UNIVERSAL PROJECTION THEOREMS WITH APPLICATIONS TO MULTIFRACTAL ANALYSIS AND THE DIMENSION OF EVERY ERGODIC MEASURE ON SELF-CONFORMAL SETS SIMULTANEOUSLY

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Abstract. We prove a *universal projection theorem*, which gives conditions on a parametrized family of maps $\Pi_\lambda : X \to \mathbb{R}^d$ and a collection $\mathcal{M}$ of measures on X under which for almost every λ equality $\dim_H \Pi_\lambda \mu = \min\{d, \dim_H \mu\}$ holds for all measures $\mu \in \mathcal{M}$ **simultaneously** (i.e. on a full measure set of λ's independent of μ). We require family Π_λ to satisfy a transversality condition and collection $\mathcal{M}$ to satisfy a new condition called *relative dimension separability*. Under the same assumptions, we also prove that if the Assouad dimension of X is smaller than d, then for almost every λ, projection Π_λ is nearly bi-Lipschitz (i.e. pointwise α-Hölder for every $\alpha \in (0,1)$) at μ-a.e. x, for all measures $\mu \in \mathcal{M}$ simultaneously. Our setting is general enough to include families of orthogonal projections, natural projections corresponding to conformal iterated functions systems, as well as non-autonomous or random IFS.

As applications, we study families of ergodic measures on self-conformal sets. We prove that given a parametrized family of contracting conformal $C^{1+\theta}$ IFS $\mathcal{F}_\lambda = \{f_i^\lambda \colon \mathbb{R}^d \to \mathbb{R}^d\}_{i\in\mathcal{A}}$ satisfying the transversality condition, for almost every parameter λ one has $\dim_H \Pi_\lambda \mu = \min\{d, \frac{h(\mu)}{\chi(\lambda,\mu)}\}$ for all ergodic shift-invariant measures on $\mathcal{A}^\mathbb{N}$ simultaneously, i.e. with a common exceptional set of parameters λ of zero measure (here $h(\mu)$ is the entropy of μ and $\chi(\lambda,\mu)$ is its Lyapunov exponent).

As a second application, we prove that given a self-similar system $\{x \mapsto \lambda_i x + t_i\}_{i\in\mathcal{A}}$ on the line with similarity dimension $s_0 < 1$, Lebesgue almost every choice of translations t_i has the property that the multifractal formalism holds on the full spectrum interval $\left[\min\limits_{i\in\mathcal{A}} \frac{\log p_i}{\log|\lambda_i|}, \max\limits_{i\in\mathcal{A}} \frac{\log p_i}{\log|\lambda_i|}\right]$ for every self-similar measure corresponding to a probability vector $(p_i)_{i\in\mathcal{A}}$ simultaneously.

We also prove that the dimension part of the Marstrand-Mattila projection theorem holds simultaneously for the collection of all ergodic measures on a self-conformal set satisfying the strong separation condition and for the collection of all Gibbs measures on a self-conformal set (without any separation).

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1. Introduction

Determining the dimension of certain objects, like sets and measures, plays a crucial role in geometric measure theory and fractal geometry, just like understanding how different actions change the value of the dimension. The classical projection theorem, originated to Marstrand [Mar54] and later generalized in several ways e.g. by Falconer [Fal82], Kaufman [Kau68], Mattila [Mat75], claims that the dimension of the orthogonal projection of a set in a typical direction (in some proper sense) does not drop with respect to the natural upper bound (the minimum of the dimension of the space, where we project, and the dimension of the projected set). For us, the most relevant is a version for measures [HT94, HK97, SY97], which states that for every finite Borel measure μ on $\mathbb{R}^n$

$$\overline{\dim}_H(P_V\mu) = \min\{d, \overline{\dim}_H\mu\} \text{ and } \underline{\dim}_H(P_V\mu) = \min\{d, \underline{\dim}_H\mu\} \text{ for } \gamma\text{-a.e. } V \in \mathrm{Gr}(d,n), \tag{1.1}$$

where $\overline{\dim}_H$ and $\underline{\dim}_H$ denote the upper and lower Hausdorff dimensions (see Section 4 for definitions), $\mathrm{Gr}(d,n)$ is the Grassmannian manifold of d-dimensional linear subspaces in $\mathbb{R}^N$ endowed with the unique $O(n)$-invariant probability measure γ (see [Mat95, Section 3.9]) and P_V denotes the orthogonal projection onto $V \in \mathrm{Gr}(d,n)$. These works have been the "precursor" of the transversality method to handling the overlaps for iterated function systems.

Let $\mathcal{A}$ be a finite collection of indices and let $\mathcal{F} = \{f_i : \mathbb{R}^d \to \mathbb{R}^d\}_{i\in\mathcal{A}}$ be a finite collection of contracting maps, called iterated function system (IFS), and let $p = (p_i)_{i\in\mathcal{A}}$ be a probability measure on $\mathcal{A}$. Then there exists a unique non-empty compact set Λ and a unique compactly supported probability measure ν such that X is invariant with respect to $\mathcal{F}$ in the sense that $\Lambda = \bigcup_{i\in\mathcal{A}} f_i(\Lambda)$ and ν is stationary, i.e. $\nu = \sum_{i\in\mathcal{A}} p_i (f_i)_*\nu$, see Hutchinson [Hut81]. There is a natural correspondence between the points of Λ and infinite sequences formed by $\mathcal{A}$. More precisely, there is a Lipschitz map $\Pi : \mathcal{A}^{\mathbb{N}} \to \Lambda$ defined by $\Pi(u_1, u_2, \ldots) = \lim_{n\to\infty} f_{u_1} \circ \cdots \circ f_{u_n}(0)$, called the natural projection. It satisfies $\Lambda = \Pi(\mathcal{A}^{\mathbb{N}})$ and stationary measure ν can be obtained as the projection of the Bernoulli measure $p^{\mathbb{N}}$ on $\mathcal{A}^{\mathbb{N}}$, i.e. $\nu = \Pi(p^{\mathbb{N}})$. Calculating the dimension of Λ and ν is a challenging problem in general, even in the case, when the maps of $\mathcal{F}$ are similarities or conformal maps. In case of a separation condition, for example if the open set condition holds, then the dimension (both box and Hausdorff) of Λ equals to the similarity dimension, and also, the dimension of ν is given by the ratio of the entropy and the Lyapunov exponents - see Example 3.3 for details and precise definitions.

The transversality method was first applied by Pollicott and Simon [PS95] to the case of iterated function systems, and later several generalizations have appeared, see for instance Peres and Solomyak [PS96], Solomyak [Sol98], Peres and Schlag [PS00], Simon, Solomyak and Urbański [SSU01a, SSU01b], Bárány, Simon, Solomyak and Śpiewak [BSSŚ22, BSSŚ24]. See also [Sol23] for a recent survey and [BSS23] for an in-depth discussion. Roughly speaking, the dimensional parts of these statements can be summarized as follows. Let $\mathcal{F}_\lambda = \{f_i^\lambda : \mathbb{R}^d \to \mathbb{R}^d\}_{i\in\mathcal{A}}$, $\lambda \in U$ be a family of parametrized conformal iterated function systems, which satisfies the transversality condition with respect to a probability measure η on U (see precise definition later). Let Π_λ be the natural projection map corresponding to $\mathcal{F}_\lambda$ and let μ be an ergodic left-shift invariant measure $\mathcal{A}^{\mathbb{N}}$. Then

$$\dim_H(\Pi_\lambda\mu) = \min\left\{d, \frac{h(\mu)}{\chi(\mu,\lambda)}\right\} \text{ for } \eta\text{-a.e. } \lambda \tag{1.2}$$

where $h(\mu)$ denotes the entropy of μ and $\chi(\mu,\lambda)$ denotes the Lyapunov exponent of μ with respect to $\mathcal{F}_\lambda$. As explained in Example 3.3, any result of the type (1.2) can be seen as an IFS analogue of the Marstrand-Mattila projection theorem (1.1).

The goal of this paper is study conditions under which (1.1) and (1.2) can be proved to hold for *all* measures in a given class of measures $\mathcal{M}$ *simultaneously*, i.e. with a measure zero set of exceptional projections independent of measure $\mu \in \mathcal{M}$. For instance, in the context of orthogonal projections we look for assumptions on a set $\mathcal{M} \subset \mathcal{M}_{\text{fin}}(\mathbb{R}^n)$ of finite Borel measures on $\mathbb{R}^n$ so that (1.1) can be improved to

$$\gamma\left(\{V \in \mathrm{Gr}(d,n) : \text{ there exists } \mu \in \mathcal{M} \text{ such that } \dim_H(P_V\mu) < \min\{d, \dim_H \mu\}\}\right) = 0. \tag{1.3}$$

Clearly, this question is relevant only for uncountable families $\mathcal{M}$ and it is easy to find examples of collections $\mathcal{M}$ for which (1.3) does not hold (see Example 3.2). Our main result (Theorem 2.7) implies that (1.3) holds if $\mathcal{M}$ satisfies a condition which we call *relative dimension separability* - see Definition 2.3. We can verify it in two important cases of dynamical origin, giving that

- (i) (1.3) holds if $\mathcal{M}$ is the family of all ergodic invariant measures on a self-conformal set satisfying the strong separation condition - see Theorem 3.5,
- (ii) (1.3) holds if $\mathcal{M}$ is the family of all Gibbs measures (corresponding to all Hölder continuous potentials) on a self-conformal set (without any separation conditions) - see Theorem 3.6.

For parametrized families of $C^{1+\theta}$ conformal IFS $\mathcal{F}_\lambda$ satisfying the transversality condition, we obtain a version of (1.4) which holds simultaneously for all ergodic measures on $\mathcal{A}^{\mathbb{N}}$:

$$\eta\left(\left\{\lambda \in U : \text{ there exists } \mu \in \mathcal{E}_\sigma(\mathcal{A}^{\mathbb{N}}) \text{ such that } \dim_H(\Pi_\lambda\mu) < \min\left\{d, \frac{h(\mu)}{\chi(\mu,\lambda)}\right\}\right\}\right) = 0, \tag{1.4}$$

where $\mathcal{E}_\sigma(\mathcal{A}^{\mathbb{N}})$ is the set of all shift-invariant ergodic Borel probability measures on $\mathcal{A}^{\mathbb{N}}$. See Theorem 3.11 for the precise statement.

