On exponential separation of analytic self-conformal sets on the real line

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Abstract

In a recent article, Rapaport showed that the there is no dimension drop for exponentially separated analytic IFSs on the real line. We show that the set of such exponentially separated IFSs in the space of analytic IFSs contains an open and dense set in the $\mathcal{C}^2$ topology. Moreover, we give a sufficient condition for the IFS to be exponentially separated which allows us to construct explicit examples which are exponentially separated. The key technical tool is the introduction of the *dual IFS* which we believe has significant interest in its own right. As an application we also characterise when an analytic IFS can be conjugated to a self-similar IFS.

1 Introduction and main results

The geometric properties of attractors of iterated function systems have been extensively studied in recent decades. An iterated function system (IFS) Φ is a finite collection of strictly contracting self-maps $(f_i)_{i\in\mathcal{I}}$ on a complete separable metric space X. By a result of Hutchinson [19], there exists a unique non-empty compact set Λ satisfying the invariance

$$\Lambda = \bigcup_{i\in\mathcal{I}} f_i(\Lambda). \tag{1.1}$$

Similarly, one can consider measures invariant under this relation in the following sense. Given an IFS and a non-degenerate probability vector $\mathbf{p} = (p_i)_{i\in\mathcal{I}}$, *i.e.* $\sum_{i\in\mathcal{I}} p_i = 1$ and all $p_i > 0$, there exists a unique Borel probability measure $\mu_{\mathbf{p}}$ supported on Λ such that

$$\mu_{\mathbf{p}} = \sum_{i\in\mathcal{I}} p_i \cdot \mu_{\mathbf{p}} \circ f_i^{-1}. \tag{1.2}$$

The size of such sets and measures, as measured through dimension, is one of the main focus points of fractal geometry. Answering questions in such generality is generally unfeasible and one often restricts to the simpler setting of $X = \mathbb{R}^d$ and where the f_i are simpler mappings such as similarities, affinities, or conformal maps.

Hutchinson [19] considered the Hausdorff dimension of the attractor of IFSs consisting of similarities on $\mathbb{R}^d$ under a separation condition, the open set condition (OSC), and provided a formula for the dimension of the sets depending solely on the contraction ratios of the similarities. In particular, the OSC holds under the stronger assumption of the strong separation condition (SSC). We say that the IFS $\Phi = (f_i)_{i\in\mathcal{I}}$ satisfies the SSC if

$$f_i(\Lambda) \cap f_j(\Lambda) = \varnothing \quad \text{whenever} \quad i \neq j.$$

Hutchinson also showed that this value provides an upper bound for the dimension regardless of the overlaps of images of Λ under the f_i. Bowen [10] and Ruelle [27] extended this result to attractors of $\mathcal{C}^{1+\alpha}$ conformal mappings, with the natural upper bound known as the conformality dimension. Similar question can be asked for the measure, see Cawley and Mauldin [11], and Patschke [22], for the self-similar and self-conformal setting.

The actual value of the Hausdorff dimension of the attractor may drop below this natural upper bound. This occurs, for instance, when the maps have exact overlaps and it is an open problem (the dimension drop conjecture) whether this is the only mechanism for a dimension drop to occur, see Simon [31]. It was since verified that the conformality dimension coincides with the Hausdorff dimension, at least typically, for several notions of typicality.

Simon, Solomyak, and Urbański [29, 30] considered parametrised families of $C^{1+\alpha}$ IFSs and showed that under some technical assumptions (transversality condition) there is no dimension drop for the set and measure for almost every parameter with respect to the Lebesgue measure. Relying on transversality methods, Simon and Solomyak [32] showed that for self-similar sets the set of exceptions where a dimension drop occurs is a meagre set in the Baire category sense.

Other approaches, involving different notions of dimension or separation conditions were also considered, see Zerner [36]; Lau and Ngai [20]; Ngai and Wang [21]; Fraser, Henderson, Olson, and Robinson [15]; and Angelevska, Käenmäki, and Troscheit [4]; and references therein.

A major breakthrough was made by Hochman [16] who showed that no dimension drop occurs for self-similar sets and measures on the line under the assumption of the exponential separation condition (ESC). The ESC is a condition that is satisfied under many natural assumptions and also holds for many typical systems including those described above. For instance, it was shown in [16] that if the parameters defining the self-similar IFS Φ are algebraic and Φ has no exact overlaps then the ESC holds. The algebraic condition has since been relaxed in certain settings, see [14, 24, 25, 35]. Generalisations have also been made to higher dimensions by Hochman [17] and for special non-linear maps, Möbius transformations, by Hochman and Solomyak [18].

Recently, Rapaport [23] extended the concept of the exponential separation condition to general analytic self-conformal iterated function systems. Furthermore, under this condition, Rapaport showed that the dimension of the set and the measure does not drop. For analytically parametrised systems of analytic self-conformal IFSs, Rapaport verified that the ESC holds for almost every choice of parameter in the sense of Hausdorff dimension. However, he could not provide any concrete examples other than those already known.

The main purpose of this article is to provide verifiable sufficient conditions that guarantee that the ESC holds. Further, we show that this property holds for an open and dense set of IFSs with respect to the $\mathcal{C}^2$ topology. We will also explore when analytic self-conformal IFS can be conjugated to self-similar systems. The methods involve constructing a *dual* IFS, which we believe is of independent interest.

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1.1 Notations and main results

For convenience we write $I = [0,1]$. Throughout, we fix $\epsilon > 0$ and we define the class $\mathcal{S}^{\omega}_{\epsilon}(I)$ consisting of maps $f : \mathbb{R} \to \mathbb{R}$ with the following properties:

(A) f is complex analytic in the open ϵ complex neighbourhood $\mathcal{B}_{2\epsilon}$ of I,

(B) $f(I) \subseteq I$ and $f(\overline{\mathcal{B}_{\epsilon}}) \subseteq \mathcal{B}_{\epsilon}$,

(C) $0 < |f'(x)| < 1$ for all $x \in \overline{\mathcal{B}_{\epsilon}}$.

We write

$$d_2(f,g) = \sup_{x\in I} |f(x) - g(x)| + \sup_{x\in I} |f'(x) - g'(x)| + \sup_{x\in I} |f''(x) - g''(x)|$$

for the $\mathcal{C}^2$ metric and equip the space $\mathcal{S}^{\omega}_{\epsilon}(I)$ with the $\mathcal{C}^2$ topology induced by d_2. We note that this space is Polish, *i.e.* a complete and separable space. We also consider the space $\mathfrak{S}_N$ of cardinality N IFSs $\Phi = (f_i)_{i=1}^N$ of maps $f_i \in \mathcal{S}^{\omega}_{\epsilon}([0,1])$. With slight abuse of notation, let

$$d_2((f_i)_{i=1}^N, (g_i)_{i=1}^N) \coloneqq \max_{i\in\mathcal{I}} d_2(f_i, g_i)$$

be the $\mathcal{C}^2$ metric on the space of IFSs $\mathfrak{S}_N$. Note that we consider an IFS to be an ordered tuple of maps, hence the single maximising index.

From the finite alphabet $\mathcal{I} \coloneqq \{1, \dots, N\}$ we construct infinite words $\mathbf{i} = (i_1, i_2, i_3, \dots) \in \Sigma \coloneqq \mathcal{I}^{\mathbb{N}}$ and finite words $\mathbf{i} = (i_1, \dots, i_k) \in \Sigma_k \coloneqq \mathcal{I}^k$ of length $k \in \mathbb{N}$, where $\Sigma_0 = \{\emptyset\}$ is just the empty word. The length of $\mathbf{i} \in \Sigma_k$ is $|\mathbf{i}| = k$ and for $\mathbf{i} \in \Sigma$ it is $|\mathbf{i}| = \infty$. We denote the set of all finite words by $\Sigma_* = \bigcup_{k=0}^{\infty} \Sigma_k$. For $\mathbf{i} \neq \mathbf{j} \in \Sigma \cup \Sigma_*$ we write $\mathbf{i} \wedge \mathbf{j}$ to denote the longest $\mathbf{k} \in \Sigma_*$ such that $\mathbf{k} = (i_1, \dots, i_{|\mathbf{k}|}) = (j_1, \dots, j_{|\mathbf{k}|})$. For any finite word $(i_1, \dots, i_n) \in \Sigma_*$, we write

$$f_{i_1,\dots,i_n} \coloneqq f_{i_1} \circ \dots \circ f_{i_n},$$

where by convention $f_{\emptyset} = \mathrm{Id}$. The natural projection $\pi \colon \Sigma \to \mathbb{R}$ defined by

$$\pi(\mathbf{i}) \coloneqq \lim_{n\to\infty} f_{i_1,\dots,i_n}(0) \tag{1.3}$$

satisfies $\pi(\Sigma) = \Lambda$, where Λ is the attractor satisfying the invariance in Eq. (1.1). Following Rapaport [23], we formally define the exponential separation condition for analytic self-conformal IFSs.

Definition 1.1. We say that the IFS $\Phi = (f_i)_{i\in\mathcal{I}}$ satisfies the *exponential separation condition (ESC)* if there exists $c > 0$ such that for infinitely many $n \in \mathbb{N}$,

$$\sup_{x\in[0,1]} |f_{\mathbf{i}}(x) - f_{\mathbf{j}}(x)| \geq c^n$$

for all distinct $\mathbf{i}, \mathbf{j} \in \Sigma_n$. If the ESC does not hold, then we say that Φ has *super-exponential condensation.*

Definition 1.2. We say that the IFS $\Phi = (f_i)_{i\in\mathcal{I}}$ satisfies the *strong exponential separation condition (SESC)* if there exists $c > 0$ such that

$$\sup_{x\in[0,1]} |f_{\mathbf{i}}(x) - f_{\mathbf{j}}(x)| \geq c^n$$

for all $\mathbf{i}, \mathbf{j} \in \Sigma_n$ and $n \in \mathbb{N}$, where $\mathbf{i} \neq \mathbf{j}$. If the SESC does not hold, *i.e.* there exist a sequence $(\eta_n)_n$ and a subsequence $n_\ell \in \mathbb{N}$ and distinct words $\mathbf{i}, \mathbf{j} \in \Sigma_{n_\ell}$ such that $\log(\eta_n)/n \to -\infty$ and

$$\sup_{x \in [0,1]} |f_{\mathbf{i}}(x) - f_{\mathbf{j}}(x)| \leq \eta_{n_\ell},$$

then we say that Φ has *weak super-exponential condensation*

Definition 1.3. We say that the IFS $\Phi = (f_i)_{i \in \mathcal{I}}$ has *exact overlaps* if there exist distinct $\mathbf{i}, \mathbf{j} \in \Sigma_*$ such that $f_{\mathbf{i}}(x) = f_{\mathbf{j}}(x)$ for every $x \in \Lambda$.

We remark that the separation conditions satisfy the implications

$$\text{SSC} \Rightarrow \text{SESC} \Rightarrow \text{ESC} \Rightarrow \text{no exact overlaps.}$$

The main objective of this article is to give generic and explicit conditions under which the ESC holds for analytic IFSs. Roughly speaking, our first main result says that the property that an analytic IFS satisfies the SESC is a generic property in a topological sense.

Theorem 1.4. *The set of IFSs $\{\Phi \colon \Phi$ satisfies SESC$\} \subseteq \mathfrak{S}_N$ contains an open and dense subset in the $\mathcal{C}^2$ topology.*

Our second main result provides a sufficient condition under which an analytic IFS satisfies the SESC. We demonstrate in Section 1.2.2 that it can be used to construct completely explicit examples of analytic IFSs that satisfy the SESC. In order to state the result, we introduce more notation. Since we will often write finite words "backwards", we adopt the convention that for $n \neq m \in \mathbb{N}$,

$$\mathbf{i}_m^n := \begin{cases} (i_m, i_{m+1}, \ldots, i_{n-1}, i_n), & \text{if } m < n; \\ (i_m, i_{m-1}, \ldots, i_{n+1}, i_n,) & \text{if } m > n, \end{cases}$$

where $|\mathbf{i}| \geq \max\{m, n\}$. This will most often be used in the form $\mathbf{i}_1^n = (i_1, \ldots, i_n)$ or $\mathbf{i}_n^1 = (i_n, \ldots, i_1)$. Thus for compositions of maps $f_{\mathbf{i}_n^1} = f_{i_n} \circ \cdots \circ f_{i_1}$. Sometimes either m or n is 0. In these cases, the convention is that both $\mathbf{i}_0^n$ and $\mathbf{i}_m^0$ are the empty word $\emptyset$. By default, if we simply write $\mathbf{i} \in \Sigma_* \cup \Sigma$ then the subscripts are understood to be in increasing order starting from 1 until $|\mathbf{i}|$. The concatenation of two finite words is $\mathbf{ij}$, while slightly abusing notation $\mathbf{i}^\infty \in \Sigma$ is the infinite word obtained by concatenating $\mathbf{i} \in \Sigma_*$ infinitely many times. For any $\mathbf{i} \in \Sigma \cup \Sigma_*$, we introduce the function

$$H_{\mathbf{i}}(x) = H_{\mathbf{i}_1^{|\mathbf{i}|}}(x) := \sum_{n=1}^{|\mathbf{i}|} \frac{f''_{i_n}}{f'_{i_n}}(f_{\mathbf{i}_{n-1}^1}(x)) \cdot f'_{\mathbf{i}_{n-1}^1}(x), \tag{1.4}$$

where the order of the indices is important. At this point the motivation for $H_{\mathbf{i}}(x)$ may be unclear, but will be made apparent in Section 2. We can now state our second main result.

Theorem 1.5. *Let $\Phi \in \mathfrak{S}_N$. If for all distinct $\mathbf{i}, \mathbf{j} \in \Sigma \cup \Sigma_*$ with $|\mathbf{i}| = |\mathbf{j}|$ we have*

$$\sup_{x \in [0,1]} |H_{\mathbf{i}}(x) - H_{\mathbf{j}}(x)| > 0, \tag{1.5}$$

then Φ satisfies the SESC.

Apart from its practical use, Theorem 1.5 is also a crucial step in proving Theorem 1.4 and shows that the $\mathcal{C}^2$ topology is the natural choice for typicality. This is because Eq. (1.5) cannot be satisfied for linear maps, having $f_i'' = 0$, and the $\mathcal{C}^2$ is the finest topology for which we can distinguish strictly conformal maps sufficiently, see also Section 1.2.3.

We shall see in Theorem 2.2 that Eq. (1.5) has an elegant interpretation as an analog of the SSC on a larger space of IFSs which we further elaborate on in Section 2. Theorem 1.5 is proved in Section 3, while the proof of Theorem 1.4 is postponed until Section 4.

1.2 Discussion

We give further context to our main results. We first discuss the dimension theoretic implications of our main results and give explicit examples of IFSs that satisfy the SESC. We end the section by discussing a link to conjugation with self-similar IFSs.

