}_0 - \overline{p}_1$; that is, all $p_i \in (\overline{p}_1, \overline{p}_0]$.

From (16) and (18) it follows that

$$n_m = [f_{m-1}] \cdot [f_{m-2}] \cdots [f_0] n_0, \qquad f_m \geq f_{m-1}^{c_1-1} f_{m-2}^{c_1-1} \cdots f_1^{c_1-1} f_0.$$

Taking account of the fact that $f_{\overline{p}_1}(n) \geq f_{p_m}(n)$, we prove Lemma 6.

Also, from (13), proved for any $c_2 > 0$, we can show that $f_{\overline{p}_1}(n)$ increases faster than a linear-fractional function. Taking account of this and applying (14) to the interval $\Delta t = \overline{p}_1 - \sum_1^\infty \Delta_i$, we prove that $f_{\overline{p}_1}(n)$ increases almost exponentially, from which the assertion of Theorem 1 follows.

We omit the proof of Theorem 2. Let us turn to the proof of Theorem 3. From Condition 3 it follows that we can find a γ_2, $0 < \gamma_2 < 1$, and an N_1 such that for all $n > N_1$ we have $|y_n(v)| < \exp\{n^{\gamma_2}\}$ for any $v \in V$. The following chain of inequalities is obvious:

$$\mathbf{P}_p(D_n(v)) \geq \mathbf{P}_p(|W(v)| > |y_n|) \geq \mathbf{P}_p(|W(v)| > \exp\{n^{\gamma_2}\}). \tag{19}$$

Thus, $\mathbf{P}_p(|W(v)| > l) \leq \mathbf{P}_p(D_{[(\ln l)^{1/\gamma_2}]})$.

By Theorem 1, for any $p < p_H$ and γ, $0 < \gamma < 1$, we can find an N_1 such that for $l > N_1$

$$\mathbf{P}_p(D_{[(\ln l)^{1/\gamma_2}]}) < \exp\{-(\ln l)^{\gamma/\gamma_2}\}.$$

Consequently, for sufficiently large l we have

$$\mathbf{P}_p\{|W(v)| > l\} \leq \exp\{-(\ln l)^{\gamma/\gamma_2}\}, \qquad \gamma/\gamma_2 > 1, \tag{20}$$

and so for any $p < p_H$

$$\mathbf{E}_p\{|W(v)|\} = \sum_{l=1}^{\infty} l\mathbf{P}\{|W(v)| = l\} < \infty. \tag{21}$$

From (21) it follows that $p_T = p_H$.

In [3] we gave an algorithm for obtaining arbitrarily precise estimates for p_T. Hence Theorem 3 makes it possible to obtain arbitrarily precise estimates for p_H. In addition, similar theorems can be formulated for the so-called problem of connections, and also for many-parameter problems of both vertices and connections.

Moscow State University

Received 22/APR/85

BIBLIOGRAPHY

1. Harry Kesten, *Percolation theory for mathematicians*, Birkhäuser, 1982.
2. Lucio Russo, Z. Wahrsch. Verw. Gebiete **56** (1981), 229–237.
3. M. V. Men'shikov, Dokl. Akad. Nauk SSSR **284** (1985), 36–39; English transl. in Soviet. Math. Dokl. **32** (1985).

Translated by E. J. F. PRIMROSE

Докл. Акад. Наук СССР
Том 288 (1986), № 6

Soviet Math. Dokl.
Vol. 33 (1986), No. 3

ON A METHOD OF SOLVING THE KOLMOGOROV-FOKKER-PLANCK EQUATIONS IN THE THEORY OF RANDOM OSCILLATIONS

UDC 517.9

NGUEN DONG AN′ [NGUYEN DONG ANH]

At the present time the study of the joint influence of periodic and random effects on the oscillations of mechanical systems is of major importance (see [1]–[4]). In this area the method of Markov processes in combination with the Krylov-Bogolyubov-Mitropol′skiĭ asymptotic methods [3]–[5] is effective. However, as is known, this approach leads to a difficult problem: solution of the Kolmogorov-Fokker-Planck (KFP) equations.

1. A sufficient condition for integrability of the KFP equations. We consider a mechanical system with one degree of freedom whose equation of motion has the form

$$\ddot{x} + \nu^2 x = \varepsilon f(t,x,\dot{x}) + \sqrt{\varepsilon}\sigma g(t,x,\dot{x})\dot{\xi}(t), \tag{1}$$

where $\dot{\xi}(t)$ is "white noise" with unit intensity, and f and g are differentiable functions of their arguments which are periodic in t; $\varepsilon, \nu, \sigma = \text{const}$, and ε is a small positive parameter. Making the change

$$x = a\cos\psi, \qquad \dot{x} = -a\nu\sin\psi, \quad \psi = \nu t + \theta, \tag{2}$$

by means of Itô's formula [6], we transform equation (1) to the standard form [3]

$$\begin{aligned} da &= \left[-\frac{\varepsilon}{\nu} f(t,x,\dot{x})\sin\psi + \frac{\varepsilon g^2(t,x,\dot{x})}{2\nu^2 a}\cos^2\psi\right] dt \\ &\quad - \left(\sqrt{\varepsilon}/\nu\right) g(t,x,\dot{x})\sin\psi\, d\xi(t), \\ d\theta &= \left[-\frac{\varepsilon}{a\nu} f(t,x,\dot{x})\cos\psi - \frac{\varepsilon g^2(t,x,\dot{x})}{\nu^2 a^2}\sin\psi\cos\psi\right] dt \\ &\quad - \left(\sqrt{\varepsilon}/a\nu\right) g(t,x,\dot{x})\cos\psi\, d\xi(t). \end{aligned} \tag{3}$$

The KFP equation formed for the steady-state probability density of the amplitude and phase $W(a,\theta)$ of a solution of system (3), after averaging [3], [7], has the form

$$\frac{\partial}{\partial a}(K_1 W) + \frac{\partial}{\partial\theta}(K_2 W) = \frac{1}{2}\left[\frac{\partial^2}{\partial a^2}(K_{11}W) + 2\frac{\partial^2}{\partial a\partial\theta}(K_{12}W) + \frac{\partial^2}{\partial\theta^2}(K_{22}W)\right], \tag{4}$$

where the drift and diffusion coefficients are computed from the formula

$$\begin{aligned} K_1(a,\theta) &= M\left\{-\frac{1}{\nu} f(t,x,\dot{x})\sin\psi + \frac{g^2(t,x,\dot{x})}{2\nu^2 a}\cos^2\psi\right\}, \\ K_2(a,\theta) &= M\left\{-\frac{1}{a\nu} f(t,x,\dot{x})\cos\psi - \frac{g^2(t,x,\dot{x})}{\nu^2 a^2}\sin\psi\cos\psi\right\}, \\ K_{11}(a,\theta) &= M\left\{(1/\nu^2) g^2(t,x,\dot{x})\sin^2\psi\right\}, \\ K_{12}(a,\theta) &= M\left\{(1/a\nu^2) g^2(t,x,\dot{x})\sin\psi\cos\psi\right\}, \\ K_{22}(a,\theta) &= M\left\{(1/a^2\nu^2) g^2(t,x,\dot{x})\cos^2\psi\right\}, \end{aligned} \tag{5}$$

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 70L05.