Our motivation for studying simultaneous projection result like (1.4) is twofold. First and foremost, this approach has novel applications to multifractal analysis. We verify that the multifractal formalism holds on the full spectrum interval for typical self-similar measures with overlaps for almost every translation parameter - see Theorems 3.14 and 8.3. In dimension one, we prove that given a self-similar system $\{x \mapsto \lambda_i x + t_i\}_{i\in\mathcal{A}}$ on the line with similarity dimension s_0 smaller than one, then Lebesgue almost every choice of translations t_i has the property that the multifractal formalism holds on the full spectrum interval $\left[\min\limits_{i\in\mathcal{A}} \frac{\log p_i}{\log|\lambda_i|}, \max\limits_{i\in\mathcal{A}} \frac{\log p_i}{\log|\lambda_i|}\right]$ for every self-similar measure corresponding to a probability vector $p = (p_i)_{i\in\mathcal{A}}$ simultaneously.According to our best knowledge, it is the first result on the multifractal spectrum of self-similar measures without separation conditions which covers *the full spectrum* and improves in this regard recent results of Barral and Feng [BF21]. See Sections 3.4 and 8 for details. It is easy to extend our results to the Birkhoff spectra of continuous potentials. Our proofs of the multifractal results are based on a study of a separation property called the exponential distance from the enemy (EDE), which in terms of the natural projection Π_λ can be seen as a nearly bi-Lipschitz (α-Hölder for every $\alpha \in (0,1)$) continuity property of the inverse to Π_λ at almost every point - see Theorem 3.11 and Proposition 3.10. As the latter property is obtained in a general version which can be applied also to orthogonal projections in terms of the Assouad dimension, we obtain a simultaneous version of [BGŚ23, Theorem 1.7.(iii)].

The second goal of our study is to bridge a gap between results obtained with the use of transversality technique and more recent approach using additive combinatorics. In the case when the IFS consists of similarity mappings and the measure μ is Bernoulli (i.e. $\Pi\mu$ is a self-similar measure), the breakthrough result of Hochman [Hoc14] significantly strengthened results like (1.2). Namely, Hochman showed that if the IFS $\Phi = \{f_i(x) = \lambda_i x + t_i\}_{i\in\mathcal{A}}$ satisfies the exponential separation condition (ESC) then (1.2) holds for every Bernoulli measure μ. Furthermore, under sufficiently smooth (analytic) and non-degenerate parametrization, the ESC holds for almost every parameter (and often outside of a set of parameters of non-full dimension). Later, Jordan and Rapaport [JR21], relying on the result of Shmerkin [Shm19], showed that equality in (1.2) holds for every ergodic shift-invariant measure simultaneously provided that the system satisfies ESC. Our result (1.4) shows that one can achieve the same simultaneity among ergodic measures using the transversal technique. While transversality remains inferior to additive combinatorics methods in some regards (e.g. transversality is rarely able to cover the whole parameter range), it is still essentially the only technique which is able to produce similar results like (1.2) for families of general non-linear IFS. As for orthogonal projections, Bruce and Jin showed in [BJ19, Theorem 1.3] that given a self-conformal IFS on $\mathbb{R}^n$ which satisfies certain irrationality condition for the derivatives, one has $\dim_H P_V \mu = \min\{d, \dim_H \mu\}$ for every $V \in \mathrm{Gr}(d, n)$ and every Gibbs measure μ. This extends results of Hochman and Shmerkin [HS12]. Our Theorem 3.6 does not require any irrationality condition at the cost of allowing the dimension formula for a zero-measure set of projections.

We prove our main result, which we call a *universal projection theorem* in a general setting of a family of projection maps which can encompass both orthogonal projections and natural projections corresponding to a family of iterated projection schemes, as well as some other families like non-autonomous IFS, where in each iterate we choose different IFS with the same parametrization, see Nakajima [Nak24] or when in each iterate we add an independent, identically distributed error, see Jordan, Pollicott and Simon [JPS07], Koivusalo [Koi14] and Liu and Wu [LW03]. This setting is also capable of covering certain suitable parametrizations of delay-coordinate maps (at least for aperiodic systems [Rob11, Chapter 14]), which are used for proving time-delayed embeddings of dynamical systems [SYC91, Rob11], studied also in the probabilistic context (see e.g. [SY97, BGŚ20, BGŚ24]) but we do not pursue this direction in this work. Our general setting is similar to, but slightly different than, some other generalized projection schemes that appeared in the literature, e.g. in [Sol98, PS00] or [BSS23, Section 6.6]. Finally, the name *universal projection theorem* is inspired by seminal results and constructions known in information theory as *universal source codings*, which give compression algorithms operating in an optimal rate for any source distribution in a given class, without any prior knowledge of this distribution, see e.g. [CT06, Chapter 13]. A famous example is the Lempel-Ziv coding [ZL77, ZL78] achieving optimal rate for any stationary ergodic source distribution [CT06, Theorem 13.5.3].

2. Main result: a universal projection theorem

We shall now describe the setting in which we will prove a universal projection theorem. Let X be a compact topological space (the *phase space*) and let U be a hereditary Lindelöf topological space[1] (the *parameter space*). Let η be a locally finite Borel measure on U. For $\lambda \in U$ let ρ_λ be a metric on X compatible with its topology and let $B_\lambda(x, r)$ denote the open r-ball in metric ρ_λ. Fix $d \in \mathbb{N}$ and for each $\lambda \in U$ consider a map $\Pi_\lambda : X \to \mathbb{R}^d$ (*projection*). Let $\mathcal{M}_{\mathrm{fin}}(X)$ denote the set of all finite Borel measures on X and consider a collection $\mathcal{M} \subset \mathcal{M}_{\mathrm{fin}}(X)$. Our goal is to study projections $\Pi_\lambda \mu, \lambda \in U, \mu \in \mathcal{M}$ and provide projection theorems which hold for η-a.e. $\lambda \in U$ and *all* measures $\mu \in \mathcal{M}$ simultaneously. A crucial property needed for the proofs is a certain separability-like condition for the set $\mathcal{M}$ and a transversality condition for the family Π_λ. For the former one, we make the following definitions.

[1]topological space U is hereditary Lindelöf if every subset has the property that its every open cover has a countable subcover. Every separable metric space is hereditary Lindelöf, see e.g. [Eng89].

Definition 2.1. Let μ and ν be finite Borel measures on a metric space (X, ρ). The **relative dimension** of μ with respect to ν is

$$\dim(\mu||\nu, \rho) := \inf\{\varepsilon > 0 : -\varepsilon < \operatorname*{essinf}_{x\sim\mu} \liminf_{r\to 0} \frac{\log \frac{\mu(B(x,r))}{\nu(B(x,r))}}{\log r} \leq \operatorname*{esssup}_{x\sim\mu} \limsup_{r\to 0} \frac{\log \frac{\mu(B(x,r))}{\nu(B(x,r))}}{\log r} < \varepsilon\},$$

where $B(x, r)$ denotes the open r-ball in metric ρ.

Remark 2.2. We adopt the convention that $\log \frac{\mu(B(x,r))}{0} = +\infty$ if $\mu(B(x, r)) > 0$. In particular $\dim(\mu||\nu, \rho) = \infty$ if $\mu(X \setminus \operatorname{supp}(\nu)) > 0$.

Definition 2.3. Let (X, ρ) be a metric space. Let $\mathcal{M} \subset \mathcal{M}_{\text{fin}}(X)$ be a collection of finite Borel measures on X. We say that $\mathcal{M}$ is **relative dimension separable** (with respect to ρ) if there exists a countable set $\mathcal{V} \subset \mathcal{M}_{\text{fin}}(X)$ such that for every $\mu \in \mathcal{M}$ and $\varepsilon > 0$ there exists $\nu \in \mathcal{V}$ with $\dim(\mu||\nu, \rho) < \varepsilon$.

We will also require measures in $\mathcal{M}$ to satisfy a mild regularity condition.

Definition 2.4. A finite Borel measure μ on a metric space (X, ρ) is **weakly diametrically regular** if

$$\lim_{r\to 0} \frac{\log \frac{\mu(B(x,r))}{\mu(B(x,2r))}}{\log r} = 0 \text{ for } \mu\text{-a.e. } x \in X.$$

Remark 2.5. It is easy to see that every measure for which the local dimension exists at μ-a.e. point (see Definition 4.1) is weakly diametrically regular. Moreover, any finite Borel measure on $\mathbb{R}^n$ is weakly diametrically regular (with respect to the Euclidean metric) - see [BS01, Lemma 1].

Our principal assumptions on the families $\{\rho_\lambda : \lambda \in U\}$, $\{\Pi_\lambda : \lambda \in U\}$ and measure η are as follows:

(A1) for every $\lambda_0 \in U$ and every $\xi \in (0, 1)$ there exists a neighbourhood U' of λ_0 and a constant $0 < H = H(\xi, \lambda_0) < \infty$ such that for every $\lambda \in U$

$$H^{-1}\rho_\lambda(x, y)^{1+\xi} \leq \rho_{\lambda_0}(x, y) \leq H\rho_\lambda(x, y)^{1-\xi} \text{ holds for every } x, y \in X,$$

(A2) for each $\lambda \in U$, map Π_λ is Lipschitz in ρ_λ,

(A3) for every $\lambda_0 \in U$ and $\varepsilon > 0$ there exist a neighbourhood U' of λ_0 and a constant $K = K(\lambda_0, \varepsilon)$ such that for every $x, y \in X$, $r > 0, \delta > 0$

$$\eta(\{\lambda \in U' : |\Pi_\lambda(x) - \Pi_\lambda(y)| < \rho_\lambda(x, y)r \text{ and } \rho_\lambda(x, y) \geq \delta\}) \leq K\delta^{-\varepsilon}r^{d-\varepsilon}.$$

Assumption (A3) is a generalization of the classical transversality condition (see e.g. [BSS23, Section 14.4] or [Sol23]). We will provide an embedding theorem with the following regularity property for the embedding map.

Definition 2.6. Let (X, ρ_X) and (Y, d_Y) be metric spaces. Let $\Pi : X \to Y$ be a Lipschtiz map and let μ be a finite Borel measure on X. We say that Π is **μ-nearly bi-Lipschitz** if μ-a.e. $x \in X$ has the property that for every $\alpha \in (0, 1)$ there exists $C = C(x, \alpha)$ such that

$$\rho_X(x, y) \leq C\rho_Y(\Pi(x), \Pi(y))^\alpha \text{ for every } y \in X.$$

The main result of this paper is the following. For the definitions of Hausdorff and Assouad dimensions see Section 4.