1.2.1 Dimension theoretic consequences

Given a non-degenerate probability vector $\mathbf{p}$ and a self-conformal IFS $\Phi \in \mathfrak{S}_N$, we define the entropy of $\mathbf{p}$ by

$$H(\mathbf{p}) := -\sum_{i\in\mathcal{I}} p_i \log p_i;$$

the Lyapunov exponent associated to $\mathbf{p}$ and Φ by

$$\chi = \chi(\Phi, \mathbf{p}) := -\sum_{i\in\mathcal{I}} p_i \int \log \left|f_i'(x)\right| \mathrm{d}\mu_{\mathbf{p}}(x),$$

where $\mu_{\mathbf{p}}$ is the self-conformal measure defined in Eq. (1.2). For $t \geq 0$, we define the pressure function

$$P(t) = P_\Phi(t) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{\mathbf{i}\in\mathcal{I}^n} \Big(\sup_{x\in[0,1]} \left|(f_{\mathbf{i}_1^n})'(x)\right| \Big)^t,$$

which is well defined by sub-additivity. It is convex, strictly decreasing and continuous, moreover, there exists a unique real $s(\Phi)$ for which $P(s(\Phi)) = 0$. Following [9, Chapter 14], we call $s(\Phi)$ the *conformality dimension* associated to Φ.

For any self-conformal set and self-conformal measure supported on it, the bounds

$$\dim_{\mathrm{H}} \Lambda \leq \min\{1, s(\Phi)\} \quad \text{and} \quad \dim \mu_{\mathbf{p}} \leq \min\{1, H(\mathbf{p})/\chi\} \text{ for every } \mathbf{p}, \tag{1.6}$$

hold, regardless of possible overlaps between the pieces $f_i(\Lambda)$. Here, we denote by $\dim_{\mathrm{H}}$ the Hausdorff dimension of a set and measure, see [13] for definitions and basic properties.

Rapaport's main result of [23] can be stated in the following way.

Theorem 1.6 ([23]). *Let $\Phi \in \mathfrak{S}_N$ be such that its attractor is not a singleton. If Φ satisfies the ESC, then there is equality in Eq. (1.6).*

We note that the assumption that the attractor is not a singleton is equivalent to the existence of $f, g \in \Phi$ with distinct fixed points. Combining our Theorem 1.4 with Theorem 1.6 immediately gives the following corollary.

Corollary 1.7. *The set* $\{\Phi\colon \Phi$ *satisfies equality in Eq.* (1.6)$\} \subseteq \mathfrak{S}_N$ *contains an open and dense subset in the* $\mathcal{C}^2$ *topology.*

This can be interpreted as follows; a typical analytic self-conformal IFS, in a strong topological sense, has no dimension drop.

Another potential direction to consider is that of L^q dimensions, a more fine-grained notion that captures the measure's regularity. Building on the methods of [16], Shmerkin [28] showed that a natural analogue of Theorem 1.6 exists for the L^q dimension of self-similar measures. Our typicality result does not extend to the L^q dimension of self-conformal measures in general. Below, we give an explicit example of an IFS that satisfies the SESC, but the natural measure exhibits a dimension drop of the L^q dimension for large q.

1.2.2 Concrete examples

Similar to the self-similar setting, it would be desirable to have conditions under which one could determine whether a concrete IFS or a parametrised family of IFSs satisfies the (S)ESC or not also in the analytic setting. Rapaport [23, Corollary 1.4], based on a result of Solomyak and Takahashi [34], showed that under a mild non-degeneracy condition, given a one-parameter family of analytic IFSs, the set of parameters for which the ESC fails has zero Hausdorff dimension. Concrete families of IFSs can be constructed to which this result applies, however, it still does not explicitly say whether an IFS is in the exceptional set of parameters or not. In the following, we give an easy to verify sufficient condition and demonstrate on an explicit analytic IFS that it satisfies the SESC.

Given an IFS $\Phi = (f_i)_{i\in\mathcal{I}} \in \mathfrak{S}_N$ let

$$c_{\min} := \inf_{\substack{x\in[0,1]\\ i\in\mathcal{I}}} |f_i'(x)| \quad\text{and}\quad c_{\max} := \sup_{\substack{x\in[0,1]\\ i\in\mathcal{I}}} |f_i'(x)|. \tag{1.7}$$

We note that by compactness arguments, $0 < c_{\min} \le c_{\max} < 1$.

Proposition 1.8. *Let* $\Phi = (f_i)_{i\in\mathcal{I}} \in \mathfrak{S}_N$ *be an IFS. Suppose that there exists* $\alpha > 0$ *such that for each* $i \neq j \in \mathcal{I}$ *there exists* $x_{i,j} \in [0,1]$ *with*

$$\left| \frac{f_i''(x_{i,j})}{f_i'(x_{i,j})} - \frac{f_j''(x_{i,j})}{f_j'(x_{i,j})} \right| \ge \alpha.$$

Let

$$\beta := \sup_{\substack{x\in[0,1]\\ i\in\mathcal{I}}} \left| \frac{f_i''(x)}{f_i'(x)} \right|.$$

Then $\beta > 0$ *and if* $\alpha > 2\beta \cdot c_{\max}/(1 - c_{\max})$*, we have* $\sup_{x\in[0,1]} |H_{\mathbf{i}}(x) - H_{\mathbf{j}}(x)| > 0$ *for all distinct* $\mathbf{i}, \mathbf{j} \in \Sigma \cup \Sigma_*$ *with* $|\mathbf{i}| = |\mathbf{j}|$*. In particular,* Φ *satisfies the SESC by Theorem 1.5.*

Proof. First, let us show that for every $\mathbf{i}, \mathbf{j} \in \Sigma \cup \Sigma_*$ with $i_1 \neq j_1$ we have for every non-degenerate closed interval $J \subseteq [0,1]$

$$\sup_{x\in J} |H_{\mathbf{i}}(x) - H_{\mathbf{j}}(x)| > 0. \tag{1.8}$$

Since $i_1 \neq j_1$, for the particular choice of x_{i_1,j_1}, we can bound

$$|H_{\mathbf{i}}(x_{i_1,j_1}) - H_{\mathbf{j}}(x_{i_1,j_1})| \ge \left| \frac{f_{i_1}''(x_{i_1,j_1})}{f_{i_1}'(x_{i_1,j_1})} - \frac{f_{j_1}''(x_{i_1,j_1})}{f_{j_1}'(x_{i_1,j_1})} \right| - 2\sum_{n=k+2}^{\infty} \sup_{x\in[0,1]} \left| f_{\mathbf{i}_{n-1}^1}'(x) \right| \cdot \sup_{x\in[0,1]} \left| \frac{f_{i_n}''(x)}{f_{i_n}'(x)} \right|$$

$$\geq \alpha - 2\frac{c_{\max}}{1-c_{\max}} \cdot \beta > 0$$

by our assumption. Hence, $H_{\mathbf{i}} \not\equiv H_{\mathbf{j}}$ and using the analyticity of the maps (which will be verified later in Lemma 2.4) Eq. (1.8) follows.

Now, let $\mathbf{i}, \mathbf{j} \in \Sigma \cup \Sigma_*$ be distinct words such that $|\mathbf{i}| = |\mathbf{j}|$ and $|\mathbf{i} \wedge \mathbf{j}| = k$. It follows from Eq. (1.4) that

$$H_{\mathbf{i}}(x) - H_{\mathbf{j}}(x) = f'_{\mathbf{i}_k^1}(x) \cdot \left(H_{\mathbf{i}_{k+1}^{|\mathbf{i}|}}(f_{\mathbf{i}_k^1}(x)) - H_{\mathbf{j}_{k+1}^{|\mathbf{j}|}}(f_{\mathbf{i}_k^1}(x)) \right)$$

for every $x \in [0,1]$. As a result,

$$|H_{\mathbf{i}}(x) - H_{\mathbf{j}}(x)| \geq c_{\min}^k \cdot \left| H_{\mathbf{i}_{k+1}^{|\mathbf{i}|}}(f_{\mathbf{i}_k^1}(x)) - H_{\mathbf{j}_{k+1}^{|\mathbf{j}|}}(f_{\mathbf{i}_k^1}(x)) \right|.$$

Taking supremum over $x \in [0,1]$ in both sides, the claim of the proposition follows by Eq. (1.8). □

We demonstrate Proposition 1.8 on an example. Consider the IFS $\Phi = (f_1, f_2, f_3)$ with

$$f_1(x) \coloneqq \frac{x}{8}, \quad f_2(x) \coloneqq \frac{x}{8} + \frac{x^2}{32}, \text{ and } \quad f_3(x) \coloneqq \frac{x}{16} + \frac{x^2}{32} + \frac{29}{32}.$$

Cylinder sets certainly overlap heavily since f_1 and f_2 have common fixed point at $x = 0$, while f_3 has fixed point at $x = 1$. Simple calculations show that $c_{\max} = 3/16$, moreover,

$$\sup_{\substack{x\in[0,1]\\ i\in\mathcal{I}}} \left| \frac{f_i''(x)}{f_i'(x)} \right| \leq \sup_{x\in[0,1]} \max\left\{ \frac{1}{1+x}, \frac{1}{2+x} \right\} = 1 \quad \text{and} \quad \min_{i\neq j\in\mathcal{I}} \left| \frac{f_i''(0)}{f_i'(0)} - \frac{f_j''(0)}{f_j'(0)} \right| \geq \frac{1}{2}.$$

Since $1/2 - 2 \cdot 1 \cdot 3/13 > 0$, it follows from Proposition 1.8 that Φ satisfies the SESC, and by Corollary 1.7, we obtain $\dim_{\mathrm{H}} \Lambda = s(\Phi)$, where $s(\Phi)$ is the conformality dimension of Φ, and $\dim \mu_{\mathbf{p}} = H(\mathbf{p})/\chi$ for every $\mathbf{p}$, defined in the previous section.

Let us note that Solomyak [33] has already given an example of a non-linear conformal IFS of linear fractional transformations with such a common fixed point structure. However, he studied the dimension of the attractor via sufficiently large subsystems, and it was not verified that the IFS itself satisfies the ESC.

We conclude this section by remarking that the L^q dimension of the natural measure in the example above drops for large q. In particular, one can show that the local dimension at 0 of the natural measure is strictly smaller than the conformality dimension $s(\Phi)$. However, if the L^q dimension did not drop for every $q > 0$ then by [28, Lemma 1.7] the local dimension at every point would be at least $s(\Phi)$, which is a contradiction. We leave the details for the interested reader.

1.2.3 Conjugation to self-similar systems

As we remarked above, assumption (1.5) of Theorem 1.5 cannot be satisfied for linear systems since $H_{\mathbf{i}}(x) = 0$ for all $\mathbf{i} \in \Sigma$ and $x \in I$. Crucially, the functions $H_{\mathbf{i}}(x)$ have a stronger relation to the linearity and linearisability of an analytic IFS, although the definition may not at first glance reveal this. The functions $H_{\mathbf{i}}$ can be used to determine whether an analytic IFS can be transformed into a self-similar one through a change of coordinates. The problem of being conjugated to a linear IFS played a significant role in the study of Fourier decay of self-conformal measures, see [1, 7] and [3, Corollary 1.2 part 3.].

Definition 1.9. We say that the IFS $\Phi = (f_i)_{i\in\mathcal{I}} \in \mathfrak{S}_N$ is *conjugated* to another IFS Ψ if there exists an analytic, invertible $g : [0,1] \to \mathbb{R}$ such that $\Psi = (g \circ f_i \circ g^{-1})_{i\in\mathcal{I}}$.

In particular, Φ is conjugated to a self-similar IFS if there exist $\lambda_i \in (-1,1) \setminus \{0\}$, $t_i \in \mathbb{R}$ with $i \in \mathcal{I}$ such that

$$f_j(x) = g^{-1}(\lambda_j g(x) + t_j)$$

for all $j \in \mathcal{I}$.

Definition 1.10. We say that the IFS $\Phi = (f_i)_{i\in\mathcal{I}} \in \mathfrak{S}_N$ is *sub-conjugated* to a self-similar IFS if there exist distinct words $\mathbf{i}, \mathbf{j} \in \Sigma_*$ of the same length such that $(f_{\mathbf{i}}, f_{\mathbf{j}})$ is conjugated to a self-similar IFS.

Remark 1.11. Note that we assume the conjugating function g to be analytic in Definition 1.9. One could impose the weaker condition that $f \in \mathcal{C}^r([0,1])$ for all $f \in \Phi$ for some $2 \leq r \leq \infty$ instead. Under this assumption, the authors of [2] show the existence of a $\mathcal{C}^r$-smooth IFS which is not $\mathcal{C}^r$-conjugate to self-similar even though $f' \equiv c_\Phi$ on Λ and $f'' \equiv 0$ for every $f \in \Phi$. This behaviour is not possible in the analytic setting since the assumption that $f''(x) = 0$ on Λ together with analyticity already forces f to be an affine function.

We give a characterisation of when an analytic IFS is (sub-)conjugated to a self-similar IFS using the function $H_{\mathbf{i}}$ introduced in Eq. (1.4).

Theorem 1.12. *Any $\Phi \in \mathfrak{S}_N$ is conjugated to another analytic IFS which has at least one similarity map. Moreover, assuming that the attractor of Φ is not a singleton then*

(a) Φ is conjugated to a self-similar IFS if and only if for every $\mathbf{i}, \mathbf{j} \in \Sigma$,

$$H_{\mathbf{i}}(x) \equiv H_{\mathbf{j}}(x)$$

for all $x \in [0,1]$;

(b) Φ is sub-conjugated to a self-similar IFS if and only if there exist distinct $\mathbf{i}, \mathbf{j} \in \Sigma_*$ of the same length such that

$$H_{(\mathbf{i})^\infty}(x) \equiv H_{(\mathbf{j})^\infty}(x)$$

for all $x \in [0,1]$.

Theorem 1.12 is proved in Section 5.

Let us return to the discussion of the dimension drop conjecture. Recall that the ESC implies that the IFS has no exact overlaps. For some time it was an important open problem whether there exists a self-similar IFS which has no exact overlaps but has super exponential condensation. Independently of each other, using different methods, Baker [5] and Bárány–Käenmäki [8] showed that such examples do exist. The idea of Baker was further developed in [6] and [12]. It follows from the work of Rapaport [24] that the examples in [5, 6, 12] further support the dimension drop conjecture. It is natural to ask whether analytic IFSs exist which have super-exponential condensation but no exact overlaps. We conjecture that this only occurs if the IFS is sub-conjugated to a self-similar IFS.

Conjecture 1.13. *Any analytic IFS which has super-exponential condensation but no exact overlaps must be sub-conjugated to a self-similar IFS.*

We remark that any analytic IFS with an exact overlap is sub-conjugated to a self-similar IFS, see Theorem 2.3 and Section 5.

2 The key idea: the dual IFS induced by analytic functions

The central idea of our work is to construct a 'dual' IFS on the space of analytic functions, derived from the mappings of the original IFS on $[0,1]$. We believe this notion is of general interest in its own right and a systematic study of it could assist in tackling other problems in the future as well.