Theorem 2.7. *Let X be a compact topological space and let U be a hereditary Lindelöf topological space. Let $\{\rho_\lambda : \lambda \in U\}$ be a family of metrics on X compatible with its topology and let $\{\Pi_\lambda : \lambda \in U\}$ be a family of maps $\Pi_\lambda : X \to \mathbb{R}^d$ satisfying assumptions (A1) - (A3). Let $\mathcal{M} \subset \mathcal{M}_{\text{fin}}(X)$ be a collection of finite Borel measures on X, such that for every ρ_λ, $\lambda \in U$, the family $\mathcal{M}$ is relative dimension separable and each $\mu \in \mathcal{M}$ is weakly diametrically regular. Then for η-a.e. $\lambda \in U$, the following holds simultaneously for all $\mu \in \mathcal{M}$:*

(1) $\underline{\dim}_H \Pi_\lambda \mu = \min\{d, \underline{\dim}_H(\mu, \rho_\lambda)\}$ *and* $\overline{\dim}_H \Pi_\lambda \mu = \min\{d, \overline{\dim}_H(\mu, \rho_\lambda)\}$,

(2) if $\dim_A(X, \rho_\lambda) < d$, *then* Π_λ *is* μ*-nearly bi-Lipschitz in metric* ρ_λ.

Here and in the rest of the paper, the precise meaning of *simultaneously for all* $\mu \in \mathcal{M}$ is, for instance in the case of the first part of point (1),

$$\eta\left(\left\{\lambda \in U : \underset{\mu \in \mathcal{M}}{\exists}\ \underline{\dim}_H \Pi_\lambda \mu \neq \min\{d, \underline{\dim}_H(\mu, \rho_\lambda)\}\right\}\right) = 0$$

and likewise for the other statements

3. Applications

Let us begin with listing several families of maps Π_λ and metrics ρ_λ satisfying conditions (A1) - (A3).

3.1. **Orthogonal projections.**

Example 3.1 (Orthogonal projections)**.** Fix $1 \leq d < n$. Let $\text{Gr}(d, n)$ be the Grassmannian of d-dimensional linear subspaces in $\mathbb{R}^n$. Let η be the unique Borel probability measure on $\text{Gr}(d, n)$ which is invariant under the action of the orthogonal group $O(d)$, see [Mat95, Section 3.9]. Given a compact set $X \subset \mathbb{R}^n$, for $V \in \text{Gr}(d, n)$ let $P_V : X \to \mathbb{R}^d$ denote the orthogonal projection onto V (which one can identify with $\mathbb{R}^d$). Let ρ be the Euclidean metric on X. Setting $U = \text{Gr}(d, n)$ and $\rho_V = \rho$ we obtain families $\{P_V : V \in U\}$, $\{\rho_V : V \in U\}$ satisfying (A1) - (A3), with (A3) following from [Mat95, Lemma 3.11]. That is, there exists a constant $c > 0$

$$\eta\left(\{V \in \text{Gr}(d, n) : \|P_V(x) - P_V(y)\| \leq r\|x - y\|\}\right) \leq c \cdot \min\{1, r^d\} \tag{3.1}$$

for every $x, y \in X$ and $r > 0$. ■

Unfortunately, given a compact set $X \subset \mathbb{R}^n$ one cannot expect the family of all finite Borel measures on X to be relative dimension separable, as conclusions of Theorem 2.7 might fail for it.

Example 3.2. Let X be the closed unit disc in $\mathbb{R}^2$ centred at zero. For $V \in \text{Gr}(1, 2)$, let μ_V be the 1-dimensional Hausdorff measure restricted to the unit interval passing through the origin and perpendicular to the subspace V. Then clearly $\dim_H \mu_V = 1$ but $\dim_H P_V \mu_V = 0$. Therefore the family $\{\mu_V : V \in \text{Gr}(1, 2)\}$ is not relative dimension separable and hence neither is the larger family $\mathcal{M}_{\text{fin}}(X)$. ■

3.2. **Conformal IFS.**

One may find relative dimension separable families within measures of dynamical origin. Our main focus is on ergodic measures on self-conformal sets. Let us now describe those.

For a compact connected set $V \subset \mathbb{R}^n$ with $V = \overline{\text{Int}(V)}$, a function $f : V \to V$ is called a conformal $C^{1+\theta}$ map if it extends to a diffeomorphism $f : W \to W$ of an open connected set $W \supset V$ such that for every $x \in V$ the differential $f'(x) = D_x f$ is a non-singular similitude and the map $V \ni x \mapsto D_x f$ is θ-Hölder for some $\theta > 0$. Note that in this case the operator norm $\|D_x f\|$ is simply the corresponding coefficient of similarity.

Example 3.3 (Conformal IFS)**.** Let $\mathcal{A}$ be a finite set and let $V \subset \mathbb{R}^n$ be a compact connected set with $V = \overline{\mathrm{Int}(V)}$. For each $i \in \mathcal{A}$, let $f_i : V \to V$ be a conformal $C^{1+\theta}$ map such that $0 < \|f_i'(x)\| < 1$ for every $x \in V$. We call the collection $\mathcal{F} = (f_i)_{i\in\mathcal{A}}$ a **conformal** $C^{1+\theta}$ **IFS**. Let $\Sigma = \mathcal{A}^{\mathbb{N}}$ be the symbolic space over the alphabet $\mathcal{A}$. We can associate to $\mathcal{F}$ a **natural projection map** $\Pi_{\mathcal{F}} : \Sigma \to V$ defined as

$$\Pi_{\mathcal{F}}(\omega) = \lim_{n\to\infty} f_{\omega_1} \circ f_{\omega_2} \circ \cdots \circ f_{\omega_n}(x), \tag{3.2}$$

where $\omega = (\omega_1, \omega_2, \ldots)$ and x is any point in V. The set $\Lambda = \Pi_{\mathcal{F}}(\Sigma)$ is called the **attractor** of $\mathcal{F}$ and it is the unique non-empty compact set X satisfying

$$\Lambda = \bigcup_{i\in\mathcal{A}} f_i(\Lambda).$$

Any set Λ of this form is called a **self-conformal set**. An **ergodic measure on** Λ is a measure of the form $\Pi_{\mathcal{F}}\mu$, where μ is an ergodic shift-invariant Borel probability measure on $\mathcal{A}^{\mathbb{N}}$. If μ is additionally a Gibbs measure corresponding to a Hölder continuous potential (see Definition 7.11), then $\Pi_{\mathcal{F}}\mu$ is called a **Gibbs measure on** X, while if μ is a Bernoulli measure then $\Pi_{\mathcal{F}}\mu$ is called a **self-conformal measure**. We say that $\mathcal{F}$ satisfies the **Strong Separation Condition** if sets $f_i(\Lambda), i \in \mathcal{A}$ are pairwise disjoint. We denote by $\mathcal{E}_\sigma(\Sigma)$ and $G_\sigma(\Sigma)$ the collections of all, respectively, ergodic and Gibbs measures on Σ. If $\mathcal{F}$ consists of similarity maps, i.e. $f_i(x) = \lambda_i O_i x + t_i$, where $\lambda_i \in (0,1), t_i \in \mathbb{R}^n$ and O_i are orthogonal $n \times n$ matrices, then the corresponding attractor is called a **self-similar set** and $\Pi_{\mathcal{F}}\mu$ is called a **self-similar measure** if μ is Bernoulli.

Given a conformal $C^{1+\theta}$ IFS $\mathcal{F}$, it will be convenient for us to consider an associated metric $\rho_{\mathcal{F}}$ on Σ defined as follows. Given two infinite sequences $\omega, \tau \in \Sigma$, let $\omega \wedge \tau$ denote the longest common prefix of ω and τ. For a finite word $\omega \in \Sigma^* = \bigcup_{k=0}^{\infty} \mathcal{A}^k$, let $f_\omega = f_{\omega_1} \circ \cdots \circ f_{\omega_k}$, where $\omega = (\omega_1, \ldots, \omega_k)$. If now $\omega, \tau \in \Sigma$ and $\omega \neq \tau$, we set

$$\rho_{\mathcal{F}}(\omega, \tau) = \|f'_{\omega\wedge\tau}\|, \tag{3.3}$$

where $\|\cdot\|$ is the supremum norm on V. It is easy to check that $\rho_{\mathcal{F}}$ is a metric on Σ and the natural projection map $\Pi_{\mathcal{F}} : \Sigma \to \mathbb{R}^n$ is Lipschitz in $\rho_{\mathcal{F}}$. If $\mathcal{F}$ satisfies the Strong Separation Condition, then $\Pi_{\mathcal{F}}$ is bi-Lipschitz in $\rho_{\mathcal{F}}$ (see [BSS23, Section 14.2]). The metric $\rho_{\mathcal{F}}$ is also significant as natural dynamical invariants of the system can be expressed in terms of dimensions calculated with respect to $\rho_{\mathcal{F}}$. Namely, an ergodic shift-invariant measure μ on Σ is exact-dimensional with respect to $\rho_{\mathcal{F}}$ and satisfies

$$\dim_H(\mu, \rho_{\mathcal{F}}) = \frac{h(\mu)}{\chi(\mu, \mathcal{F})},$$

where $h(\mu)$ is the Kolmogorov-Sinai entropy of μ with respect to the left shift $\sigma : \Sigma \to \Sigma$ and $\chi(\mu, \mathcal{F})$ is the Lyapunov exponent of μ defined as

$$\chi(\mu, \mathcal{F}) = -\int \log \|f'_{\omega_1}(\Pi_{\mathcal{F}}(\sigma\omega))\| d\mu(\omega).$$

Similarly, let us define the pressure function $P_{\mathcal{F}} : (0, \infty) \to \mathbb{R}$ as

$$P_{\mathcal{F}}(s) = \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{\omega\in\mathcal{A}^n} \|f'_\omega\|^s \right). \tag{3.4}$$

It is well-known that there exists a unique solution $s(\mathcal{F})$ of the equation (the so-called Bowen's formula)

$$P_{\mathcal{F}}(s(\mathcal{F})) = 0, \tag{3.5}$$

which satisfies

$$\dim_H(\Sigma, \rho_{\mathcal{F}}) = \dim_A(\Sigma, \rho_{\mathcal{F}}) = s(\mathcal{F}).$$

The Hausdorff dimension being equal to $s(\mathcal{F})$, and in particular, Σ being $s(\mathcal{F})$-Ahlfors regular, follows by Bedford [Bed88] and the assertion on the Assouad dimension (denoted by $\dim_A$) follows by the $s(\mathcal{F})$-Ahlfors regularity and [Fra21, Theorem 6.4.1]. ■

Our basic example of a relative dimension separable collection of measures is the following one.