2.1 Basic definitions and properties

Let $\mathcal{C}_\epsilon^\omega([0,1])$ be the set of complex analytic maps f on $\mathcal{B}_\epsilon$ such that $f\colon I \to \mathbb{R}$. We equip $\mathcal{C}_\epsilon^\omega([0,1])$ with the supremum norm $\|\cdot\|_\infty$ over $\mathcal{B}_\epsilon$. Given an analytic IFS $\Phi = (f_i)_{i=1}^N \in \mathfrak{S}_N$, we 'lift' each map f_i to an operator $F_i : \mathcal{C}_\epsilon^\omega([0,1]) \to \mathcal{C}_\epsilon^\omega([0,1])$ acting on the space of analytic functions by the formula

$$(F_i h)(x) \coloneqq f_i'(x) \cdot h(f_i(x)) + \frac{f_i''}{f_i'}(x). \tag{2.1}$$

We call $\Phi^* \coloneqq (F_i)_{i=1}^N$ the *dual IFS* of Φ. The operator F_i can be considered as a contractive affinity map and Φ^* as self-affine IFS on $\mathcal{C}_\epsilon^\omega([0,1])$. Indeed, F_i is a translation of the linear operator $h \mapsto f_i'(x) \cdot h(f_i(x))$, and each F_i is clearly a strict contraction in the supremum norm since $\|F_i g - F_i h\|_\infty \le \|f_i'\|_\infty \cdot \|g - h\|_\infty$, where $\|f_i'\|_\infty < 1$ by our assumption (C).

Our objective now is to establish some basic properties about the dual IFS. We first justify calling Φ^* an IFS by showing that it has an attractor. We use the convention that if $A \subset \mathcal{C}_\epsilon^\omega([0,1])$, then

$$F_i A \coloneqq \{F_i h : h \in A\}.$$

Lemma 2.1. *Let Φ^* be the dual IFS of an analytic IFS $\Phi \in \mathfrak{S}_N$. There exists a unique, non-empty, compact set $\Lambda^* \subset \mathcal{C}_\epsilon^\omega([0,1])$, which we call the* attractor *of Φ^*, that satisfies*

$$\Lambda^* = \bigcup_{i \in \mathcal{I}} F_i \Lambda^*.$$

Proof. For $L > 0$, we define

$$\mathcal{C}_{\epsilon,L}^\omega([0,1]) \coloneqq \big\{ g \in \mathcal{C}_\epsilon^\omega([0,1]) : |g(x)| \le L \text{ for every } x \in \mathcal{B}_\epsilon \big\},$$

and $\mathcal{C}_\epsilon^\omega([0,1]) = \bigcup_{L=1}^\infty \mathcal{C}_{\epsilon,L}^\omega([0,1])$. It is well-known that the space of continuous and bounded maps is complete with respect to the supremum distance. By applying Morera's Theorem [26, Theorem 10.17], it follows that $\mathcal{C}_{\epsilon,L}^\omega([0,1])$ is a complete and separable metric space for every $L > 1$.

Since $\|F_i h\|_\infty \le \|f_i''/f_i'\|_\infty + \|f_i'\|_\infty \|h\|_\infty$, there exists $L > 0$ sufficiently large such that $\|F_i h\|_\infty \le L$ if $\|h\|_\infty \le L$. Hence, the claim of the lemma follows by [19]. □

Let us recall the strong separation condition (SSC) in the context of dual IFS Φ^*. The SSC holds if $F_i \Lambda^* \cap F_j \Lambda^* = \varnothing$ for every $i \neq j$ in $\mathcal{I}$, that is, there is no $h \in \Lambda^*$ such that $h \in F_i \Lambda^*$ and $h \in F_j \Lambda^*$. Using the dual IFS Φ^* and the attractor Λ^* of the dual IFS, Theorem 1.5 can be restated in the following elegant form.

Theorem 2.2. *An IFS $\Phi \in \mathfrak{S}_N$ satisfies the SESC if its dual IFS Φ^* satisfies the SSC.*

Theorem 1.12 also has an equivalent formalisation as follows:

Theorem 2.3. *An IFS $\Phi \in \mathfrak{S}_N$ can be conjugated to a self-similar IFS if and only if the attractor of its dual IFS Φ^* is a singleton.*

Moreover, an IFS $\Phi \in \mathfrak{S}_N$ can be sub-conjugated to a self-similar IFS if and only if its dual IFS Φ^* has an exact overlap.

We postpone the explicit proof of Theorem 2.2 until the end of Section 3 and Theorem 2.3 until Section 5.

For $\mathbf{i} \in \Sigma_*$, an induction argument readily gives that the composition $F_{\mathbf{i}}h = F_{i_1} \circ \ldots \circ F_{i_{|\mathbf{i}|}} h$ is equal to

$$(F_{\mathbf{i}}h)(x) = f'_{\mathbf{i}_{|\mathbf{i}|}^1}(x) \cdot h(f_{\mathbf{i}_{|\mathbf{i}|}^1}(x)) + \sum_{n=1}^{|\mathbf{i}|} f'_{\mathbf{i}_{n-1}^1}(x) \cdot \frac{f''_{i_n}}{f'_{i_n}}(f_{\mathbf{i}_{n-1}^1}(x)).$$

Observe that for every $\mathbf{i} \in \Sigma_*$, taking h equal to the constant 0 function gives $(F_{\mathbf{i}}0)(x) \equiv H_{\mathbf{i}}(x)$, which was introduced in Eq. (1.4). Moreover, for every $\mathbf{i} \in \Sigma$

$$H_{\mathbf{i}}(x) = \lim_{n\to\infty} (F_{\mathbf{i}_1^n} h)(x)$$

in the uniform sense for every $h \in \mathcal{C}_\epsilon^\omega([0,1])$. With this interpretation the function $H_{\mathbf{i}}(x)$ is an analog of the natural projection $\pi(\mathbf{i})$ from Eq. (1.3), and this motivates us to call $H_{\mathbf{i}}(x)$ the *dual natural projection.* The following further justifies this nomenclature.

Lemma 2.4. *For every $\mathbf{i} \in \Sigma$, we have $H_{\mathbf{i}} \in \mathcal{C}_\epsilon^\omega([0,1])$. Moreover, the map $\mathbf{i} \mapsto H_{\mathbf{i}}$ is Hölder continuous, that is, there exists $K > 0$ such that for every distinct $\mathbf{i}, \mathbf{j} \in \Sigma$,*

$$\|H_{\mathbf{i}} - H_{\mathbf{j}}\|_\infty \leq c_{\max}^{|\mathbf{i}\wedge\mathbf{j}|} K.$$

In particular, $\Lambda^* = \{H_{\mathbf{i}}(x) : \mathbf{i} \in \Sigma\}$.

Proof. From the properties in (A) to (C), it follows that the maps

$$\frac{f''_j}{f'_j}, \quad f_{\mathbf{i}}, \quad \text{and} \quad f'_{\mathbf{i}}$$

are analytic on $\mathcal{B}_{2\epsilon}$ for all $\mathbf{i} \in \Sigma_*$ and $j \in \mathcal{I}$. Hence, there exists $C > 0$ such that

$$\left|\frac{f''_j}{f'_j}(z)\right| \leq C \text{ and } 0 < c_{\min} \leq |f'_j(z)| \leq c_{\max} < 1$$

for all $j \in \mathcal{I}$ and $z \in \mathcal{B}_\epsilon$, $H_{\mathbf{i}_1^n}$ converges uniformly to $H_{\mathbf{i}}$ on $\mathcal{B}_\epsilon$ and $H_{\mathbf{i}}$ is analytic on $\mathcal{B}_\epsilon$ by Morera's theorem [26, Theorem 10.17]. In particular, there exists $K > 0$ such that $\|H_{\mathbf{i}}\|_\infty \leq K$ for every $\mathbf{i} \in \Sigma$. Hence,

$$\|H_{\mathbf{i}} - H_{\mathbf{j}}\|_\infty \leq c_{\max}^{|\mathbf{i}\wedge\mathbf{j}|} 2K.$$

Using the Hölder-continuity, $\{H_{\mathbf{i}}(x) : \mathbf{i} \in \Sigma\}$ is compact, invariant with respect to the dual IFS Φ^*, and by the uniqueness of the attractor, Lemma 2.1, the last assertion follows. ☐

We now define cylinder sets for the dual IFS. For two real numbers a, b, their convex hull is the interval $\operatorname{conv}(a,b) = [\min\{a,b\}, \max\{a,b\}]$. Slightly abusing notation, a constant $k \in \mathbb{R}$ also denotes the constant function on $\mathcal{C}_\epsilon^\omega([0,1])$. A *cylinder set* on $\mathcal{C}_\epsilon^\omega([0,1])$ is given by

$$(k, K) := \big\{g \in \mathcal{C}_\epsilon^\omega([0,1]) : k < g(x) < K \text{ for every } x \in [0,1]\big\}.$$

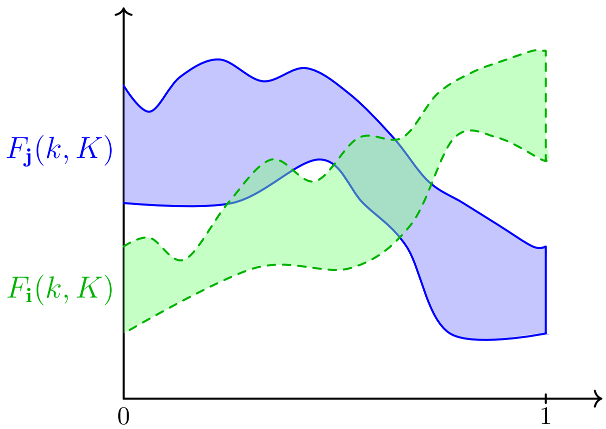

Figure 1: Illustration of disjoint cylinders on the left and ones which are not disjoint on the right.

Since each F_i is a contraction, there exists $k < K$ such that

$$k < \min_{i\in\mathcal{I}}\{\min_{x\in[0,1]}(F_ik)(x), \min_{x\in[0,1]}(F_iK)(x)\} \text{ and } K > \max_{i\in\mathcal{I}}\{\max_{x\in[0,1]}(F_ik)(x), \max_{x\in[0,1]}(F_iK)(x)\},$$

moreover, $F_i(\overline{(k,K)}) \subseteq (k,K)$ for every $i \in \mathcal{I}$. The image of any cylinder (k,K) under $F_{\mathbf{i}}$ for any $\mathbf{i} \in \Sigma_*$ has width

$$\max_{x\in[0,1]} |f'_{\mathbf{i}}(x)| \cdot (K-k) < c_{\max}^{|\mathbf{i}|}(K-k),$$

where $c_{\max}$ is as in Eq. (1.7).

We say that two cylinder sets $F_{\mathbf{i}}(k,K)$ and $F_{\mathbf{j}}(k,K)$ are *disjoint*, which we denote by $F_{\mathbf{i}}(k,K) \cap F_{\mathbf{j}}(k,K) = \varnothing$, if there exists $x \in [0,1]$ such that

$$\operatorname{conv}\big((F_{\mathbf{i}}k)(x),(F_{\mathbf{i}}K)(x)\big) \cap \operatorname{conv}\big((F_{\mathbf{j}}k)(x),(F_{\mathbf{j}}K)(x)\big) = \varnothing. \tag{2.2}$$

If they are not disjoint, we write $F_{\mathbf{i}}(k,K) \cap F_{\mathbf{j}}(k,K) \neq \varnothing$. See Fig. 1 for an illustration.

Lemma 2.5. *The following statements are equivalent:*

(a) the dual IFS Φ^* satisfies the SSC;

(b) there exist $k < K$ and $n \geq 1$ such that $F_i\overline{(k,K)} \subseteq (k,K)$ for every $i \in \mathcal{I}$ and for every $\mathbf{i},\mathbf{j} \in \Sigma_n$ with $i_1 \neq j_1$ we have

$$F_{\mathbf{i}}(k,K) \cap F_{\mathbf{j}}(k,K) = \varnothing;$$

(c) there exists $\delta > 0$ such that for all $\mathbf{i},\mathbf{j} \in \Sigma$ with $i_1 \neq j_1$ we have $\sup_{x\in[0,1]} |H_{\mathbf{i}}(x) - H_{\mathbf{j}}(x)| > \delta$;

(d) there exists $\delta > 0$ such that for all $\mathbf{i},\mathbf{j} \in \Sigma_*$ with $i_1 \neq j_1$ we have $\sup_{x\in[0,1]} |H_{\mathbf{i}}(x) - H_{\mathbf{j}}(x)| > \delta$.

Proof. (b)⇒(a): Let $k < K$ be such that $F_i\overline{(k,K)} \subseteq (k,K)$, and so, $\Lambda^* \subseteq (k,K)$. Thus, the implication is clear. For the other direction, (a)⇒(b), let us argue by contradiction. That is, for every $n \geq 1$ there exist $\mathbf{i},\mathbf{j} \in \Sigma_n$ with $i_1 \neq j_1$ such that $F_{\mathbf{i}}(k,K) \cap F_{\mathbf{j}}(k,K) \neq \varnothing$. By the compactness of Σ, there exist a subsequence n_ℓ and $\mathbf{i},\mathbf{j} \in \Sigma$ with $i_1 \neq j_1$ such that $F_{\mathbf{i}_1^{n_\ell}}(k,K) \cap F_{\mathbf{j}_1^{n_\ell}}(k,K) \neq \varnothing$

for every $\ell \geq 1$. In particular, Eq. (2.2) implies that for every $x \in [0,1]$ there exists $y \in \mathbb{R}$ such that $y \in \mathrm{conv}\big((F_{\mathbf{i}_1^{n_\ell}}k)(x), (F_{\mathbf{i}_1^{n_\ell}}K)(x)\big) \cap \mathrm{conv}\big((F_{\mathbf{j}_1^{n_\ell}}k)(x), (F_{\mathbf{j}_1^{n_\ell}}K)(x)\big)$. Hence,

$$|F_{\mathbf{i}_1^{n_\ell}}K(x) - F_{\mathbf{j}_1^{n_\ell}}K(x)| \leq |F_{\mathbf{i}_1^{n_\ell}}K(x) - y| + |y - F_{\mathbf{j}_1^{n_\ell}}K(x)| \leq 2c_{\max}^{n_\ell}(K-k).$$

Thus, $H_{\mathbf{i}} = \lim_{\ell\to\infty} F_{\mathbf{i}_1^{n_\ell}}K = \lim_{\ell\to\infty} F_{\mathbf{j}_1^{n_\ell}}K = H_{\mathbf{j}}$ uniformly, which contradicts to SSC.