Proposition 3.4. *Let $\mathcal{F}$ be a conformal $C^{1+\theta}$ IFS on $\mathbb{R}^n$ and let $\rho_{\mathcal{F}}$ be the corresponding metric on the symbolic space Σ. Then the collection of all ergodic shift-invariant Borel probability measures on Σ is relative dimension separable with respect to $\rho_{\mathcal{F}}$.*

The above proposition follows from a more general Proposition 7.10. An immediate consequence, as the regular dimension separability is invariant under bi-Lipschitz mappings (see Proposition 5.1), is the following application of Theorem 2.7 with the use of transversality of orthogonal projections in Example 3.1.

Theorem 3.5. *Let $\mathcal{F}$ be a $C^{1+\theta}$ conformal IFS on $\mathbb{R}^n$ with attractor Λ, satisfying the Strong Separation Condition. Then the family of all ergodic measures on Λ is relative dimension separable and hence for every $1 \le d < n$, for almost every $V \in \mathrm{Gr}(d, n)$*

$$\dim_H P_V\mu = \min\{d, \dim_H \mu\} \text{ simultaneously for all ergodic measures } \mu \text{ on } \Lambda.$$

A less trivial application is the result saying that without assuming any separation, the same holds for all Gibbs measures on a self-conformal set.

Theorem 3.6. *Let Λ be a self-conformal set on $\mathbb{R}^n$. The collection of all Gibbs measures on Λ is relative dimension separable. Consequently, for every $1 \le d < n$, for almost every $V \in \mathrm{Gr}(d, n)$*

$$\dim_H P_V\mu = \min\{d, \dim_H \mu\} \text{ simultaneously for all Gibbs measures } \mu \text{ on } \Lambda.$$

The above theorem follows from Theorem 2.7, Example 3.1 and Propositions 5.5, 7.12.

Let us now turn to *parametrized* families of conformal IFS. Even in the presence of overlaps, one can obtain projection results for typical parameters if the transversality condition holds.

Example 3.7 (Parametrized conformal IFS)**.** Let $\mathcal{A}$ be a finite set and let $V \subset \mathbb{R}^d$ be a compact connected set with $V = \overline{\mathrm{Int}(V)}$. For each $\lambda \in U$, where U is a (hereditary Lindelöf) topological space, let $\mathcal{F}^\lambda = (f_i^\lambda)_{i \in \mathcal{A}}$ be a conformal $C^{1+\theta}$ IFS on V. Assume that there exist $0 < \gamma_1 < \gamma_2 < 1$ with $\gamma_1 \le \|(f_i^\lambda)'(x)\| \le \gamma_2$ for every $x \in V, \lambda \in U$. Moreover, assume that the map $U \ni \lambda \mapsto \mathcal{F}_\lambda$ is continuous, where the distance between $\mathcal{F}^{\lambda_1}$ and $\mathcal{F}^{\lambda_2}$ is defined as $\max_{i \in \mathcal{A}} \left(\|f_i^{\lambda_1} - f_i^{\lambda_2}\| + \|(f_i^{\lambda_1})' - (f_i^{\lambda_2})'\| + \sup_{x \neq y \in V} \frac{\|(f_i^{\lambda_1})'(x) - (f_i^{\lambda_2})'(y)\|}{|x-y|^\theta} \right)$. We call a parametrized family $\mathcal{F}^\lambda, \lambda \in U$ a **continuous family of $C^{1+\theta}$ conformal IFS**. Let $\Sigma = \mathcal{A}^{\mathbb{N}}$ be the symbolic space, and for each $\lambda \in U$, let $\Pi_\lambda := \Pi_{\mathcal{F}^\lambda} : \Sigma \to \mathbb{R}^d$ be the corresponding natural projection map (3.2). For each $\lambda \in U$, let $\rho_\lambda := \rho_{\mathcal{F}^\lambda}$ be the metric on Σ corresponding to $\mathcal{F}^\lambda$, as defined in (3.3). Under these assumptions families $\{\Pi_\lambda : \lambda \in U\}$ and $\{\rho_\lambda : \lambda \in U\}$ satisfy assumptions (A1) and (A2). Property (A1) follows by the bounded distortion property and the distortion continuity, see for example Nakajima [Nak24, Section 3] in the more general non-autonomous case, while (A2) follows directly from the definition of metric ρ_λ.

It is straightforward to check that the transversality condition (A3) with respect to the family of metrics ρ_λ follows from a stronger (and classical) condition: there exists K such that

$$\eta\left(\{\lambda \in U : \|\Pi_\lambda(\omega) - \Pi_\lambda(\tau)\| \leq r\}\right) \leq K \min\{1, r^d\} \text{ for every } \omega, \tau \in \Sigma \text{ with } \omega_1 \neq \tau_1. \tag{3.6}$$

Indeed, by the bounded distortion property, there exists a continuous function $K \colon U \to (0, \infty)$ such that for every $x, y \in V$ and $\omega \in \Sigma_*$

$$\|f_\omega^\lambda(x) - f_\omega^\lambda(y)\| \geq K(\lambda)^{-1} \|(f_\omega^\lambda)'\| \|x - y\|,$$

see Nakajima [Nak24, Lemma 4.1]. And so,

$$\begin{aligned}\eta\left(\{\lambda \in U : \|\Pi_\lambda(\omega) - \Pi_\lambda(\tau)\| \leq \rho_\lambda(\omega, \tau) r\}\right) &\leq \eta\left(\left\{\lambda \in U : \|\Pi_\lambda(\sigma^{|\omega \wedge \tau|}\omega) - \Pi_\lambda(\sigma^{|\omega \wedge \tau|}\tau)\| \leq K(\lambda) r\right\}\right) \\ &\leq \min\left\{1, r^d \sup_{\lambda \in U} K(\lambda)^d\right\}.\end{aligned}$$

■

It is usually a non-trivial task to check whether a given parametrized IFS the transversality condition (A3) holds. We refer to [BSS23] for an overview of the technique and to [Sol23] for a recent survey. Let us give a one simple construction which leads to a transversal family of conformal IFS.

Example 3.8 (Translation family)**.** Let $\mathcal{F} = (f_i)_{i \in \mathcal{A}}$ be a conformal $C^{1+\theta}$ IFS on a compact connected set $V \subset \mathbb{R}^d$ as described in Example 3.3 and assume that $\max\limits_{i \neq j \in \mathcal{A}} \|f_i'\| + \|f_j'\| < 1$. Let $U = \{(\lambda_i)_{i \in \mathcal{A}} \in (\mathbb{R}^d)^{\mathcal{A}} : f_i(V) + \lambda_i \subset V\}$, and let η be the normalized Lebesgue measure on U. Then $\mathcal{F}^\lambda = \{f_i^\lambda = f_i + \lambda_i\}_{i \in \mathcal{A}}, \lambda \in U$ is a continuous family of $C^{1+\theta}$ conformal IFS. A family of natural projections $\{\Pi_\lambda : \lambda \in U\}$ and corresponding metrics $\{\rho_\lambda : \lambda \in U\}$ as in Example 3.7 satisfies (A1) - (A3) with measure η. See for example [BSS23, Theorem 14.5.2] for the proof in the case $d = 1$, which extends in a straightforward manner to higher dimensions. ■

Before formulating our main result on parametrized families of IFS satisfying the transversality condition, let us interpret the nearly bi-Lipschitz condition for natural projection maps as a separation condition for cylinders. Given an infinite word $\omega = (\omega_1, \omega_2, \ldots) \in \Sigma$, let $\omega|_n = (\omega_1, \ldots, \omega_n)$ denote its restriction to the first n coordinates, while given a finite word $\omega \in \Sigma^*$, let $|\omega|$ denote its length and let $[\omega] = \{\tau \in \Sigma : \tau|_n = \omega\}$ be the corresponding cylinder.

Definition 3.9. Let $\mathcal{F}$ be a $C^{1+\theta}$ conformal IFS on $\mathbb{R}^d$. We say that $\omega \in \Sigma$ has **exponential distance from the enemy (EDE)** if for every $\varepsilon > 0$ there exists $C = C(\omega, \varepsilon) > 0$ such that for every $n \in \mathbb{N}$

$$\operatorname{dist}\left(\Pi_{\mathcal{F}}(\omega), \bigcup_{\substack{|\tau| = n \\ \tau \neq \omega|_n}} \Pi_{\mathcal{F}}([\tau])\right) > C \operatorname{diam}(\Pi_{\mathcal{F}}([\omega|_n]))^{1+\varepsilon}, \tag{3.7}$$

Proposition 3.10. *Let $\mathcal{F}$ be a $C^{1+\theta}$ conformal IFS on $\mathbb{R}^d$. Then $\omega \in \Sigma$ has exponential distance from the enemy if and only if for every $\alpha \in (0, 1)$ there exists $C = C(\omega, \alpha)$ such that*

$$\rho_{\mathcal{F}}(\omega, \tau) \leq C |\Pi_{\mathcal{F}}(\omega) - \Pi_{\mathcal{F}}(\tau)|^\alpha \text{ for every } \tau \in \Sigma. \tag{3.8}$$

Consequently, for $\mu \in \mathcal{M}_{\text{fin}}(\Sigma)$, we have that μ-a.e. $\omega \in \Sigma$ has exponential distance from the enemy if and only if the natural projection map $\Pi_{\mathcal{F}}$ is μ-nearly bi-Lipschitz in metric $\rho_{\mathcal{F}}$.

Therefore, for a measure $\mu \in \mathcal{M}_{\text{fin}}(\Sigma)$, μ-a.e. $\omega \in \Sigma$ satisfies EDE if and only if $\Pi_{\mathcal{F}}$ is μ-nearly bi-Lipschitz in metric $\rho_{\mathcal{F}}$. For the proof of Proposition 3.10 see Section 7.3. Now we can state our main result on transversal families of conformal IFS.

Theorem 3.11. *Let $\mathcal{F}^\lambda, \lambda \in U$ be a continuous family of $C^{1+\theta}$ conformal IFS on $V \subset \mathbb{R}^n$. Assume that η is a measure on U such that the transversality condition (A3) is satisfied (for the family of corresponding metrics ρ_λ as defined in (3.3); it suffices if (3.6) holds). Then for η-a.e. $\lambda \in U$*

(1) $\dim_H \Pi_\lambda \mu = \min\left\{n, \frac{h(\mu)}{\chi(\mu,\mathcal{F}^\lambda)}\right\}$ *holds simultaneously for all ergodic measures μ on Σ,*
(2) for every $1 \leq d < n$, for almost every $V \in \mathrm{Gr}(d,n)$, the equality $\dim_H(P_V \Pi_\lambda \mu) = \min\{d, \frac{h(\mu)}{\chi(\mu,\mathcal{F}^\lambda)}\}$ *holds simultaneously for all ergodic measures μ on Σ,*
(3) if $s(\mathcal{F}^\lambda) < n$, then simultaneously for all ergodic measures μ on Σ, μ-a.e. $\omega \in \Sigma$ has exponential distance from the enemy.