The implications (a)⇔(c) and (a)⇔(d) follow by the compactness of Λ^* Lemma 2.1 and the Hölder continuity of the dual natural projection Lemma 2.4. We leave the details for the reader. □

Lemma 2.6. *Let Φ^* be the dual IFS of an analytic IFS $\Phi \in \mathfrak{S}_N$ such that Φ^* satisfies the SSC. Then there exists $\varepsilon > 0$ such that for every $\Psi \in \mathfrak{S}_N$ with $d_2(\Phi, \Psi) < \varepsilon$, the dual IFS Ψ^* of Ψ satisfies the SSC.*

Proof. Let $\Phi = (f_i)_{i=1}^N \in \mathfrak{S}_N$. Let $c_{\min}$ and $c_{\max}$ as in Eq. (1.7) for Φ. Moreover, let $C > 0$ be such that $|f_i''(x)| < \left|\frac{f_i''}{f_i'}(x)\right| < C$ and $\left|\frac{f_i'''(x)f_i'(x) - f_i''(x)}{f_i'(x)}\right| < C$ for every $i \in \mathcal{I}$ and $x \in [0,1]$. By using the continuity of the maps and its derivatives, one can choose $\varepsilon > 0$ sufficiently small such that for every $\Psi = (g_i)_{i=1}^N \in \mathfrak{S}_N$ with $d_2(\Phi, \Psi) < \varepsilon$,

$$c_{\min} \leq |g_i'(x)| \leq c_{\max} \quad \text{and} \quad |g_i''(x)| < \left|\frac{g_i''}{g_i'}(x)\right| < C$$

for every $i \in \mathcal{I}$ and $x \in [0,1]$. Moreover, for every $x \in [0,1]$

$$\left|\frac{f_i''}{f_i'}(x) - \frac{g_i''}{g_i'}(x)\right| \leq \frac{c_{\max} + C}{c_{\min}^2} d_2(\Phi, \Psi). \tag{2.3}$$

Observe that for every $\mathbf{i} \in \Sigma_*$ and $x \in [0,1]$

$$|f_{\mathbf{i}}(x) - g_{\mathbf{i}}(x)| \leq \|f_{i_1}'\|_\infty \cdot |f_{\mathbf{i}_2^{|\mathbf{i}|}}(x) - g_{\mathbf{i}_2^{|\mathbf{i}|}}(x)| + \sup_{y\in[0,1]} |f_{i_1}(y) - g_{i_1}(y)|.$$

Thus,

$$\sup_{x\in[0,1]} |f_{\mathbf{i}}(x) - g_{\mathbf{i}}(x)| \leq \frac{1}{1 - c_{\max}} d_2(\Phi, \Psi). \tag{2.4}$$

On the other hand, for every $\mathbf{i} \in \Sigma_*$ and $x \in [0,1]$

$$\begin{aligned}
|f_{\mathbf{i}}'(x) - g_{\mathbf{i}}'(x)| &\leq c_{\max}|f'_{\mathbf{i}_2^{|\mathbf{i}|}}(x) - g'_{\mathbf{i}_2^{|\mathbf{i}|}}(x)| + c_{\max}^{|\mathbf{i}|-1}|f_{i_1}'(f_{\mathbf{i}_2^{|\mathbf{i}|}}(x)) - g_{i_1}'(g_{\mathbf{i}_2^{|\mathbf{i}|}}(x))| \\
&\leq c_{\max}|f'_{\mathbf{i}_2^{|\mathbf{i}|}}(x) - g'_{\mathbf{i}_2^{|\mathbf{i}|}}(x)| + c_{\max}^{|\mathbf{i}|-1}\left(d_2(\Phi, \Psi) + C|f_{\mathbf{i}_2^{|\mathbf{i}|}}(x) - g_{\mathbf{i}_2^{|\mathbf{i}|}}(x)|\right) \\
&\leq c_{\max}|f'_{\mathbf{i}_2^{|\mathbf{i}|}}(x) - g'_{\mathbf{i}_2^{|\mathbf{i}|}}(x)| + c_{\max}^{|\mathbf{i}|-1}\frac{C+1}{1 - c_{\max}} d_2(\Phi, \Psi).
\end{aligned}$$

Thus, by induction

$$\sup_{x\in[0,1]} |f_{\mathbf{i}}'(x) - g_{\mathbf{i}}'(x)| \leq c_{\max}^{|\mathbf{i}|-1}\frac{C+1}{1 - c_{\max}} d_2(\Phi, \Psi). \tag{2.5}$$

Combining Eqs. (2.3) to (2.5), we get that for every $\mathbf{i} \in \Sigma_*$

$$\begin{aligned}
&\left| f'_{\mathbf{i}^1_{|\mathbf{i}|-1}}(x) \cdot \frac{f''_{i_{|\mathbf{i}|}}}{f'_{i_{|\mathbf{i}|}}}(f_{\mathbf{i}^1_{|\mathbf{i}|-1}}(x)) - g'_{\mathbf{i}^1_{|\mathbf{i}|-1}}(x) \cdot \frac{g''_{i_{|\mathbf{i}|}}}{g'_{i_{|\mathbf{i}|}}}(g_{\mathbf{i}^1_{|\mathbf{i}|-1}}(x)) \right| \\
&\leq C|f'_{\mathbf{i}^1_{|\mathbf{i}|-1}}(x) - g'_{\mathbf{i}^1_{|\mathbf{i}|-1}}(x)| + c_{\max}^{|\mathbf{i}|-1} \left| \frac{f''_{i_{|\mathbf{i}|}}}{f'_{i_{|\mathbf{i}|}}}(f_{\mathbf{i}^1_{|\mathbf{i}|-1}}(x)) - \frac{g''_{i_{|\mathbf{i}|}}}{g'_{i_{|\mathbf{i}|}}}(g_{\mathbf{i}^1_{|\mathbf{i}|-1}}(x)) \right| \\
&\leq C|f'_{\mathbf{i}^1_{|\mathbf{i}|-1}}(x) - g'_{\mathbf{i}^1_{|\mathbf{i}|-1}}(x)| + c_{\max}^{|\mathbf{i}|-1} \left(C \left| f_{\mathbf{i}^1_{|\mathbf{i}|-1}}(x) - g_{\mathbf{i}^1_{|\mathbf{i}|-1}}(x) \right| + \left| \frac{f''_i}{f'_i}(g_{\mathbf{i}^1_{|\mathbf{i}|-1}}(x)) - \frac{g''_i}{g'_i}(g_{\mathbf{i}^1_{|\mathbf{i}|-1}}(x)) \right| \right) \\
&\leq \left(\frac{C(C+1)+C}{1-c_{\max}} + \frac{c_{\max}+C}{c_{\min}} \right) c_{\max}^{|\mathbf{i}|-1} d_2(\Phi, \Psi).
\end{aligned}$$

In particular, there exists $C' > 0$ depending on Φ such that

$$|H_{\mathbf{i}}(x) - \hat{H}_{\mathbf{i}}(x)| \leq C' d_2(\Phi, \Psi), \tag{2.6}$$

where $H_{\mathbf{i}}$ denotes the dual projection of Φ^* and $\hat{H}_{\mathbf{i}}$ denotes the dual projection of Ψ^*.

Now, if the dual IFS Φ^* of $\Phi = (f_i)_{i=1}^N \in \mathfrak{S}_N$ satisfies the SSC then by Lemma 2.5 there exists $\delta > 0$ such that $\sup_{x \in [0,1]} |H_{\mathbf{i}}(x) - H_{\mathbf{j}}(x)| \geq \delta$ for every $\mathbf{i}, \mathbf{j} \in \Sigma$ with $i_1 \neq j_1$. Hence, by Eq. (2.6) and Lemma 2.5, for every Ψ with $d_2(\Phi, \Psi) < \delta/(3C')$ the dual Ψ^* satisfies the SSC. □

2.2 Further analysis of the dual natural projection $H_{\mathbf{i}}$

After establishing that $H_{\mathbf{i}}$ is analytic, we wish to obtain bounds on its derivatives. To simplify notation we write $f^{(k)}$ to refer to the k-th derivative of f. For any $\mathbf{i} \in \Sigma_*$, using the chain rule we get $f'_{\mathbf{i}^1_{|\mathbf{i}|}}(x) = \prod_{n=1}^{|\mathbf{i}|} f'_{i_n}(f_{\mathbf{i}^1_{n-1}}(x))$. From here, a simple calculation yields that for every finite word $\mathbf{i} \in \Sigma_*$, $H_{\mathbf{i}}$ reduces to

$$H_{\mathbf{i}}(x) = H_{\mathbf{i}_1^{|\mathbf{i}|}}(x) = \frac{f''_{\mathbf{i}^1_{|\mathbf{i}|}}(x)}{f'_{\mathbf{i}^1_{|\mathbf{i}|}}(x)}. \tag{2.7}$$

Another way of writing $H_{\mathbf{i}}$ for $\mathbf{i} \in \Sigma \cup \Sigma_*$ is

$$H_{\mathbf{i}}(x) = \sum_{n=1}^{|\mathbf{i}|} (\phi_{i_n} \circ f_{\mathbf{i}^1_{n-1}})'(x), \tag{2.8}$$

where $\phi_{i_k}(x) := \log |f'_{i_k}(x)|$. Recall that the k-th derivative of the composition of two functions can be calculated using Faà di Bruno's formula:

$$(f \circ g)^{(k)}(x) = \sum_{\pi \in \Pi_k} f^{(|\pi|)}(g(x)) \cdot \prod_{B \in \pi} g^{(|B|)}(x), \tag{2.9}$$

where Π_k is the set of all partitions of $\{1, \ldots, k\}$, and $B \in \pi$ refers to the elements, or blocks, of the partition π. Finally, let $g_k : \mathbb{R}^k \to \mathbb{R}$ be a k-variable polynomial such that $g_1(y_1) = y_1$ and the next one is given by the formula

$$g_{k+1}(y_{k+1}, \ldots, y_1) := \sum_{\ell=1}^{k} \frac{\partial g_k}{\partial y_\ell}(y_k, \ldots, y_1) \cdot y_{\ell+1} + g_k(y_k, \ldots, y_1) \cdot y_1.$$

In particular, $g_2(y_2, y_1) = y_2 + y_1^2$, $g_3(y_3, y_2, y_1) = y_3 + 3y_1y_2 + y_1^3$ and so on. We are now ready to give a formula for $H_{\mathbf{i}}^{(k)}$, the kth derivatives of $H_{\mathbf{i}}$.

Lemma 2.7. *For any* $\mathbf{i} \in \Sigma \cup \Sigma_*$*, the* k*-th derivative of* $H_{\mathbf{i}}$ *is given by*

$$H_{\mathbf{i}}^{(k)}(x) = \sum_{\pi \in \Pi_{k+1}} \sum_{n=1}^{|\mathbf{i}|} \phi_{i_n}^{(|\pi|)}(f_{\mathbf{i}_{n-1}^1}(x)) \cdot (f'_{\mathbf{i}_{n-1}^1}(x))^{|\pi|} \cdot \prod_{B \in \pi} g_{|B|-1}\big(H_{\mathbf{i}_1^{n-1}}^{(|B|-2)}(x), \ldots, H_{\mathbf{i}_1^{n-1}}(x)\big).$$

Proof. We first show by induction that for any $\mathbf{i} \in \Sigma_*$,

$$\frac{f_{\mathbf{i}_{|\mathbf{i}|}^1}^{(k)}(x)}{f'_{\mathbf{i}_{|\mathbf{i}|}^1}(x)} = g_{k-1}\big(H_{\mathbf{i}}^{(k-2)}(x), \ldots, H_{\mathbf{i}}(x)\big). \tag{2.10}$$

Indeed, for $k = 2$, Eq. (2.10) is the same as Eq. (2.7). Differentiating both sides of Eq. (2.10) we get

$$\frac{f_{\mathbf{i}_{|\mathbf{i}|}^1}^{(k+1)}(x)}{f'_{\mathbf{i}_{|\mathbf{i}|}^1}(x)} - \frac{f_{\mathbf{i}_{|\mathbf{i}|}^1}^{(k)}(x)}{f'_{\mathbf{i}_{|\mathbf{i}|}^1}(x)} \cdot \frac{f''_{\mathbf{i}_{|\mathbf{i}|}^1}(x)}{f'_{\mathbf{i}_{|\mathbf{i}|}^1}(x)} = \sum_{\ell=0}^{k-2} \frac{\partial g_{k-1}}{\partial y_\ell}\big(H_{\mathbf{i}}^{(k-2)}, \ldots, H_{\mathbf{i}}(x)\big) \cdot H_{\mathbf{i}}^{(l+1)}(x).$$

We use the induction hypothesis Eq. (2.10) in the second term on the left hand side for k and Eq. (2.7) to see that

$$\frac{f_{\mathbf{i}_{|\mathbf{i}|}^1}^{(k)}(x)}{f'_{\mathbf{i}_{|\mathbf{i}|}^1}(x)} \cdot \frac{f''_{\mathbf{i}_{|\mathbf{i}|}^1}(x)}{f'_{\mathbf{i}_{|\mathbf{i}|}^1}(x)} = g_{k-1}(H_{\mathbf{i}}^{(k-2)}(x), \ldots, H_{\mathbf{i}}(x)) \cdot H_{\mathbf{i}}(x).$$

Substituting this back, after rearranging the inductive step is proved for $k + 1$:

$$\begin{aligned}\frac{f_{\mathbf{i}_{|\mathbf{i}|}^1}^{(k+1)}(x)}{f'_{\mathbf{i}_{|\mathbf{i}|}^1}(x)} &= g_{k-1}(H_{\mathbf{i}}^{(k-2)}(x), \ldots, H_{\mathbf{i}}(x)) \cdot H_{\mathbf{i}}(x) + \sum_{\ell=0}^{k-2} \frac{\partial g_{k-1}}{\partial y_\ell}\big(H_{\mathbf{i}}^{(k-2)}, \ldots, H_{\mathbf{i}}(x)\big) \cdot H_{\mathbf{i}}^{(l+1)}(x) \\ &= g_k(H_{\mathbf{i}}^{(k-1)}(x), \ldots, H_{\mathbf{i}}(x)).\end{aligned}$$

We can now derive the formula for $H_{\mathbf{i}}^{(k)}$:

$$\begin{aligned}H_{\mathbf{i}}^{(k)}(x) &\overset{(2.8)}{=} \sum_{n=1}^{|\mathbf{i}|} (\phi_{i_n} \circ f_{\mathbf{i}_{n-1}^1})^{(k+1)}(x) \\ &\overset{(2.9)}{=} \sum_{\pi \in \Pi_{k+1}} \sum_{n=1}^{|\mathbf{i}|} \phi_{i_n}^{(|\pi|)}(f_{\mathbf{i}_{n-1}^1}(x)) \cdot \prod_{B \in \pi} f_{\mathbf{i}_{n-1}^1}^{(|B|)}(x) \\ &\overset{(2.10)}{=} \sum_{\pi \in \Pi_{k+1}} \sum_{n=1}^{|\mathbf{i}|} \phi_{i_n}^{(|\pi|)}(f_{\mathbf{i}_{n-1}^1}(x)) \cdot (f'_{\mathbf{i}_{n-1}^1}(x))^{|\pi|} \cdot \prod_{B \in \pi} g_{|B|-1}\big(H_{\mathbf{i}_1^{n-1}}^{(|B|-2)}(x), \ldots, H_{\mathbf{i}_1^{n-1}}(x)\big).\end{aligned}$$

□

Lemma 2.8. *For every integer* $k \geq 0$ *there exists* C_k *such that for all* $x \in I$ *and all* $\mathbf{i} \in \Sigma \cup \Sigma_*$*,*

$$\big|H_{\mathbf{i}}^{(k)}(x)\big| \leq C_k.$$

Proof. We proceed by induction. Let

$$D_k := \max_{i\in\Sigma_1} \max_{x\in[0,1]} |\phi_i^{(k)}(x)|.$$

From Eq. (2.8) we see that $|H_{\mathbf{i}}(x)| \le D_1/(1-c_{\max}) =: C_0$. Suppose that the statement is true for k and define

$$E_k := \sup_{\substack{y_{j+1}\in[-C_j,C_j]\\ 0\le j\le k-1}} g_k(y_k,\dots,y_1). \tag{2.11}$$

By Lemma 2.7,

$$\left|H_{\mathbf{i}}^{(k)}(x)\right| \le \sum_{\pi\in\Pi_{k+1}} \sum_{n=1}^{|\mathbf{i}|} D_{|\pi|}\cdot c_{\max}^{(n-1)|\pi|}\cdot \prod_{B\in\pi} E_{|B|-1} = \sum_{\pi\in\Pi_{k+1}} \frac{D_{|\pi|}\cdot\prod_{B\in\pi}E_{|B|-1}}{1-c_{\max}^{|\pi|}}$$

and the statement follows. □

Corollary 2.9. *For all $k\ge 1$, for all $x,y\in I$ and for all $\mathbf{i}\in\Sigma\cup\Sigma_*$,*

$$\left|H_{\mathbf{i}}^{(k)}(x)-H_{\mathbf{i}}^{(k)}(y)\right| \le C_{k+1}\cdot|x-y|,$$

where $C_k>0$ are as defined in Lemma 2.8.