Note that point (2) of Theorem 3.11 asserts that under the transversality condition, despite possible overlaps, one obtains for typical $\mathcal{F}_\lambda$ the same conclusion as in Theorem 3.6 under Strong Separation Condition. Point (1) follows directly from Theorem 2.7 and Proposition 3.4, while point (3) requires additionally Proposition 3.10.

Point (2) requires an additional step to show that the map $P_V \circ \Pi_\lambda \colon \Sigma \mapsto \mathbb{R}^d$ satisfies the transversality condition with respect to the parametrisation $(V, \lambda) \in \mathrm{Gr}(d,n) \times U$. Then the claim of Point (2) follows simply by Fubini's Theorem. Let us denote by γ the unique measure defined in Example 3.1 and note that (3.6) gives that $\eta(\{\lambda \in U : \Pi_\lambda(\omega) = \Pi_\lambda(\tau)\}) = 0$ whenever $\omega \neq \tau$. Therefore for every $r > 0$ by (3.1) and (3.6)

$$
\begin{aligned}
&\gamma \times \eta\left(\{(V,\lambda) : \ \|P_V\Pi_\lambda(\omega) - P_V\Pi_\lambda(\tau)\| < \rho_\lambda(\omega,\tau) r\}\right) \\
&= \int \gamma\left(\left\{V : \|P_V\Pi_\lambda(\omega) - P_V\Pi_\lambda(\tau)\| < \|\Pi_\lambda(\omega) - \Pi_\lambda(\tau)\| \frac{\rho_\lambda(\omega,\tau) r}{\|\Pi_\lambda(\omega) - \Pi_\lambda(\tau)\|}\right\}\right) d\eta(\lambda) \\
&\leq c \int \left(\frac{\rho_\lambda(\omega,\tau) r}{\|\Pi_\lambda(\omega) - \Pi_\lambda(\tau)\|}\right)^d d\eta(\lambda) = c r^d \int_0^\infty \eta\left(\left\{\lambda : \left(\frac{\rho_\lambda(\omega,\tau) r}{\|\Pi_\lambda(\omega) - \Pi_\lambda(\tau)\|}\right)^d > a\right\}\right) da \\
&\leq c' r^d \int_0^\infty \min\left\{1, a^{-n/d}\right\} da \leq c'' r^d.
\end{aligned}
$$

3.3. **Further examples.** Let us now present some more examples related to iterated function systems to which Theorem 2.7 can be applied.

Example 3.12 (Parametrized non-autonomous system)**.** Non-autonomous systems were introduced Rempe-Gillen and Urbański in [RGU16], with the so-called Open Set Condition being assumed. For the sake of studying the overlapping non-autonumous systems, Nakajima [Nak24] considered families of non-autonomous systems and introduced a transversality condition for the non-autonomous systems. In particular the following parametrized family of non-autonomous system (with overlaps) was studied by Nakajima [Nak24]: The parameter domain is $U = \{t \in \mathbb{C} : |t| < 2 \times 5^{-5/8}\} \setminus \mathbb{R}$, and let $X := \left\{z \in \mathbb{Z} : |z| \leq \frac{1}{1 - 2\cdot 5^{-5/8}}\right\}$. For every parameter $t \in U$ and $j \in \mathbb{N}$ Nakajima considered the mappings

$$\phi_{0,t}^{(j)}, \phi_{1,t}^{(j)} : X \to X, \qquad \phi_{0,t}^{(j)}(z) := tz \quad \text{and} \quad \phi_{1,t}^{(j)}(z) := tz + \frac{1}{j}. \tag{3.9}$$

Let $\Phi_t := \left(\Phi_t^{(j)}\right)_{j=1}^{\infty}$, where

$$\Phi_t^{(j)} = \left\{ z \mapsto tz, z \mapsto tz + \frac{1}{j} \right\} \tag{3.10}$$

Then Φ_t is a parametrized non-autonomous system whose attractor is the set of all limit points

$$\Lambda_t = \left\{ \lim_{n\to\infty} \phi^{(1)}_{\omega_1,t} \circ \cdots \circ \phi^{(n)}_{\omega_n,t}(z) : \ (\omega_1, \omega_2, \dots) \in \{0,1\}^{\mathbb{N}} \right\},$$

where $z \in X$. Set $\mathcal{A} = \{0,1\}$, $\Sigma = \mathcal{A}^{\mathbb{N}}$ and $\Pi_t : \Sigma \to \mathbb{C}$, $\Pi_t(\omega) = \sum_{k=1}^{\infty} \frac{\omega_k t^k}{k}$ for $\omega = (\omega_1, \omega_2, \dots) \in \Sigma$, so that $\Lambda_t = \Pi_t(\Sigma)$. Consider a family of metric on Σ defined as $\rho_t(\omega, \tau) = t^{|\omega\wedge\tau|}$ for $\omega, \tau \in \Sigma$. It is straightforward to that conditions (A1) and (A2) hold. It follows from [Nak24, Theorem B] that the transversality condition (A3) holds on every compact subset of U with η being the Lebesgue measure on $\mathbb{C}$. Indeed, by [Nak24, Theorem B], if $\omega, \tau \in \Sigma$ are such that $|\omega \wedge \tau| = n$, then for a compact set $G \subset U$

$$\eta\left(\{t \in G : |\Pi_t(\sigma^n \omega) - \Pi_t(\sigma^n \tau)| \le r\}\right) \le C_n r^2$$

with C_n satisfying $\lim_{n\to\infty} \frac{\log C_n}{n} = 0$. Therefore for every $\varepsilon > 0$ there exists K so that $C_n \le K 2^{n\varepsilon}$ and hence setting $\gamma = 2 \times 5^{-5/8}$ we have for $r > 0, \delta > 0$

$$\begin{aligned}
&\eta\left(\{t \in G : |\Pi_t(\omega) - \Pi_t(\tau)| \le \rho_t(\omega,\tau) r, \ \rho_t(\omega,\tau) \ge \delta\}\right) \\
&\quad = \eta\left(\{t \in G : |\Pi_t(\sigma^n \omega) - \Pi_t(\sigma^n \tau)| \le r, \ |t|^n \ge \delta\}\right) \\
&\quad \le \eta\left(\{t \in G : |\Pi_t(\sigma^n \omega) - \Pi_t(\sigma^n \tau)| \le r, \ \gamma^n \ge \delta\}\right) \\
&\quad \le \eta\left(\{t \in G : |\Pi_t(\sigma^n \omega) - \Pi_t(\sigma^n \tau)| \le r\}\right) \mathbb{1}_{\left\{n \le \frac{\log\delta}{\log\gamma}\right\}} \\
&\quad \le C_n r^2 \mathbb{1}_{\left\{n \le \frac{\log\delta}{\log\gamma}\right\}} \le K 2^{n\varepsilon} r^2 \mathbb{1}_{\left\{n \le \frac{\log\delta}{\log\gamma}\right\}} \le K \delta^{\frac{\varepsilon}{\log\gamma}} r^2.
\end{aligned}$$

Consequently (A3) holds in this case. Finally, the set of all ergodic shift-invariant measures on Σ is relative dimensional separable with respect to each ρ_t with $t \in U$ by Proposition 7.10, hence all assumptions of Theorem 2.7 are met in this case. ■

Example 3.13 (Random self-similar system)**.** Let $\mathcal{A}$ be a finite set of indices, and for every $i \in \mathcal{A}$, let $\theta_i \in (0,1)$, $O_i \in O(\mathbb{R}, d)$ and $t_i \in \mathbb{R}^d$. Let $I \subset \mathbb{R}^d$ be a compact domain and let ζ be the normalized Lebesgue measure on I. Let $U = I^{\Sigma_*}$. Set $\Sigma = \mathcal{A}^{\mathbb{N}}$ and for $\lambda \in U$ define $\Pi_\lambda : \Sigma \to \mathbb{R}^d$ as $\Pi_\lambda(\omega) = \sum_{k=1}^{\infty} (t_{\omega_k} + \lambda_{\omega|_k}) \theta_{\omega|_{k-1}} O_{\omega_1 \dots \omega_k}$, where $\theta_{\omega|_k} = \theta_{\omega_1} \cdots \theta_{\omega_k}$ and $O_{\omega_1 \dots \omega_k} := O_{\omega_1} \cdots O_{\omega_k}$. With the choice of a single metric $\rho_\lambda(\omega, \tau) = \rho(\omega, \tau) = \theta_{\omega\wedge\tau}$ on Σ, it is easy to check that the assumptions (A1) and (A2) hold. Choosing the probability measure $\eta = \zeta^{\Sigma_*}$, the transversality condition (A3) holds by [JPS07, Lemma 5.1] The set of all ergodic shift-invariant measures on Σ is relative dimensional separable with respect to ρ by Proposition 7.10. ■

3.4. **Applications to multifractal analysis.** Finally, let us present here an application of Theorem 3.11 in the theory of multifractal analysis. Let $\mathcal{F} = \{f_i(x) = \lambda_i x + t_i\}_{i\in\mathcal{A}}$ be a self-similar IFS on the line and let $\underline{p} = (p_i)_{i\in\mathcal{A}}$ be a probability vector, and let μ be a Bernoulli measure on Σ defined by $\underline{p}$. In this case, the quantity s_0 defined in (3.5) is given by the equation $\sum_{i\in\mathcal{A}} |\lambda_i|^{s_0} = 1$ and called the similarity dimension.

Arbeiter and Patschke [AP96] studied the multifractal spectrum of self-similar measures under the open set condition. For every $q \in \mathbb{R}$, let $T(q)$ be the unique map such that

$$\sum_{i\in\mathcal{A}} p_i^q |\lambda_i|^{T(q)} = 1.$$

Arbeiter and Patschke [AP96] showed that if the OSC holds and $(p_i)_{i\in\mathcal{A}} \neq (|\lambda_i|^{s_0})_{i\in\mathcal{A}}$ then for every $\alpha \in \left[\min_{i\in\mathcal{A}} \frac{\log p_i}{\log|\lambda_i|}, \max_{i\in\mathcal{A}} \frac{\log p_i}{\log|\lambda_i|}\right]$

$$\dim_H \{x : d(\Pi\mu, x) = \alpha\} = \inf_{q\in\mathbb{R}}(\alpha q + T(q)) =: T^*(\alpha). \tag{3.11}$$

In particular T^* is an analytic concave map.