We show the following useful Hölder type bound.

Lemma 2.10. *For all integers $k\ge 0$, $x\in I$ and $\mathbf{i},\mathbf{j}\in\Sigma\cup\Sigma_*$ with $|\mathbf{i}\wedge\mathbf{j}|<\min\{|\mathbf{i}|,|\mathbf{j}|\}$,*

$$\left|H_{\mathbf{i}}^{(k)}(x)-H_{\mathbf{j}}^{(k)}(x)\right| \le 2C_k\cdot c_{\max}^{|\mathbf{i}\wedge\mathbf{j}|},$$

where the $C_k>0$ are as defined in Lemma 2.8.

Proof. Let $m=|\mathbf{i}\wedge\mathbf{j}|$. Again, by Lemmas 2.7 and 2.8,

$$\begin{aligned}
&|H_{\mathbf{i}}^{(k)}(x)-H_{\mathbf{j}}^{(k)}(x)|\\
&=\Bigg|\sum_{\pi\in\Pi_{k+1}}\sum_{n=m+1}^{|\mathbf{i}|}\phi_{i_n}^{(|\pi|)}(f_{\mathbf{i}_{n-1}^1}(x))\cdot(f'_{\mathbf{i}_{n-1}^1}(x))^{|\pi|}\cdot\prod_{B\in\pi}g_{|B|-1}\big(H_{\mathbf{i}_1^{n-1}}^{(|B|-2)}(x),\dots,H_{\mathbf{i}_1^{n-1}}(x)\big)\\
&\quad-\sum_{\pi\in\Pi_{k+1}}\sum_{n=m+1}^{|\mathbf{j}|}\phi_{j_n}^{(|\pi|)}(f_{\mathbf{j}_{n-1}^1}(x))\cdot(f'_{\mathbf{j}_{n-1}^1}(x))^{|\pi|}\cdot\prod_{B\in\pi}g_{|B|-1}\big(H_{\mathbf{j}_1^{n-1}}^{(|B|-2)}(x),\dots,H_{\mathbf{j}_1^{n-1}}(x)\big)\Bigg|\\
&\le\sum_{\pi\in\Pi_{k+1}}\sum_{n=m+1}^{\infty}2D_{|\pi|}\cdot c_{\max}^{n|\pi|}\cdot\prod_{B\in\pi}E_{|B|-1}\le 2C_k\cdot c_{\max}^m.
\end{aligned}$$

□

3 Proof of the sufficient condition for SESC

We need one auxiliary lemma before we can proceed with the proof of Theorem 1.5.

Lemma 3.1. *Let f and g be real analytic maps on J and let $\eta > 0$ with $2\sqrt{\eta} < |J|$. Denote*

$$Q := \max\left\{\sup_{x\in J}|f''(x)|, \sup_{x\in J}|g''(x)|\right\}.$$

If $\sup_{x\in J}|f(x)-g(x)| \le \eta$, then $\sup_{x\in J}|f'(x)-g'(x)| \le (2+Q)\sqrt{\eta}$.

Proof. Let $x \in J$ be arbitrary and take $y \in J$ such that $|x-y| = \sqrt{\eta}$. By assumption $\max\{|f(x) - g(x)|, |f(y)-g(y)|\} \le \eta$. Using the second order Taylor approximation

$$f(y) = f(x) + f'(x)(y-x) + \frac{f''(\xi_1)}{2}(y-x)^2,$$

where $\xi_1 \in (x,y)$, and similarly for $g(y)$ around x we get

$$\begin{aligned}\eta \ge |f(y)-g(y)| &= \left|f(x)-g(x)+(f'(x)-g'(x))(y-x)+(f''(\xi_1)-g''(\xi_2))\frac{(y-x)^2}{2}\right| \\ &\ge |f'(x)-g'(x)|\cdot|y-x| - |f(x)-g(x)| - (|f''(\xi_1)|+|g''(\xi_2)|)\frac{(y-x)^2}{2}.\end{aligned}$$

Thus,

$$|f'(x)-g'(x)| \le \frac{\eta+\eta+Q\eta}{\sqrt{\eta}} = (2+Q)\sqrt{\eta}$$

as required. □

Proof of Theorem 1.5. We prove the theorem by contradiction. Recall Definition 1.2 and suppose that Φ has weak super-exponential condensation, *i.e.* there exists a sequence $(\eta_n)_n$ such that $\log(\eta_n)/n \to -\infty$ and there exist a subsequence $n_\ell \in \mathbb{N}$ and $\mathbf{i} \ne \mathbf{j} \in \Sigma_{n_\ell}$ such that

$$\sup_{x\in[0,1]} |f_{\mathbf{i}}(x) - f_{\mathbf{j}}(x)| \le \eta_{n_\ell}. \tag{3.1}$$

We need to show that there exist $\mathbf{i}^* \ne \mathbf{j}^* \in \Sigma \cup \Sigma_*$ with $|\mathbf{i}^*| = |\mathbf{j}^*|$ for which $H_{\mathbf{i}^*}(x) \equiv H_{\mathbf{j}^*}(x)$ for all $x \in [0,1]$, thus contradicting our main assumption. For the remainder of the proof we work with the sequence η_{n_ℓ} and $\mathbf{i} \ne \mathbf{j} \in \Sigma_{n_\ell}$ provided by Eq. (3.1). Let $m = m(n_\ell) := \max\{k \le n_\ell : i_k \ne j_k\}$ and $\mathbf{u}^{(n_\ell)} := \mathbf{i}_{m+1}^{n_\ell} \in \Sigma_{n_\ell - m}$, then $\mathbf{j} = \mathbf{j}_1^m \mathbf{u}^{(n_\ell)}$. Let us also denote $\mathbf{i}^{(n_\ell)} := \mathbf{i}_m^1 \in \Sigma_m$ and $\mathbf{j}^{(n_\ell)} := \mathbf{j}_m^1 \in \Sigma_m$, so $(\mathbf{i}^{(n_\ell)})_1 \ne (\mathbf{j}^{(n_\ell)})_1$. We first show that there exists a sequence η''_{n_ℓ} with $\log(\eta''_{n_\ell})/n_\ell \to -\infty$ such that for every $x \in I$,

$$|H_{\mathbf{i}^{(n_\ell)}}(f_{\mathbf{u}^{(n_\ell)}}(x)) - H_{\mathbf{j}^{(n_\ell)}}(f_{\mathbf{u}^{(n_\ell)}}(x))| \le \eta''_{n_\ell}. \tag{3.2}$$

For any $\mathbf{i} \in \Sigma_*$, recall from Eq. (2.7) that

$$H_{\mathbf{i}}(x) = \frac{f''_{\mathbf{i}^1_{|\mathbf{i}|}}(x)}{f'_{\mathbf{i}^1_{|\mathbf{i}|}}(x)} \quad \text{or equivalently,} \quad H_{\mathbf{i}^1_{|\mathbf{i}|}}(x) = \frac{f''_{\mathbf{i}}(x)}{f'_{\mathbf{i}}(x)}.$$

Using (2.7), we get

$$|f''_{\mathbf{i}}(x)| = \left|H_{\mathbf{i}^1_{|\mathbf{i}|}}(x)\cdot f'_{\mathbf{i}}(x)\right| \le C_0 \cdot c_{\max}^{|\mathbf{i}|}$$

by Lemma 2.8. Moreover,

$$|f_{\mathbf{i}}'''(x)| = \left|g_2(H'_{\mathbf{i}^1_{|\mathbf{i}|}}(x), H_{\mathbf{i}^1_{|\mathbf{i}|}}(x)) \cdot f_{\mathbf{i}}'(x)\right| \leq E_2 \cdot c_{\max}^{|\mathbf{i}|}$$

by the definition of E_k in Eq. (2.11). Together with Lemma 3.1 these bounds imply that for the particular choice of $\mathbf{i}, \mathbf{j}$ in Eq. (3.1) we have

$$\sup_{x\in[0,1]} |f_{\mathbf{i}}'(x) - f_{\mathbf{j}}'(x)| \leq (2 + C_0 c_{\max}^{n_\ell})\sqrt{\eta_{n_\ell}};$$
$$\sup_{x\in[0,1]} |f_{\mathbf{i}}''(x) - f_{\mathbf{j}}''(x)| \leq (2 + E_2 c_{\max}^{n_\ell})\sqrt{2 + C_0 c_{\max}^{n_\ell}} \cdot \eta_{n_\ell}^{1/4}.$$

We deduce

$$\begin{aligned}\left|\frac{f_{\mathbf{i}}''(x)}{f_{\mathbf{i}}'(x)} - \frac{f_{\mathbf{j}}''(x)}{f_{\mathbf{j}}'(x)}\right| &\leq \frac{|f_{\mathbf{i}}''(x)|}{|f_{\mathbf{i}}'(x)||f_{\mathbf{j}}'(x)|} \cdot |f_{\mathbf{i}}'(x) - f_{\mathbf{j}}'(x)| + \frac{1}{|f_{\mathbf{j}}'(x)|} \cdot |f_{\mathbf{i}}''(x) - f_{\mathbf{j}}''(x)| \\ &\leq \frac{C_0}{c_{\min}^{n_\ell}}(2 + C_0 c_{\max}^{n_\ell}) \cdot \eta_{n_\ell}^{1/2} + \frac{1}{c_{\min}^{n_\ell}}(2 + E_2 c_{\max}^{n_\ell})\sqrt{2 + C_0 c_{\max}^{n_\ell}} \cdot \eta_{n_\ell}^{1/4} \;=:\; \eta'_{n_\ell}.\end{aligned}$$

Now observe that

$$H_{\mathbf{i}^1_{n_\ell}}(x) = \frac{(f_{\mathbf{i}_1^m} \circ f_{\mathbf{u}^{(n_\ell)}})''(x)}{(f_{\mathbf{i}_1^m} \circ f_{\mathbf{u}^{(n_\ell)}})'(x)} = f'_{\mathbf{u}^{(n_\ell)}}(x) \cdot H_{\mathbf{i}^{(n_\ell)}}\big(f_{\mathbf{u}^{(n_\ell)}}(x)\big) + \frac{f''_{\mathbf{u}^{(n_\ell)}}(x)}{f'_{\mathbf{u}^{(n_\ell)}}(x)},$$

hence, $H_{\mathbf{i}^1_{n_\ell}}(x) - H_{\mathbf{j}^1_{n_\ell}}(x) = f'_{\mathbf{u}^{(n_\ell)}}(x) \cdot \big(H_{\mathbf{i}^{(n_\ell)}}\big(f_{\mathbf{u}^{(n_\ell)}}(x)\big) - H_{\mathbf{j}^{(n_\ell)}}\big(f_{\mathbf{u}^{(n_\ell)}}(x)\big)\big)$, so we can conclude

$$\left|H_{\mathbf{i}^{(n_\ell)}}\big(f_{\mathbf{u}^{(n_\ell)}}(x)\big) - H_{\mathbf{j}^{(n_\ell)}}\big(f_{\mathbf{u}^{(n_\ell)}}(x)\big)\right| \leq \frac{\left|H_{\mathbf{i}^1_{n_\ell}}(x) - H_{\mathbf{j}^1_{n_\ell}}(x)\right|}{\left|f'_{\mathbf{u}^{(n_\ell)}}(x)\right|} \leq c_{\min}^{-n_\ell} \cdot \eta'_{n_\ell} =: \eta''_{n_\ell}.$$

Having established Eq. (3.2), there are two cases to consider: whether $|\mathbf{u}^{(n_\ell)}| \to \infty$ or there exists a constant $C > 0$ and infinitely many ℓ such that $|\mathbf{u}^{(n_\ell)}| \leq C$. Let us first assume the latter. Since $|\mathbf{i}^{(n_\ell)}| + |\mathbf{u}^{(n_\ell)}| = |\mathbf{j}^{(n_\ell)}| + |\mathbf{u}^{(n_\ell)}| = n_\ell$ we conclude, by compactness, that there exists a subsequence n'_ℓ such that $\mathbf{i}^{(n'_\ell)} \to \mathbf{i}^* \in \Sigma$, $\mathbf{j}^{(n'_\ell)} \to \mathbf{j}^* \in \Sigma$ and $\mathbf{u}^{(n'_\ell)} = \mathbf{u}^* \in \Sigma_*$ with $i_1^* \neq j_1^*$. It follows from Lemma 2.10 and Eq. (3.2) that $H_{\mathbf{i}^*}(f_{\mathbf{u}^*}(x)) \equiv H_{\mathbf{j}^*}(f_{\mathbf{u}^*}(x))$ for all $x \in I$. Hence, using the analyticity of $H_{\mathbf{i}}$ from Lemma 2.4 we conclude that $H_{\mathbf{i}^*}(x) \equiv H_{\mathbf{j}^*}(x)$ for all $x \in I$ which contradicts the main assumption.