The multifractal analysis becomes significantly more difficult, if there are overlaps between the cylinder sets. Barral and Feng [BF13, Theorem 1.3, Theorem 6.4] showed that under the transversality condition, the multifractal formalim (3.11) holds on partial interval of the domain. Let us state here their result on the line under the assumption that $s_0 < 1$. That is, under these assumptions, for Lebesgue almost every $(t_i)_{i\in\mathcal{A}}$ the multifractal formalism (3.11) holds for every $\alpha \in \left[\frac{\sum_i p_i \log p_i}{\sum_i p_i \log|\lambda_i|}, \frac{\sum_i |\lambda_i|^{s_0} \log p_i}{\sum_i |\lambda_i|^{s_0}\log|\lambda_i|}\right]$. Later in [BF21, Theorem 1.2, Remark 7.3], they extended this result significantly. Namely, if $\mathcal{F}$ satisfies the exponential separation condition and $s_0 < 1$ then for every $\alpha \in \left[\min_{i\in\mathcal{A}} \frac{\log p_i}{\log|\lambda_i|}, \frac{\sum_i |\lambda_i|^{s_0} \log p_i}{\sum_i |\lambda_i|^{s_0}\log|\lambda_i|}\right]$ the multifractal formalism (3.11) holds. Obtaining the multifractal formalism on the full interval $\left[\min_{i\in\mathcal{A}} \frac{\log p_i}{\log|\lambda_i|}, \max_{i\in\mathcal{A}} \frac{\log p_i}{\log|\lambda_i|}\right]$ under exponential separation remains an open problem. Using Theorem 2.7 we can show that the multifractal formalism holds on the whole domain for almost every translation parameter.

Theorem 3.14. *Let $\mathcal{A}$ be a finite set and for each $i \in \mathcal{A}$ fix $\lambda_i \in (-1,1)\setminus\{0\}$. For $t = (t_i)_{i\in\mathcal{A}} \in \mathbb{R}^{\mathcal{A}}$ let $\mathcal{F}_t = \{f_i(x) = \lambda_i x + t_i\}_{i\in\mathcal{A}}$ be a self-similar IFS on the line. Let $s_0 = s(\mathcal{F}_t)$ be the similarity dimension of $\mathcal{F}_t$ and assume that $s_0 < 1$. Then the following holds for Lebesgue almost every $(t_i)_{i\in\mathcal{A}} \in \mathbb{R}^{\mathcal{A}}$. For every probability vector $\underline{p} = (p_i)_{i\in\mathcal{A}}$ such that $(p_i)_{i\in\mathcal{A}} \neq (|\lambda_i|^{s_0})_{i\in\mathcal{A}}$, the multifractal formalism* (3.11) *holds for the self-similar measure $\Pi_{\mathcal{F}_t}(\underline{p}^{\mathbb{N}})$ and every $\alpha \in \left[\min_{i\in\mathcal{A}} \frac{\log p_i}{\log|\lambda_i|}, \max_{i\in\mathcal{A}} \frac{\log p_i}{\log|\lambda_i|}\right]$ simultaneously.*

See Section 8 for the proof and Theorem 8.3 for a multidimensional version.

4. Preliminaries

For a map Π between metric spaces, we will denote by $\mathrm{Lip}(\Pi)$ the Lipschitz constant of Π (or $\mathrm{Lip}(\Pi,\rho)$ if we want to emphasize dependence on the metric). Given a Borel measure μ on a metric space (X,ρ), we denote by $\mathrm{supp}(\mu)$ the topological support of μ, i.e. the set of all $x \in X$ such that $\mu(B(x,r)) > 0$ for every $r > 0$.

Definition 4.1. Let (X,ρ) be a metric space. The **lower** and **upper local dimensions** of a finite Borel measure $\mu \in \mathcal{M}_{\mathrm{fin}}(X)$ at a point $x \in \mathrm{supp}\mu$ are defined as

$$\underline{d}(\mu,x) = \liminf_{r\to 0} \frac{\log\mu(B(x,r))}{\log r} \text{ and } \overline{d}(\mu,x) = \limsup_{r\to 0} \frac{\log\mu(B(x,r))}{\log r}.$$

If the limit above exists, then their common value is called the **local dimension** of μ at x and denoted as $d(\mu,x)$. The **lower** and **upper Hausdorff dimensions** of μ are defined as

$$\underline{\dim}_H\mu = \operatorname*{essinf}_{x\sim\mu} \underline{d}(\mu,x) \text{ and } \overline{\dim}_H\mu = \operatorname*{esssup}_{x\sim\mu} \underline{d}(\mu,x).$$

If $\underline{\dim}_H\mu = \overline{\dim}_H\mu$, then their common value is called the **Hausdorff dimension** of μ and denoted $\dim_H\mu$. Measure μ is called **exact-dimensional** if $d(\mu,x)$ exists and is constant μ-almost everywhere.

It is well known (see e.g. [BSS23, Section 1.9.1]) that the Hausdorff dimensions can be equivalently expressed as

$$\underline{\dim}_H \mu = \inf \{\dim_H A : A \subset X \text{ Borel with } \mu(A) > 0\}$$

and

$$\overline{\dim}_H \mu = \inf \left\{\dim_H A : A \subset X \text{ Borel with } \mu(\mathbb{R}^N \setminus A) = 0\right\}.$$

Definition 4.2. Let (X, ρ) be a metric space. For a set $Y \subset \mathbb{R}^N$ and $\delta > 0$, let $N(Y, \delta)$ denote the minimal number of balls of radius δ required to cover X. Set Y is said to be (M, s)-*homogeneous* if $N(Y \cap B(x, r), \rho) \leq M(r/\rho)^s$ for every $x \in Y$, $0 < \rho < r$, i.e. the intersection $B(x, r) \cap Y$ can be covered by at most $M(r/\rho)^s$ balls of radius ρ. The **Assouad dimension** of Y is defined as

$$\dim_A Y = \inf\{s > 0 : Y \text{ is } (M, s)\text{-homogeneous for some } M > 0\}.$$

We will repeatedly make use of the following simple lemma.

Lemma 4.3. *Let X be a metric space. Fix $r > 0$. For every set $G \subset X$ there exists a cover*

$$G \subset \bigcup_{x' \in F} B(x', r)$$

such that

$$F \subset G \text{ and } \left\{B(x', r/2) : x' \in F\right\} \text{ consists of pairwise disjoint sets.}$$

Moreover, if X is separable, then F can be taken to be countable and if X is compact, then F can be taken finite.

Proof. As F one can take a maximal packing $r/2$-packing of G, i.e. a set $F \subset G$ with the property that balls $\{B(x', r/2)\}_{x' \in F}$ are pairwise disjoint and no larger set (in the sense of inclusion) has this property (its existence follows from the Kuratowski-Zorn lemma). Then $G \subset \bigcup_{x' \in F} B(x', r)$, as otherwise F would not be a maximal $r/2$-packing. If X is separable, then it is hereditary Lindelöf [Eng89], hence one choose a countable subcover of G. If X is compact, then F must be finite, as otherwise X would have a sequence without a convergent subsequence. □

5. Relative dimension separability

We begin with simple observations providing conditions which guarantee that the relative dimension separability is preserved under a Lipschitz map.

Proposition 5.1. *Let X and Y be metric spaces and let $\Pi : X \to Y$ be a bi-Lipschitz map. Assume that $\mathcal{M} \subset \mathcal{M}_{\text{fin}}(X)$ is relative dimension separable and that for each $\mu \in \mathcal{M}$, measure $\Pi\mu$ is weakly diametrically regular. Then the collection $\{\Pi\mu : \mu \in \mathcal{M}\} \subset \mathcal{M}_{\text{fin}}(Y)$ is relative dimension separable.*

Proof. Let ρ_X and ρ_Y denote the metrics on X and Y, respectively. Let $\mathcal{V} \subset \mathcal{M}_{\text{fin}}(X)$ be a countable set witnessing relative dimension separability of $\mathcal{M}$. Fix $\varepsilon > 0$, $\mu \in \mathcal{M}$ and $\nu \in \mathcal{V}$ such that $\dim(\mu||\nu, \rho_X) < \varepsilon$. As Π is bi-Lipschitz, there exists $L > 0$ such that for every $x \in X$ and $r > 0$

$$B(x, r/L) \subset \Pi^{-1}(B(\Pi(x), r)) \subset B(x, Lr).$$

Fix $x \in X$ such that there exist $R > 0$ and $d \geq 0$ for which

$$r^{\varepsilon}\nu(B(x, r)) \leq \mu(B(x, r)) \leq r^{-\varepsilon}\nu(B(x, r))$$

and

$$\Pi\mu(B(\Pi(x), L^2 r)) \leq r^{-\varepsilon}\Pi\mu(B(\Pi(x), r))$$

hold for every $0 < r < R$. By assumptions, μ-a.e. $x \in X$ satisfies the above properties. For $0 < r < R$ we have

$$\begin{aligned}\Pi\mu(B(\Pi(x),r)) &\le L^{2\varepsilon}r^{\varepsilon}\Pi\mu(B(\Pi(x),r/L^2)) \le L^{2\varepsilon}r^{-\varepsilon}\mu(B(x,r/L)) \le L^{3\varepsilon}r^{-2\varepsilon}\nu(B(x,r/L)) \\ &\le L^{3\varepsilon}r^{-2\varepsilon}\nu(\Pi^{-1}(B(\Pi(x),r))) = L^{3\varepsilon}r^{-2\varepsilon}\Pi\nu(B(\Pi(x),r)).\end{aligned}$$

Similarly, one has for $0 < r < r/L^2$.

$$\begin{aligned}\Pi\mu(B(\Pi(x),r)) &\ge r^{\varepsilon}\Pi\mu(B(\Pi(x),L^2r)) \ge r^{\varepsilon}\mu(B(x,Lr)) \ge L^{\varepsilon}r^{2\varepsilon}\nu(B(x,Lr)) \\ &\ge L^{\varepsilon}r^{2\varepsilon}\nu(\Pi^{-1}(B(\Pi(x),r))) = L^{\varepsilon}r^{2\varepsilon}\Pi\nu(B(\Pi(x),r)).\end{aligned}$$

The two above calculations show together that

$$-2\varepsilon \le \liminf_{r\to 0}\frac{\log\frac{\Pi\mu(B(\Pi(x),r))}{\Pi\nu(B(\Pi(x),r))}}{\log r} \le \limsup_{r\to 0}\frac{\log\frac{\Pi\mu(B(\Pi(x),r))}{\Pi\nu(B(\Pi(x),r))}}{\log r} \le 2\varepsilon.$$

Recalling that the above holds for μ-a.e. $x \in X$ we obtain $\dim(\Pi\mu||\Pi\nu,\rho_Y) \le 2\varepsilon$. As the set $\{\Pi\nu : \nu \in \mathcal{V}\}$ is at most countable, we see that $\{\Pi\mu : \mu \in \mathcal{M}\}$ is relative dimension separable. $\square$

For the next preservation property we need a stronger separability condition.