Now let us assume that $|\mathbf{u}^{(n_\ell)}| \to \infty$. Again by compactness, there exists $\mathbf{u}^* \in \Sigma$ as well as $\mathbf{i}^*, \mathbf{j}^* \in \Sigma \cup \Sigma_*$ and a subsequence n'_ℓ such that $\mathbf{i}^{(n'_\ell)} \to \mathbf{i}^*$ and $\mathbf{j}^{(n'_\ell)} \to \mathbf{j}^*$ with $i_1^* \neq j_1^*$ as well as $f_{\mathbf{u}^{(n'_\ell)}}(x) \to \pi(\mathbf{u}^*)$ for all $x \in [0,1]$. Note that both $|\mathbf{i}^*|$ and $|\mathbf{j}^*|$ might be finite or infinite, however, $|\mathbf{i}^*| = |\mathbf{j}^*|$ by the construction. Combining Lemma 3.1 and Lemma 2.8 with Eq. (3.2), we deduce that for all k there exists $\widetilde{C}_k > 0$ such that for all $\ell \geq 1$ we have

$$\left|H^{(k)}_{\mathbf{i}^{(n_\ell)}}(f_{\mathbf{u}^{(n_\ell)}}(x)) - H^{(k)}_{\mathbf{j}^{(n_\ell)}}(f_{\mathbf{u}^{(n_\ell)}}(x))\right| \leq \widetilde{C}_k \cdot \frac{1}{(f'_{\mathbf{u}^{(n_\ell)}}(x))^k} \cdot \left(\eta''_{n_\ell}\right)^{2^{-k}} \leq \widetilde{C}_k \cdot \frac{1}{c_{\min}^{kn_\ell}} \cdot \left(\eta''_{n_\ell}\right)^{2^{-k}} \tag{3.3}$$

for all $x \in [0,1]$ which still tends to 0 as $\ell \to \infty$ since $\eta''_{n_\ell} \to 0$ super-exponentially fast. Combining Corollary 2.9 and Lemma 2.10 with Eq. (3.3) we see that $H^{(k)}_{\mathbf{i}^*}(\pi(\mathbf{u}^*)) = H^{(k)}_{\mathbf{j}^*}(\pi(\mathbf{u}^*))$ for all k. Since $H_{\mathbf{i}^*}$ and $H_{\mathbf{j}^*}$ are analytic by Lemma 2.4, we get $H_{\mathbf{i}^*}(x) \equiv H_{\mathbf{j}^*}(x)$ for all $x \in [0,1]$ which again contradicts our main assumption, concluding the proof of Theorem 1.5. □

Proof of Theorem 2.2. The proof follows by Lemma 2.5 and Theorem 1.5. □

4 Existence of open and dense set of IFSs with SESC

4.1 Preliminaries

Fix an arbitrary $n \geq 1$. Let $\mathcal{B}_n \coloneqq \{(\mathbf{i},\mathbf{j}) \in \Sigma_n \times \Sigma_n : i_1 < j_1\}$. We say that $(\mathbf{i},\mathbf{j}) \in \mathcal{B}_n$ is bad if the associated cylinders overlap,

$$F_{\mathbf{i}}(k,K) \cap F_{\mathbf{j}}(k,K) \neq \varnothing, \tag{4.1}$$

recall Eq. (2.2). For any $\mathbf{i} \in \Sigma_*$ and $x \in [0,1]$ define the orbit of $\mathbf{i}$ starting from x as the multiset $\mathcal{O}_{\mathbf{i}}(x) \coloneqq \{x, f_{i_1}(x), f_{\mathbf{i}_2^1}(x), \ldots, f_{\mathbf{i}_{|\mathbf{i}|}^1}(x)\}$.

Lemma 4.1. *If the IFS $(f_i)_{i\in\mathcal{I}} \in \mathfrak{S}_N$ has no exact overlaps, then there exists a collection of points $\{x_{\mathbf{i},\mathbf{j}}\}_{(\mathbf{i},\mathbf{j})\in\mathcal{B}_n} \subseteq [0,1]$ such that*

(a) $\operatorname{conv}\big((F_{\mathbf{i}}k)(x_{\mathbf{i},\mathbf{j}}), (F_{\mathbf{i}}K)(x_{\mathbf{i},\mathbf{j}})\big) \cap \operatorname{conv}\big((F_{\mathbf{j}}k)(x_{\mathbf{i},\mathbf{j}}), (F_{\mathbf{j}}K)(x_{\mathbf{i},\mathbf{j}})\big) = \varnothing$ *if $(\mathbf{i},\mathbf{j})$ is not bad;*

(b) all points in $\mathcal{O}_{\mathbf{i}}(x_{\mathbf{i},\mathbf{j}})$ and also in $\mathcal{O}_{\mathbf{j}}(x_{\mathbf{i},\mathbf{j}})$ are distinct, moreover, $\mathcal{O}_{\mathbf{i}}(x_{\mathbf{i},\mathbf{j}}) \cap \mathcal{O}_{\mathbf{j}}(x_{\mathbf{i},\mathbf{j}}) = \{x_{\mathbf{i},\mathbf{j}}\}$;

(c) $\big(\mathcal{O}_{\mathbf{i}}(x_{\mathbf{i},\mathbf{j}}) \cup \mathcal{O}_{\mathbf{j}}(x_{\mathbf{i},\mathbf{j}})\big) \cap \big(\mathcal{O}_{\mathbf{h}}(x_{\mathbf{h},\mathbf{k}}) \cup \mathcal{O}_{\mathbf{k}}(x_{\mathbf{h},\mathbf{k}})\big) = \varnothing$ *for every $(\mathbf{i},\mathbf{j}) \neq (\mathbf{k},\mathbf{h}) \in \mathcal{B}_n$.*

Proof. We first establish an order on the elements of $\mathcal{B}_n$ by setting

$$(\mathbf{i}^{(k)},\mathbf{j}^{(k)}) = (i_1^{(k)},\ldots,i_n^{(k)}; j_1^{(k)},\ldots,j_n^{(k)}),$$

where $k = 1,\ldots,\#\mathcal{B}_n$. The points are constructed inductively. If $(\mathbf{i}^{(1)},\mathbf{j}^{(1)})$ is not bad then by definition there exists an $\hat{x}_{(\mathbf{i}^{(1)},\mathbf{j}^{(1)})}$ for which (a) holds. Since (a) is an open condition, there exists an $x_{(\mathbf{i}^{(1)},\mathbf{j}^{(1)})}$ in the neighborhood of $\hat{x}_{(\mathbf{i}^{(1)},\mathbf{j}^{(1)})}$ for which (b) also holds. If this were not the case, then there would be k,ℓ such that $f_{(\mathbf{i}^{(1)})_\ell^1}(x) = f_{(\mathbf{i}^{(1)})_k^1}(x)$ for infinitely many x, but then analyticity implies that $f_{(\mathbf{i}^{(1)})_\ell^1}(x) \equiv f_{(\mathbf{i}^{(1)})_k^1}(x)$ on $[0,1]$, which contradicts the no exact overlaps assumption. If $(\mathbf{i}^{(1)},\mathbf{j}^{(1)})$ is bad, then choose $x_{(\mathbf{i}^{(1)},\mathbf{j}^{(1)})}$ to satisfy (b) (which is possible by a similar argument). Thus we have constructed the first point $x_{(\mathbf{i}^{(1)},\mathbf{j}^{(1)})}$. Condition (c) trivially holds with just the single pair $(\mathbf{i}^{(1)},\mathbf{j}^{(1)})$.

We continue by induction. Assume that $x_{(\mathbf{i}^{(1)},\mathbf{j}^{(1)})},\ldots,x_{(\mathbf{i}^{(k)},\mathbf{j}^{(k)})}$ have already been constructed (for some $k \geq 1$) so that (a) to (c) all hold. The set $\bigcup_{m=1}^k \widehat{\mathcal{O}}(x_{(\mathbf{i}^{(m)},\mathbf{j}^{(m)})})$ is finite, where we use the shorthand $\widehat{\mathcal{O}}(x_{\mathbf{i},\mathbf{j}}) \coloneqq \big(\mathcal{O}_{\mathbf{i}}(x_{\mathbf{i},\mathbf{j}}) \cup \mathcal{O}_{\mathbf{j}}(x_{\mathbf{i},\mathbf{j}})\big)$. Then the set

$$A_k \coloneqq \bigcup_{\ell=0}^{n} \left(\big(f_{(\mathbf{i}^{(k+1)})_\ell^1}\big)^{-1} \Big(\bigcup_{m=1}^{k} \widehat{\mathcal{O}}(x_{(\mathbf{i}^{(m)},\mathbf{j}^{(m)})}) \Big) \cup \big(f_{(\mathbf{j}^{(k+1)})_\ell^1}\big)^{-1} \Big(\bigcup_{m=1}^{k} \widehat{\mathcal{O}}(x_{(\mathbf{i}^{(m)},\mathbf{j}^{(m)})}) \Big) \right)$$

is also finite since all $f_{\mathbf{i}}$ are strictly monotone (for $\ell = 0$ it is defined to be identity map).

If $(\mathbf{i}^{(k+1)},\mathbf{j}^{(k+1)})$ is not bad, then using that A_k is finite and the continuity of the maps one can choose $\hat{x}_{(\mathbf{i}^{(k+1)},\mathbf{j}^{(k+1)})} \in (0,1) \setminus A_k$ for which (a) holds. By continuity of the maps, there exists a small neighbourhood of $\hat{x}_{(\mathbf{i}^{(k+1)},\mathbf{j}^{(k+1)})}$ in $(0,1) \setminus A_k$ where (a) still holds, and by the same argument as before can be used to pick a $x_{(\mathbf{i}^{(k+1)},\mathbf{j}^{(k+1)})}$ from this small neighbourhood for which (a) to (c) all hold. If $(\mathbf{i}^{(k+1)},\mathbf{j}^{(k+1)})$ is bad, then analyticity and the no exact overlaps assumption imply again the existence of $x_{(\mathbf{i}^{(k+1)},\mathbf{j}^{(k+1)})} \in (0,1) \setminus A_k$ that satisfies (b) which completes the induction. □

Proposition 4.2. *Let $f \in \mathcal{S}^\omega_\epsilon([0,1])$. There exists a constant $C > 0$ such that for any two finite collections of points $\mathcal{Y} = \{y_1 < \ldots < y_M\} \subseteq [0,1]^M$ and $\mathcal{Z} = \{z_1 < \ldots < z_Q\} \subseteq [0,1]^Q$ with $\mathcal{Y} \cap \mathcal{Z} = \varnothing$ we have the following: for every $\varepsilon > 0$ and $\delta > 0$ there exists an analytic function $g \in \mathcal{S}^\omega_\epsilon([0,1])$ such that*

(i) $g(z_i) = f(z_i)$ *for every* $z_i \in \mathcal{Z}$ *and* $g(y_i) = f(y_i)$ *for every* $y_i \in \mathcal{Y}$*, moreover,*

$$\sup_{x\in[0,1]} |g(x) - f(x)| < \varepsilon;$$

(ii) $g'(z_i) = f'(z_i)$ *for every* $z_i \in \mathcal{Z}$ *and* $g'(y_i) = f'(y_i)$ *for every* $y_i \in \mathcal{Y}$*, moreover,*

$$\sup_{x\in[0,1]} |g'(x) - f'(x)| < \varepsilon;$$

(iii) $g''(z_i) = f''(z_i)$ *for every* $z_i \in \mathcal{Z}$*, however,*

$$\left| \frac{g''(y_i)}{g'(y_i)} - \frac{f''(y_i)}{f'(y_i)} \right| \geq \delta \quad \textit{for every } y_i \in \mathcal{Y},$$

nevertheless, $\sup_{x\in[0,1]} |g''(x) - f''(x)| < C \cdot \delta + \varepsilon$.

We may assume that $\mathcal{Y} \neq \varnothing$, otherwise there is nothing to prove. Our claim is that with appropriate choices of $a_1, \ldots, a_M > 0$ and $\eta_1 \ldots, \eta_M > 0$ the analytic function

$$g(x) \coloneqq f(x) \cdot e^{\varphi(x)\cdot\psi(x)\cdot A(x)}, \tag{4.2}$$

where

$$\varphi(x) = \prod_{y_i\in\mathcal{Y}} (x - y_i)^2, \; \psi(x) = \prod_{z_i\in\mathcal{Z}} (x - z_i)^4 \; \text{ and } \; A(x) = \sum_{i=1}^{M} a_i \cdot e^{\frac{-(x-y_i)^2}{\eta_i}}$$

satisfies the conditions of Proposition 4.2. We will often use the following simple fact.

Lemma 4.3. *Let us fix constants $c, p, \varepsilon > 0$ and $q > -p/2$. Then*

$$\sup_{x\in\mathbb{R}} c \cdot \sigma^q \cdot |x|^p \cdot e^{\frac{-x^2}{\sigma}} \leq \varepsilon \quad \textit{whenever } 0 < \sigma \leq (2e/p)^{\frac{p}{2q+p}} \cdot (\varepsilon/c)^{\frac{1}{q+p/2}}. \tag{4.3}$$

Proof. It is easy to check that the global maximum of the function is at $x^2 = p\sigma/2$. Substituting back this value and using the upper bound on σ gives the claim. □

Proof of Proposition 4.2. During the proof we suppress ∞ from the norm $\|\cdot\|_\infty$. The argument is essentially a careful analysis of the function g defined in Eq. (4.2). Let us first observe that g is complex analytic on $\mathcal{B}_{2\epsilon}$, and so, satisfies the assumption (A).