Definition 5.2. Let μ and ν be finite Borel measures on a metric space (X,ρ). The **uniform relative dimension** of μ with respect to ν is

$$\dim_u(\mu||\nu,d) := \inf\{\varepsilon > 0 : \underset{R>0}{\exists}\ \underset{x\in X}{\forall}\ \underset{0<r<R}{\forall}\ r^{\varepsilon}\nu(B(x,r)) \le \mu(B(x,r)) \le r^{-\varepsilon}\nu(B(x,r))\}.$$

Definition 5.3. Let (X,ρ) be a metric space. Let $\mathcal{M} \subset \mathcal{M}_{\text{fin}}(X)$ be a collection of finite Borel measures on X. We say that $\mathcal{M}$ is **uniform relative dimension separable** (with respect to ρ) if there exists a countable set $\mathcal{V} \subset \mathcal{M}_{\text{fin}}(X)$ such that for every $\mu \in \mathcal{M}$ and $\varepsilon > 0$ there exists $\nu \in \mathcal{V}$ such that $\dim_u(\mu||\nu,d) < \varepsilon$.

Note that clearly $\dim(\mu||\nu,d) \le \dim_u(\mu||\nu,d)$, hence uniform relative dimension separability implies relative dimension separability. We shall also need a stronger diametric regularity condition.

Definition 5.4. Let (X,ρ) be a metric space. Measure $\mu \in \mathcal{M}_{\text{fin}}(X)$ is called **uniformly diametrically regular** if for every $\varepsilon > 0$ there exists $R > 0$ such that

$$\mu(B(x,2r)) \le r^{-\varepsilon}\mu(B(x,r))$$

holds for every $0 < r < R$ and $x \in X$.

Proposition 5.5. *Let X and Y be metric spaces and let $\Pi : X \to Y$ be a Lipschitz map. Assume that X is separable, $\mathcal{M} \subset \mathcal{M}_{\text{fin}}(X)$ is uniform relative dimension separable and that for every $\mu \in \mathcal{M}$, μ is uniformly diametrically regular and $\Pi\mu$ is weakly diametrically regular. Then the collection $\{\Pi\mu : \mu \in \mathcal{M}\} \subset \mathcal{M}_{\text{fin}}(Y)$ is relative dimension separable.*

Proof. Let $\mathcal{V} \subset \mathcal{M}_{\text{fin}}(X)$ be a countable set witnessing uniform relative dimension separability of $\mathcal{M}$. Fix $\varepsilon > 0$, $\mu \in \mathcal{M}$ and $\nu \in \mathcal{V}$ such that $\dim_u(\mu||\nu,\rho_X) < \varepsilon$. Let $R_1 > 0$ be such that

$$r^{\varepsilon}\nu(B(x,r)) \le \mu(B(x,r)) \le r^{-\varepsilon}\nu(B(x,r)) \tag{5.1}$$

and

$$\mu(B(x,2r)) \le r^{-\varepsilon}\mu(B(x,r)) \tag{5.2}$$

hold for every $x \in X$ and $0 < r < R_1$. Fix $x \in X$ such that there exists $R_2 > 0$ so that

$$\Pi\mu(B(\Pi(x),2r)) \le r^{-\varepsilon}\Pi\mu(B(\Pi(x),r)) \tag{5.3}$$

holds for every $0 < r < R_2$. By assumptions, μ-a.e. $x \in X$ satisfies the above properties. Set $R = \min\{R_1, R_2\}$ and $L = \mathrm{Lip}(\Pi)$. Using Lemma 4.3, take a countable cover

$$\Pi^{-1}\left(B\left(\Pi\left(x\right), r/2\right)\right) \subset \bigcup_{x' \in F} B\left(x', \frac{r}{L}\right)$$

such that

$$F \subset \Pi^{-1}\left(B\left(\Pi\left(x\right), r/2\right)\right) \text{ and } \left\{B\left(x', \frac{r}{2L}\right) : x' \in F\right\} \text{ consists of pairwise disjoint sets.}$$

Note that

$$\bigcup_{x' \in F} B\left(x', \frac{r}{2L}\right) \subset \Pi^{-1}\left(B\left(\Pi\left(x\right), r\right)\right).$$

We therefore have by (5.1) and (5.3) for $0 < r < \min\{R, LR\}$

$$\begin{aligned}
\Pi\mu\left(B\left(\Pi\left(x\right), r\right)\right) &\le 2^{\varepsilon} r^{-\varepsilon} \Pi\mu\left(B\left(\Pi\left(x\right), r/2\right)\right) \le 2^{\varepsilon} r^{-\varepsilon} \sum_{x' \in F} \mu\left(B\left(x', \frac{r}{L}\right)\right) \\
&\le 2^{2\varepsilon} L^{\varepsilon} r^{-2\varepsilon} \sum_{x' \in F} \mu\left(B\left(x', \frac{r}{2L}\right)\right) \\
&\le 2^{3\varepsilon} L^{2\varepsilon} r^{-3\varepsilon} \sum_{x' \in F} \nu\left(B\left(x', \frac{r}{2L}\right)\right) \\
&\le 2^{3\varepsilon} L^{2\varepsilon} r^{-3\varepsilon} \Pi\nu(B(\Pi(x), r)).
\end{aligned}$$

Similarly for $0 < r < \min\{R, LR\}$

$$\begin{aligned}
\Pi\nu\left(B\left(\Pi\left(x\right), r/2\right)\right) &\le \sum_{x' \in F} \nu\left(B\left(x', \frac{r}{L}\right)\right) \le L^{\varepsilon} r^{-\varepsilon} \sum_{x' \in F} \mu\left(B\left(x', \frac{r}{L}\right)\right) \\
&\le 2^{\varepsilon} L^{2\varepsilon} r^{-2\varepsilon} \sum_{x' \in F} \mu\left(B\left(x', \frac{r}{2L}\right)\right) \\
&\le 2^{\varepsilon} L^{2\varepsilon} r^{-2\varepsilon} \Pi\mu(B(\Pi(x), r)) \\
&\le 2^{2\varepsilon} L^{2\varepsilon} r^{-3\varepsilon} \Pi\mu(B(\Pi(x), r/2)).
\end{aligned}$$

The two above calculations show together that

$$-3\varepsilon \le \liminf_{r \to 0} \frac{\log \frac{\Pi\mu(B(\Pi(x),r))}{\Pi\nu(B(\Pi(x),r))}}{\log r} \le \limsup_{r \to 0} \frac{\log \frac{\Pi\mu(B(\Pi(x),r))}{\Pi\nu(B(\Pi(x),r))}}{\log r} \le 3\varepsilon.$$

Recalling that the above holds for μ-a.e. $x \in X$ we obtain $\dim(\Pi\mu || \Pi\nu, \rho_Y) \le 3\varepsilon$. As the set $\{\Pi\nu : \nu \in \mathcal{V}\}$ is at most countable, we see that $\{\Pi\mu : \mu \in \mathcal{M}\}$ is relative dimension separable. □

6. Proof of Theorem 2.7

This section is devoted to the proof of Theorem 2.7. **We assume in this section that all assumptions of Theorem 2.7 are satisfied.** We will denote by $\mathrm{Lip}(\Pi_\lambda, \rho_\lambda)$ the Lipschitz constant of $\Pi_\lambda : X \to \mathbb{R}^d$ with respect to metric ρ_λ on X.

6.1. **Preliminaries on relative dimension.** Let us begin with formulating the main technical consequence of the relative dimension separability assumption, which is a construction of sets on which one can compare μ- and ν-measures of balls for all μ which are relative dimension close to the reference measure ν. For that, given $\mu, \nu \in \mathcal{M}_{\mathrm{fin}}(X)$, $\lambda_0 \in U, q \geq 0, \varepsilon > 0$ and $R > 0$ define sets

$$A_{\lambda_0,q,\varepsilon,R}(\mu,\nu) = \Big\{ x \in X : \underset{0<r<R}{\forall} r^{\varepsilon}\nu(B_{\lambda_0}(x,r)) \leq \mu(B_{\lambda_0}(x,r)) \leq r^{-\varepsilon}\nu(B_{\lambda_0}(x,r)), \\ \mu(B_{\lambda_0}(x,r)) \leq r^{q-\varepsilon},\ \mu(B_{\lambda_0}(x,2r)) \leq r^{-\varepsilon}\mu(B_{\lambda_0}(x,r)) \Big\}$$

and

$$G_{\lambda_0,q,\varepsilon,R}(\nu) = \Big\{ x \in X : \underset{0<r<R/2}{\forall} \nu(B_{\lambda_0}(x,r)) \leq r^{q-2\varepsilon},\ \nu(B_{\lambda_0}(x,2r)) \leq 2^{-\varepsilon} r^{-3\varepsilon}\nu(B_{\lambda_0}(x,r)) \Big\}.$$

A formal corollary of definitions of $A_{\lambda_0,q,\varepsilon,R}(\mu,\nu)$ and $G_{\lambda_0,q,\varepsilon,R}(\nu)$ is the following.