Let us now calculate the derivatives of g. Clearly,

$$g' = f' \cdot e^{\varphi\cdot\psi\cdot A} + f \cdot e^{\varphi\cdot\psi\cdot A}\big(\varphi' \cdot \psi \cdot A + \varphi \cdot \psi' \cdot A + \varphi \cdot \psi \cdot A'\big),$$

where

$$\varphi'(x) = \sum_{i=1}^{M} 2(x - y_i) \prod_{y_j\in\mathcal{Y}\setminus\{y_i\}} (x - y_j)^2,$$

$$\psi'(x) = \sum_{i=1}^{Q} 4(x - z_i)^3 \prod_{z_j \in \mathcal{Z} \setminus \{z_i\}} (x - z_j)^4,$$

$$A'(x) = -2 \cdot \sum_{i=1}^{M} a_i \frac{x - y_i}{\eta_i} e^{\frac{-(x-y_i)^2}{\eta_i}}.$$

Moreover,

$$g'' = f'' e^{\varphi\psi A} + 2f' e^{\varphi\psi A}\big(\varphi'\psi A + \varphi\psi' A + \varphi\psi A'\big) + f e^{\varphi\psi A}\big(\varphi'\psi A + \varphi\psi' A + \varphi\psi A'\big)^2 \\ + f e^{\varphi\psi A}\big(\varphi''\psi A + \varphi\psi'' A + \varphi\psi A'' + 2\varphi'\psi' A + 2\varphi'\psi A' + 2\varphi\psi' A'\big),$$

where

$$\varphi''(x) = 2\sum_{i=1}^{M} \prod_{\substack{y_j \in \mathcal{Y} \\ y_j \neq y_i}} (x - y_j)^2 + 4\sum_{i=1}^{M}(x - y_i)\sum_{\substack{k=1 \\ k \neq i}}^{M}(x - y_k) \prod_{\substack{y_j \in \mathcal{Y} \\ y_j \notin \{y_i, y_k\}}} (x - y_j)^2,$$

$$\psi''(x) = 12\sum_{i=1}^{Q}(x - z_i)^2 \prod_{\substack{z_j \in \mathcal{Z} \\ z_j \neq z_i}} (x - z_j)^4 + 16\sum_{i=1}^{Q}(x - z_i)^3\sum_{\substack{k=1 \\ k \neq i}}^{Q}(x - z_k)^3 \prod_{\substack{z_j \in \mathcal{Z} \\ z_j \notin \{z_i, z_k\}}} (x - z_j)^4,$$

$$A''(x) = 2 \cdot \sum_{i=1}^{M} a_i \Big(\frac{2(x - y_i)^2}{\eta_i^2} - \frac{1}{\eta_i}\Big) e^{\frac{-(x-y_i)^2}{\eta_i}}.$$

By construction $\varphi(y_i) = \varphi'(y_i) = 0$ for every $y_i \in \mathcal{Y}$ and $\psi(z_i) = \psi'(z_i) = \psi''(z_i) = 0$ for every $z_i \in \mathcal{Z}$, hence,

$$g(y_i) = f(y_i) \text{ and } g'(y_i) = f'(y_i) \text{ for every } y_i \in \mathcal{Y},$$

furthermore,

$$g(z_i) = f(z_i),\ g'(z_i) = f'(z_i) \text{ and } g''(z_i) = f''(z_i) \text{ for every } z_i \in \mathcal{Z}.$$

Since $\varphi''(y_i) \neq 0$, let us evaluate

$$g''(y_i) = f''(y_i) + f(y_i)\varphi''(y_i)\psi(y_i)A(y_i) \\ = f''(y_i) + 2f(y_i) \prod_{\substack{y_j \in \mathcal{Y} \\ y_j \neq y_i}} (y_i - y_j)^2 \prod_{z_j \in \mathcal{Z}} (y_i - z_j)^4 \Bigg(a_i + \sum_{\substack{j=1 \\ j \neq i}}^{M} a_j \cdot e^{\frac{-(y_i - y_j)^2}{\eta_j}}\Bigg).$$

Dividing both sides by $f'(y_i) = g'(y_i)$ and rearranging we get

$$\left|\frac{g''(y_i)}{g'(y_i)} - \frac{f''(y_i)}{f'(y_i)}\right| \geq 2a_i \cdot \frac{|f(y_i)|}{|f'(y_i)|} \prod_{y_j \in \mathcal{Y} \setminus \{y_i\}} (y_i - y_j)^2 \prod_{z_j \in \mathcal{Z}} (y_i - z_j)^4 \geq \delta$$

for all $y_i \in \mathcal{Y}$ as required if we choose

$$a_i \coloneqq \frac{\delta \cdot |f'(y_i)|}{2 \cdot |f(y_i)|}\Big(\prod_{y_j \in \mathcal{Y} \setminus \{y_i\}} (y_i - y_j)^2 \cdot \prod_{z_j \in \mathcal{Z}} (y_i - z_j)^4\Big)^{-1}. \tag{4.4}$$

It remains to bound the norms $\|g-f\|$, $\|g'-f'\|$ and $\|g''-f''\|$. Using that $|x-z_i|\le 1$, we have the trivial bounds

$$\|\psi\|\le 1,\quad \|\psi'\|\le 4Q \quad \text{and} \quad \|\psi''\|\le 16Q^2+12Q.$$

Similarly, using that $|x-y_i|\le 1$, we also have

$$\|\varphi\|\le \min_{y_i\in\mathcal{Y}}|x-y_i|^2,\quad \|\varphi'\|\le 2M\cdot\min_{y_i\in\mathcal{Y}}|x-y_i|$$

and

$$\|\varphi''\|\le 4M^2\cdot\min_{y_i\in\mathcal{Y}}|x-y_i|+\left\|2\sum_{i=1}^{M}\prod_{\substack{y_j\in\mathcal{Y}\\ y_j\ne y_i}}(x-y_j)^2\right\|.$$

Choose η_i so small such that

$$\eta_i\le\min\Bigg\{\underbrace{2e\cdot\left(\frac{\varepsilon}{2M^2(16Q^2+12Q)a_i}\right)^2}_{(C1)},\underbrace{\left(\frac{\varepsilon\cdot e^{3/2}}{8MQa_i(3/2)^{3/2}}\right)^2}_{(C2)}\Bigg\}.$$

Several norms can be handled simultaneously:

$$\max\{\|\varphi\psi A\|,\|\varphi'\psi A\|,\|\varphi\psi' A\|,\|\varphi\psi'' A\|,\|\varphi'\psi' A\|\}$$
$$\le\sum_{i=1}^{M}2M(16Q^2+12Q)a_i|x-y_i|\cdot e^{\frac{-(x-y_i)^2}{\eta_i}}\le\varepsilon$$

by Eq. (4.3) and $(C1)$ (with the choice $p=1,q=0$ and $c=2M(16Q^2+12Q)a_i$). Two more norms can be handled together:

$$\max\{\|\varphi\psi A'\|,\|\varphi\psi' A'\|\}\le\sum_{i=1}^{M}8Qa_i\frac{|x-y_i|^3}{\eta_i}\cdot e^{\frac{-(x-y_i)^2}{\eta_i}}\le\varepsilon$$

by Eq. (4.3) and $(C2)$ (with the choice $p=3,q=-1$ and $c=8Qa_i$). These bounds already imply that $\|g-f\|\le(e^{\varepsilon}-1)\cdot\|f\|$ and $\|g'-f'\|\le(e^{\varepsilon}-1)\cdot\|f'(x)\|+3\varepsilon e^{\varepsilon}\cdot\|f(x)\|$. Also, by choosing the values of η_i possibly smaller, one can ensure that $g(\overline{\mathcal{B}_\epsilon})\subseteq\mathcal{B}_\epsilon$ and $0<|g'(x)|<1$ for every $x\in\overline{\mathcal{B}_\epsilon}$, hence, g satisfies (B) and (C), and in particular, $g\in\mathcal{S}^{\omega}_{\epsilon}([0,1])$.

The remaining three norms, $\|\varphi''\psi A\|,\|\varphi\psi A''\|$ and $\|\varphi'\psi A'\|$ require additional care. The trivial bounds can not be blindly used in some of the expressions when x is too close to one of the y_i. We demonstrate this on $\|\varphi''\psi A\|$ and leave the other two to the reader since the arguments are analogous.

Besides $\eta_i\le\min\{(C1),(C2)\}$, we need further restrictions on η_i. Assume that

$$\eta_i\le\min\Bigg\{\underbrace{\left(\frac{\varepsilon\sqrt{2e}}{4M^3a_i}\right)^2}_{(C3)},\underbrace{\frac{\varepsilon e}{2M^2a_i}}_{(C4)}\Bigg\},\qquad \eta_i^{1/3}<\frac{1}{2}\min\{y_{i+1}-y_i,y_i-y_{i-1}\}\tag{4.5}$$

and

$$\sum_{i=1}^{M}2a_i\cdot e^{-\eta_i^{-1/3}}<\varepsilon.\tag{4.6}$$

Clearly all these conditions can be simultaneously satisfied. Using the bound on $\|\varphi''\|$,

$$\left|\varphi''(x)\cdot\psi(x)\cdot A(x)\right| \leq \left|2\psi(x)\cdot A(x)\cdot\sum_{i=1}^{M}\prod_{\substack{y_j\in\mathcal{Y}\\ y_j\neq y_i}}(x-y_j)^2\right| + 4M^2\cdot\sum_{i=1}^{M}a_i|x-y_i|\cdot e^{\frac{-(x-y_i)^2}{\eta_i}}.$$

The second term is $\leq \varepsilon$ because we can apply Eq. (4.3) and $(C3)$. The first term is a double sum which we split into two parts

$$\underbrace{2\psi(x)\cdot\sum_{i=1}^{M}a_i e^{\frac{-(x-y_i)^2}{\eta_i}}\prod_{\substack{y_j\in\mathcal{Y}\\ y_j\neq y_i}}(x-y_j)^2}_{=:I(x)} + 2\sum_{k=1}^{M}a_k e^{\frac{-(x-y_k)^2}{\eta_k}}\underbrace{\sum_{\substack{i=1\\ i\neq k}}^{M}\prod_{\substack{y_j\in\mathcal{Y}\\ y_j\neq y_i}}(x-y_j)^2}_{\leq M(x-y_k)^2}.$$

We can apply Eq. (4.3) again to the second term and then $(C4)$ to see that the second term is bounded above by ε. What remains is to bound $I(x)$. This is where we distinguish whether x is close to a y_i or not. Recall, we assume Eq. (4.5). If x is not too close to any of the y_i in the sense that $x \in \bigcap_{i=1}^{M}(y_i - \eta_i^{1/3}, y_i + \eta_i^{1/3})^C$, then we use the trivial bounds

$$I(x) \leq \sum_{i=1}^{M} 2a_i e^{-\eta_i^{-1/3}} \overset{(4.6)}{<} \varepsilon.$$

So assume $x \in (y_i - \eta_i^{1/3}, y_i + \eta_i^{1/3})$ for some $y_i \in \mathcal{Y}$. Since x is still far enough from the other y_j, we just use the same bound there:

$$I(x) \leq 2a_i\psi(x)\prod_{y_j\in\mathcal{Y}\setminus\{y_i\}}(x-y_j)^2 + \sum_{\substack{j=1\\ j\neq i}}^{M}2a_j e^{-\eta_j^{-1/3}} \overset{(4.6)}{<} \varepsilon + 2a_i\psi(x)\prod_{y_j\in\mathcal{Y}\setminus\{y_i\}}(x-y_j)^2.$$

In the final term, we substitute the value of a_i from Eq. (4.4) and $\psi(x)$ to get

$$\begin{aligned} I(x) &\leq \varepsilon + \delta\cdot\frac{|f'(y_i)|}{|f(y_i)|}\prod_{\substack{y_j\in\mathcal{Y}\\ y_j\neq y_i}}\frac{(x-y_j)^2}{(y_i-y_j)^2}\prod_{z_j\in\mathcal{Z}}\frac{(x-z_j)^4}{(y_i-z_j)^4} \\ &\overset{(4.5)}{\leq} \varepsilon + \delta\cdot\frac{|f'(y_i)|}{|f(y_i)|}\prod_{\substack{y_j\in\mathcal{Y}\\ y_j\neq y_i}}\left(1+\frac{\eta_i^{1/3}}{|y_i-y_j|}\right)^2\prod_{z_j\in\mathcal{Z}}\left(1+\frac{\eta_i^{1/3}}{|y_i-z_j|}\right)^4. \end{aligned}$$

We may assume by choosing η_i even smaller if necessary that the product of the final two products is at most say 2. Since we also assume that $f([0,1]) \subset (0,1)$ and $0 < |f'(x)| < 1$ for every x, we have shown that $I(x) \leq C\cdot\delta + \varepsilon$ for some constant $C > 0$ depending only f. This completes the bound for $\|\varphi''\psi A\|$. ☐

4.2 Proof of Theorem 1.4

The main idea of the proof of Theorem 1.4 is to apply Proposition 4.2 to each map f_i of the IFS with appropriately chosen collections of points $\mathcal{Y}_i$ and $\mathcal{Z}_i$ using Lemma 4.1 to get the maps $(g_i)_{i\in\mathcal{I}}$. We then lift the IFS $(g_i)_{i\in\mathcal{I}}$ as in Eq. (2.1) to obtain the dual IFS $(G_i)_{i\in\mathcal{I}}$ and show that this IFS satisfies the SSC. Then Theorem 1.4 follows immediately from Theorem 2.2 and Lemma 2.6.

Proof of Theorem 1.4. By Theorem 2.2, it is enough to show that $\{\Phi \in \mathfrak{S}_N : \Phi^* \text{ satisfies the SSC}\}$ is open and dense in $\mathfrak{S}_N$ with respect to the metric d_2. The set is open by Lemma 2.6 and so it is enough to show that it is dense.

Let $\delta > 0$ and $\Phi = (f_i)_{i\in\mathcal{I}} \in \mathfrak{S}_N$ be arbitrary but fixed. Choose $n \geq 1$ such that $c_{\max}^n(K-k) < \delta/3$, where $k < K$ is chosen such that $F_i(k,K) \subseteq (k,K)$ for every $i \in \mathcal{I}$, where $\Phi^* = (F_i)_{i\in\mathcal{I}}$ is the dual IFS of Φ. Recall that, $\mathcal{B}_n = \{(\mathbf{i},\mathbf{j}) \in \Sigma_n \times \Sigma_n : i_1 < j_1\}$. Let $\{x_{\mathbf{i},\mathbf{j}} : (\mathbf{i},\mathbf{j}) \in \mathcal{B}_n\}$ be points as in Lemma 4.1.

Recall $\mathcal{O}_{\mathbf{i}}(x) = \{x, f_{i_1}(x), f_{i_2 i_1}(x), \ldots, f_{i_{|\mathbf{i}|}\ldots i_1}(x)\}$. For every $i \in \mathcal{I}$, we define a partition of $\bigcup_{(\mathbf{i},\mathbf{j})\in\mathcal{B}_n} \big(\mathcal{O}_{\mathbf{i}}(x_{\mathbf{i},\mathbf{j}}) \cup \mathcal{O}_{\mathbf{j}}(x_{\mathbf{i},\mathbf{j}})\big)$ consisting of two elements $\{\mathcal{Y}_i, \mathcal{Z}_i\}$ as follows:

- $x_{\mathbf{i},\mathbf{j}} \in \mathcal{Y}_{i_1}$ and $x_{\mathbf{i},\mathbf{j}} \in \mathcal{Z}_{j_1}$ if $(\mathbf{i},\mathbf{j}) \in \mathcal{B}_n$ is bad and
$$(F_{\mathbf{i}}k)(x_{\mathbf{i},\mathbf{j}}) \in \operatorname{conv}\left((F_{\mathbf{j}}k)(x_{\mathbf{i},\mathbf{j}}), (F_{\mathbf{j}}K)(x_{\mathbf{i},\mathbf{j}})\right);$$
- $x_{\mathbf{i},\mathbf{j}} \in \mathcal{Z}_{i_1}$ and $x_{\mathbf{i},\mathbf{j}} \in \mathcal{Y}_{j_1}$ if $(\mathbf{i},\mathbf{j}) \in \mathcal{B}_n$ is bad and
$$(F_{\mathbf{i}}k)(x_{\mathbf{i},\mathbf{j}}) \notin \operatorname{conv}\left((F_{\mathbf{j}}k)(x_{\mathbf{i},\mathbf{j}}), (F_{\mathbf{j}}K)(x_{\mathbf{i},\mathbf{j}})\right);$$
- $x_{\mathbf{i},\mathbf{j}} \in \mathcal{Z}_{i_1}$ and $x_{\mathbf{i},\mathbf{j}} \in \mathcal{Z}_{j_1}$ if $(\mathbf{i},\mathbf{j}) \in \mathcal{B}_n$ is not bad;
- $y \in \mathcal{Z}_{i_1}$ and $y \in \mathcal{Z}_{j_1}$ for every $y \in \bigcup_{(\mathbf{i},\mathbf{j})\in\mathcal{B}_n} \big(\mathcal{O}_{\mathbf{i}}(x_{\mathbf{i},\mathbf{j}}) \cup \mathcal{O}_{\mathbf{j}}(x_{\mathbf{i},\mathbf{j}})\big) \setminus \{x_{\mathbf{i},\mathbf{j}}\}$.

Recall that by Eq. (4.1) either

$$(F_{\mathbf{i}}k)(x_{\mathbf{i},\mathbf{j}}) \in \operatorname{conv}\left((F_{\mathbf{j}}k)(x_{\mathbf{i},\mathbf{j}}), (F_{\mathbf{j}}K)(x_{\mathbf{i},\mathbf{j}})\right) \quad \text{or} \quad (F_{\mathbf{j}}k)(x_{\mathbf{i},\mathbf{j}}) \in \operatorname{conv}\left((F_{\mathbf{i}}k)(x_{\mathbf{i},\mathbf{j}}), (F_{\mathbf{i}}K)(x_{\mathbf{i},\mathbf{j}})\right).$$

Hence, the sets $\mathcal{Y}_i$ and $\mathcal{Z}_i$ are well defined. We are now ready to apply Proposition 4.2.

To each f_i and $\mathcal{Y}_i, \mathcal{Z}_i$ we obtain a map $g_i \in \mathcal{S}_\epsilon^\omega([0,1])$ which satisfies the properties listed in Proposition 4.2 with the choice $\varepsilon = \delta$, and let $\Psi = (g_i)_{i\in\mathcal{I}} \in \mathfrak{S}_N$. We construct the dual IFS $\Psi^* = (G_i)_{i\in\mathcal{I}}$ of Ψ as in Eq. (2.1), i.e.

$$(G_i h)(x) := g_i'(x) \cdot h(g_i(x)) + \frac{g_i''(x)}{g_i'(x)}.$$

By Proposition 4.2, if $(\mathbf{i},\mathbf{j}) \in \mathcal{B}_n$ is not bad then $x_{\mathbf{i},\mathbf{j}} \in \mathcal{Z}_{i_1}$ and $x_{\mathbf{i},\mathbf{j}} \in \mathcal{Z}_{j_1}$. Hence,

$$(G_{\mathbf{i}}h)(x_{\mathbf{i},\mathbf{j}}) = (F_{\mathbf{i}}h)(x_{\mathbf{i},\mathbf{j}}) \text{ and } (G_{\mathbf{j}}h)(x_{\mathbf{i},\mathbf{j}}) = (F_{\mathbf{j}}h)(x_{\mathbf{i},\mathbf{j}}) \text{ for every } h \in \mathcal{C}_\epsilon^\omega([0,1]),$$

and in particular, $\operatorname{conv}\big((G_{\mathbf{i}}k)(x_{\mathbf{i},\mathbf{j}}), (G_{\mathbf{i}}K)(x_{\mathbf{i},\mathbf{j}})\big) \cap \operatorname{conv}\big((G_{\mathbf{j}}k)(x_{\mathbf{i},\mathbf{j}}), (G_{\mathbf{j}}K)(x_{\mathbf{i},\mathbf{j}})\big) = \varnothing$, or equivalently, $G_{\mathbf{i}}(k,K) \cap G_{\mathbf{j}}(k,K) = \varnothing$.

Let us now suppose that $(\mathbf{i},\mathbf{j}) \in \mathcal{B}_n$ is bad. Without loss of generality, we may assume that $x_{\mathbf{i},\mathbf{j}} \in \mathcal{Y}_{i_1}$ and $x_{\mathbf{i},\mathbf{j}} \in \mathcal{Z}_{j_1}$. Then

$$(F_{\mathbf{i}_2^{|\mathbf{i}|}}h)(f_{i_1}(x_{\mathbf{i},\mathbf{j}})) = (G_{\mathbf{i}_2^{|\mathbf{i}|}}h)(g_{i_1}(x_{\mathbf{i},\mathbf{j}})) \text{ and } (F_{\mathbf{j}}h)(x_{\mathbf{i},\mathbf{j}}) = (G_{\mathbf{j}}h)(x_{\mathbf{i},\mathbf{j}}) \text{ for every } h \in \mathcal{C}_\epsilon^\omega([0,1]),$$

and so,

$$\left|(G_{\mathbf{i}}h)(x_{\mathbf{i},\mathbf{j}}) - (F_{\mathbf{i}}h)(x_{\mathbf{i},\mathbf{j}})\right| = \left|\frac{g_{i_1}''(x_{\mathbf{i},\mathbf{j}})}{g_{i_1}'(x_{\mathbf{i},\mathbf{j}})} - \frac{f_{i_1}''(x_{\mathbf{i},\mathbf{j}})}{f_{i_1}'(x_{\mathbf{i},\mathbf{j}})}\right| \geq \delta \text{ for every } h \in \mathcal{C}_\epsilon^\omega([0,1]). \tag{4.7}$$

Since $\operatorname{conv}((F_{\mathbf{j}}k)(x_{\mathbf{i},\mathbf{j}}),(F_{\mathbf{j}}K)(x_{\mathbf{i},\mathbf{j}})) = \operatorname{conv}((G_{\mathbf{j}}k)(x_{\mathbf{i},\mathbf{j}}),(G_{\mathbf{j}}K)(x_{\mathbf{i},\mathbf{j}}))$ has length strictly less than $\delta/3$, and $(F_{\mathbf{i}}h)(x_{\mathbf{i},\mathbf{j}}) \in \operatorname{conv}((F_{\mathbf{j}}k)(x_{\mathbf{i},\mathbf{j}}),(F_{\mathbf{j}}K)(x_{\mathbf{i},\mathbf{j}}))$, Eq. (4.7) implies that

$$\operatorname{dist}((G_{\mathbf{i}}k)(x_{\mathbf{i},\mathbf{j}})), \operatorname{conv}((G_{\mathbf{j}}k)(x_{\mathbf{i},\mathbf{j}}),(G_{\mathbf{j}}K)(x_{\mathbf{i},\mathbf{j}}))) > 2\delta/3.$$

On the other hand, $|(G_{\mathbf{i}}k)(x_{\mathbf{i},\mathbf{j}})) - (G_{\mathbf{i}}K)(x_{\mathbf{i},\mathbf{j}}))| \leq c_{\max}^n(K-k) < \delta/3$, and so

$$\operatorname{dist}((G_{\mathbf{i}}K)(x_{\mathbf{i},\mathbf{j}})), \operatorname{conv}((G_{\mathbf{j}}k)(x_{\mathbf{i},\mathbf{j}}),(G_{\mathbf{j}}K)(x_{\mathbf{i},\mathbf{j}}))) > \delta/3.$$

This clearly implies that $G_{\mathbf{i}}(k,K) \cap G_{\mathbf{j}}(k,K) = \varnothing$. Finally, Theorem 1.4 concludes by Lemma 2.5. □

5 Conjugation to self-similar IFS

Let $f \in \mathcal{S}_\epsilon^\omega([0,1])$ be arbitrary but fixed. Along the lines of Eq. (1.4), let us define the map

$$\hat{H}_f(x) := \sum_{k=0}^{\infty} \frac{f''}{f'}(f^{\circ k}(x)) \cdot (f^{\circ k})'(x), \tag{5.1}$$

where $f^{\circ k}$ denotes the self-composition of f k-times. Analogously to the proof of Lemma 2.4, $\hat{H} \in \mathcal{C}_\epsilon^\omega([0,1])$.

Let us begin the proof of Theorem 1.12 with the following observation.

Lemma 5.1. *Let $f \in \mathcal{S}_\epsilon^\omega([0,1])$. Then for every $a, b \in \mathbb{R}$ there exists an invertible map $g \in \mathcal{C}_\epsilon^\omega([0,1])$ such that $g''(x) = \hat{H}_f(x)g'(x)$, $g(p) = a$ and $g'(p) = b$, where p is the unique fixed point of f in $[0,1]$ and $\hat{H}_f$ is defined in Eq. (5.1).*

Moreover, for every $g\colon [0,1] \to \mathbb{R}$ such that $g''(x) \equiv \hat{H}_f(x)g'(x)$

$$g(f(x)) = f'(p)g(x) + g(p)(1-f'(p)).$$

Proof. Using the analyticity of $\hat{H}$, we get that the map $g(x) = b\int_p^x e^{\int_p^z \hat{H}(y)dy}dz + a$ is in $\mathcal{C}_\epsilon^\omega([0,1])$ with $g(p) = a$ and $g'(p) = b$.

Now, integrating $\hat{H}$

$$\int_p^x \hat{H}(y)dy = \sum_{k=0}^{\infty}\left(\log\left(f'(f^{\circ k}(x))\right) - \log f'(p)\right) = \log\left(\prod_{k=0}^{\infty}\frac{f'(f^{\circ k}(x))}{f'(p)}\right).$$

Moreover, letting

$$\hat{g}(x) := \int_p^x e^{\int_p^z \hat{H}(y)dy}dz,$$

and using the previous equation, we get that

$$\hat{g}(x) = \lim_{n\to\infty}\frac{(f^{\circ n})(x) - p}{(f'(p))^n} \tag{5.2}$$

is also analytic.

Using Eq. (5.2), it is easy to see that $\hat{g}(f(x)) \equiv f'(p)\hat{g}(x)$. Also, for any map g with $g''(x) \equiv \hat{H}_f(x)g'(x)$, we have $g(x) \equiv g'(p)\hat{g}(x) + g(p)$ for every $x \in [0,1]$. Thus,

$$g(f(x)) = g'(p)\cdot\hat{g}(f(x)) + g(p) = g'(p)\cdot f'(p)\hat{g}(x) + g(p) = f'(p)\cdot g(x) + g(p)(1-f'(p)),$$

which had to be proven. □

Proof of Theorem 1.12. Let $\Phi = (f_i)_{i\in\mathcal{I}} \in \mathfrak{S}_N$. The first assertion of Theorem 1.12 follows by applying Lemma 5.1 for f_1 and considering the IFS $(g \circ f_i \circ g^{-1})_{i\in\mathcal{I}}$.

The assertion (b) of Theorem 1.12 clearly follows by (a), so we finish the proof by showing (a).

Let $\Phi = (f_i)_{i\in\mathcal{I}} \in \mathfrak{S}_N$. First, suppose that $H(x) \coloneqq H_{\mathbf{i}}(x) \equiv H_{\mathbf{j}}(x)$ for every $\mathbf{i}, \mathbf{j} \in \Sigma$, where $H_{\mathbf{i}}$ is the dual natural projection defined in Eq. (1.4). Then $H(x) = H_{(i)^\infty} = \hat{H}_{f_i}$ for every $i \in \mathcal{I}$. Hence, by Lemma 5.1 there exists $g\colon [0,1] \to \mathbb{R}$ analytic such that $g''(x) \equiv H(x)g'(x)$, and $g(f_i(x)) = f_i'(p_i)g(x) + g(p_i)(1 - f_i'(p_i))$ for every $i \in \mathcal{I}$, where $f_i(p_i) = p_i$.

Finally, let us suppose that $\Phi = (f_i)_{i\in\mathcal{I}} \in \mathfrak{S}_N$ is conjugated to a self-similar IFS $(x \mapsto \lambda_i x + t_i)_{i\in\mathcal{I}}$ by the invertible analytic map $g\colon [0,1] \to \mathbb{R}$. Let $p_{\mathbf{i}}$ be the fixed point of $f_{\mathbf{i}}$ for every $\mathbf{i} \in \Sigma_*$. Then,

$$g(f_{\mathbf{i}}(x)) = \lambda_{\mathbf{i}} g(x) + t_{\mathbf{i}} \quad \text{and} \quad g(p_{\mathbf{i}}) = \frac{t_{\mathbf{i}}}{1 - \lambda_{\mathbf{i}}}.$$

We have

$$g'(f_{\mathbf{i}}(x)) \cdot f_{\mathbf{i}}'(x) = \lambda_{\mathbf{i}} g'(x) \quad \text{and} \quad |g'(p_{\mathbf{i}})||f_{\mathbf{i}}'(p_{\mathbf{i}}) - \lambda_{\mathbf{i}}| = 0.$$

Since $|g'(p_{\mathbf{i}})| > 0$ we must have $f_{\mathbf{i}}'(p_{\mathbf{i}}) = \lambda_{\mathbf{i}}$. Differentiating again we get

$$g''(f_{\mathbf{i}}(x)) \cdot f_{\mathbf{i}}'(x)^2 + g'(f_{\mathbf{i}}(x)) \cdot f_{\mathbf{i}}''(x) = \lambda_{\mathbf{i}} g''(x) \text{ for every } x \in [0,1]$$

and by Using Eq. (2.7), we have that

$$\frac{g''(p_{\mathbf{i}})}{g'(p_{\mathbf{i}})} = \frac{f_{\mathbf{i}}''(p_{\mathbf{i}})}{f_{\mathbf{i}}'(p_{\mathbf{i}})(1 - f_{\mathbf{i}}'(p_{\mathbf{i}}))} = \frac{H_{\mathbf{i}_{|\mathbf{i}|}^1}(p_{\mathbf{i}})}{1 - f_{\mathbf{i}}'(p_{\mathbf{i}})}.$$

Now, let $\mathbf{i}, \mathbf{j} \in \Sigma$ be arbitrary but fixed, and let $\mathbf{k}_n = \mathbf{i}_1^n \mathbf{j}_n^1$. Then $p_{\mathbf{k}_n} \to \pi(\mathbf{i})$ as $n \to \infty$, where we recall that $\pi\colon \Sigma \to \mathbb{R}$ is the natural projection of Φ defined in Eq. (1.3). Furthermore, by Lemma 2.4 and by $f_{\mathbf{k}_n}'(p_{\mathbf{k}_n}) \to 0$ as $n \to \infty$

$$\frac{g''(p_{\mathbf{k}_n})}{g'(p_{\mathbf{k}_n})} = \frac{H_{\mathbf{j}_1^n \mathbf{i}_n^1}(p_{\mathbf{k}_n})}{1 - f_{\mathbf{k}_n}'(p_{\mathbf{k}_n})} \to H_{\mathbf{j}}(\pi(\mathbf{i})) \quad \text{as} \quad n \to \infty.$$

However, the left-hand side converges to $g''(\pi(\mathbf{i}))/g'(\pi(\mathbf{i}))$, and so, we get

$$\frac{g''}{g'}(\pi(\mathbf{i})) = H_{\mathbf{j}}(\pi(\mathbf{i}))$$

for every $\mathbf{i}, \mathbf{j} \in \Sigma$. In particular, $H_{\mathbf{j}}(\pi(\mathbf{i})) = H_{\mathbf{k}}(\pi(\mathbf{i}))$ for every $\mathbf{i}, \mathbf{j}, \mathbf{k} \in \Sigma$. Since the attractor of Φ is not a singleton (i.e. uncountable), the maps $H_{\mathbf{j}}$ are analytic, we have that $H_{\mathbf{j}}(x) \equiv H_{\mathbf{k}}(x)$ for all $\mathbf{j}, \mathbf{k} \in \Sigma$. □

Proof of Theorem 2.3. The claim follows by Theorem 1.12 and Lemma 2.4. □

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