Lemma 6.1. *For every $\mu, \nu \in \mathcal{M}_{\mathrm{fin}}(X)$, $\lambda_0 \in U, q \geq 0, \varepsilon > 0$ and $R > 0$*

$$A_{\lambda_0,q,\varepsilon,R}(\mu,\nu) \subset G_{\lambda_0,q,\varepsilon,R}(\nu) \tag{6.1}$$

and inequality

$$\mu(B_{\lambda_0}(x,r) \cap A_{\lambda_0,q,\varepsilon,R}(\mu,\nu)) \leq 2^{-\varepsilon} r^{-5\varepsilon}\nu(B_{\lambda_0}(x,r)) \tag{6.2}$$

holds for every $x \in G_{\lambda_0,q,\varepsilon,R}(\nu)$ and $0 < r < R/2$. Moreover, there exists $M = M(R,\varepsilon)$ such that

$$\mu(A_{\lambda_0,q,\varepsilon,R}(\mu,\nu)) \leq M\nu(X). \tag{6.3}$$

Proof. Containment (6.1) follows directly from the definitions of $A_{\lambda_0,q,\varepsilon,R}(\mu,\nu)$ and $G_{\lambda_0,q,\varepsilon,M}(\nu)$. For (6.2), fix $x \in G_{\lambda_0,q,\varepsilon,M}(\nu)$ and $0 < r < R/2$. If $B_{\lambda_0}(x,r) \cap A_{\lambda_0,q,\varepsilon,R}(\mu,\nu) = \emptyset$, then (6.2) holds trivially. Otherwise, choose $y \in B_{\lambda_0}(x,r) \cap A_{\lambda_0,q,\varepsilon,R}(\mu,\nu)$. Then

$$\mu(B_{\lambda_0}(x,r) \cap A_{\lambda_0,q,\varepsilon,R}(\mu,\nu)) \leq \mu(B_{\lambda_0}(y,2r)) \leq r^{-\varepsilon}\mu(B_{\lambda_0}(y,r)) \leq r^{-2\varepsilon}\nu(B_{\lambda_0}(y,r)) \\ \leq r^{-2\varepsilon}\nu(B_{\lambda_0}(x,2r)) \leq 2^{-\varepsilon}r^{-5\varepsilon}\nu(B_{\lambda_0}(x,r)),$$

where the second and third inequality follow from the definition of $A_{\lambda_0,q,\varepsilon,R}(\mu,\nu)$, the last inequality follows from the definition of $G_{\lambda_0,q,\varepsilon,M}(\nu)$ and the remaining ones follow from $B_{\lambda_0}(x,r) \subset B_{\lambda_0}(y,2r)$ and $B_{\lambda_0}(y,r) \subset B_{\lambda_0}(x,2r)$.

For (6.3) consider a countable cover

$$A_{\lambda_0,q,\varepsilon,R}(\mu,\nu) \subset \bigcup_{x' \in F} B_{\lambda_0}(x', R/4),$$

so that

$$F \subset A_{\lambda_0,q,\varepsilon,R}(\mu,\nu) \text{ and } \big\{B_{\lambda_0}\big(x', R/8\big) : x' \in F\big\} \text{ consists of pairwise disjoint balls.}$$

Then by (6.1), (6.2) and definition of $G_{\lambda_0,q,\varepsilon,M}(\nu)$

$$\mu(A_{\lambda_0,q,\varepsilon,R}(\mu,\nu)) \leq \sum_{x' \in F} \mu(B_{\lambda_0}(x', R/4) \cap A_{\lambda_0,q,\varepsilon,R}(\mu,\nu)) \leq 2^{-\varepsilon}(R/4)^{-5\varepsilon} \sum_{x' \in F} \nu(B_{\lambda_0}(x', R/4)) \\ \leq 2^{-2\varepsilon}(R/4)^{-5\varepsilon}(R/8)^{-3\varepsilon} \sum_{x' \in F} \nu(B_{\lambda_0}(x, R/8) \leq 2^{17\varepsilon}R^{-8\varepsilon}\nu(X).$$

This proves (6.3) with $M = 2^{17\varepsilon}R^{-8\varepsilon}$. □

In order to make use of the Lemma 6.1, we need to prove that sets $A_{\lambda_0,q,\varepsilon,R}(\mu,\nu)$ have large (or at least positive) μ-measure. This is achieved in the following lemma.

Lemma 6.2. *Fix $\lambda_0 \in U, q \geq 0$ and $\varepsilon > 0$. Let $\mathcal{V}_{\lambda_0} \subset \mathcal{M}_{\text{fin}}(X)$ be a countable set witnessing the relative dimension separability of $\mathcal{M}$ with respect to ρ_{λ_0}. For $\nu \in \mathcal{V}_{\lambda_0}$ set*

$$\underline{V}_{\lambda_0,q,\varepsilon}(\nu) = \{\mu \in \mathcal{M} : \underline{\dim}_H(\mu, \rho_{\lambda_0}) \geq q,\ \dim(\mu||\nu, \rho_{\lambda_0}) < \varepsilon\}$$

and

$$\overline{V}_{\lambda_0,q,\varepsilon}(\nu) = \{\mu \in \mathcal{M} : \overline{\dim}_H(\mu, \rho_{\lambda_0}) \geq q,\ \dim(\mu||\nu, \rho_{\lambda_0}) < \varepsilon\}.$$

Then

$$\{\mu \in \mathcal{M} : \underline{\dim}_H(\mu, \rho_{\lambda_0}) \geq q\} \subset \bigcup_{\nu \in \mathcal{V}_{\lambda_0}} \underline{V}_{\lambda_0,q,\varepsilon}(\nu) \tag{6.4}$$

and

$$\{\mu \in \mathcal{M} : \overline{\dim}_H(\mu, \rho_{\lambda_0}) \geq q\} \subset \bigcup_{\nu \in \mathcal{V}_{\lambda_0}} \overline{V}_{\lambda_0,q,\varepsilon}(\nu). \tag{6.5}$$

Moreover

$$\lim_{R \to 0} \mu\left(X \setminus A_{\lambda_0,q,\varepsilon,R}(\mu, \nu)\right) = 0 \text{ for every } \mu \in \underline{V}_{\lambda_0,q,\varepsilon}(\nu) \tag{6.6}$$

and

$$\lim_{R \to 0} \mu\left(A_{\lambda_0,q,\varepsilon,R}(\mu, \nu)\right) > 0 \text{ for every } \mu \in \overline{V}_{\lambda_0,q,\varepsilon}(\nu) \tag{6.7}$$

Proof. Equalities (6.4) and (6.5) follow from $\mathcal{M} \subset \bigcup\limits_{\nu \in \mathcal{V}_{\lambda_0}} \{\mu \in \mathcal{M} : \dim(\mu||\nu, \rho_{\lambda_0}) < \varepsilon\}$, which is a consequence of the relative dimension separability. For (6.6), note that if $\dim(\mu||\nu, \rho_{\lambda_0}) < \varepsilon$, then for μ-a.e. $x \in X$, there exists $R(x) > 0$ such that for all $0 < r < R(x)$ inequalities

$$r^{\varepsilon}\nu(B_{\lambda_0}(x, r)) \leq \mu(B_{\lambda_0}(x, r)) \leq r^{-\varepsilon}\nu(B_{\lambda_0}(x, r)) \tag{6.8}$$

hold. Similarly, if $\underline{\dim}_H(\mu, \rho_{\lambda_0}) \geq q$, then for μ-a.e. $x \in X$ there exists $R(x)$ such that

$$\mu(B_{\lambda_0}(x, r)) \leq r^{q-\varepsilon} \text{ for all } 0 < r < R(x) \tag{6.9}$$

and if μ is weakly diametrically regular with respect to ρ_{λ_0}, then for μ-a.e. $x \in X$ there exists $R(x) > 0$ such that

$$\mu(B_{\lambda_0}(x, 2r)) \leq r^{-\varepsilon}\mu(B_{\lambda_0}(x, r)) \text{ for all } 0 < r < R(x). \tag{6.10}$$

Combining (6.8), (6.9), (6.10) proves (6.6). For (6.7) it suffices to note that if $\overline{\dim}_H\mu \geq q$ then there is a set of positive μ-measure (rather than full) such that for μ-a.e. $x \in X$ there exists $R(x) > 0$ for which (6.9) holds. □

6.2. **Theorem 2.7 - Hausdorff dimension.** The following proposition is the main step of the proof of the Hausdorff dimension part of Theorem 2.7. Recall that for a finite Borel measure μ on $\mathbb{R}^d$, its s-energy for $s > 0$ is defined as

$$\mathcal{E}_s(\mu) = \int\int |x - y|^{-s} d\mu(x) d\mu(y).$$

Proposition 6.3. *Fix $L > 0$. Let $U_L = \{\lambda \in U : \mathrm{Lip}(\Pi_\lambda, \rho_\lambda) \leq L\}$. Fix $\lambda_0 \in U_L$, $q, \varepsilon > 0$ and $0 < \xi \leq 1$. There exists an open neighbourhood U' of λ_0 in U_L with the following property: for every $\nu \in \mathcal{V}_{\lambda_0}$ and every $R > 0$*

$$\int\limits_{U'} \sup_{\mu \in \underline{V}_{\lambda_0,q,\varepsilon}(\nu)} \mathcal{E}_s(\Pi_\lambda(\mu|_{A_{\lambda_0,q,\varepsilon,R}(\mu,\nu)}))d\eta(\lambda) < \infty$$

if $0 < s < \min\left\{\frac{d-\varepsilon}{1+\xi} - 16\varepsilon,\ (1+\xi)\left(q - 18\varepsilon - (d-\varepsilon)\left(\frac{1}{1-\xi} - \frac{1}{1+\xi}\right) - \frac{\varepsilon}{1-\xi}\right)\right\}$, *where $\underline{V}_{\lambda_0,q,\varepsilon}(\nu)$ is as defined in Lemma 6.2.*

Proof. Fix $\lambda_0 \in U, q > \varepsilon > 0,\ \xi > 0, R > 0, s > 0$ and $\nu \in \mathcal{V}_{\lambda_0}$. Let us denote for short $V = \underline{V}_{\lambda_0,q,\varepsilon}(\nu)$, $A_\mu = A_{\lambda_0,q,\varepsilon,R}(\mu,\nu)$, $G = G_{\lambda_0,q,\varepsilon,M}(\nu)$ and $\tilde{\mu} = \mu|_{A_\mu}$. We will use the following notation: $A \lesssim B$ means that there exists a (finite) constant $C = C(\lambda_0, q, \varepsilon, \xi, R, s, \nu)$ such that $A \leq CB + C$ (here A, B are allowed to depend on all these parameters and possibly some others). In particular, if $A \lesssim B$, then $B < \infty$ implies $A < \infty$.

Let $U' \subset U_L$ be a neighbourhood of λ_0 such that $\eta(U') < \infty$ and for every $x, y \in X$ inequalities

$$H^{-1}\rho_\lambda(x,y)^{1+\xi} \leq \rho_{\lambda_0}(x,y) \leq H\rho_\lambda(x,y)^{1-\xi} \tag{6.11}$$

and

$$\eta(\{\lambda \in U' : |\Pi_\lambda(x) - \Pi_\lambda(y)| < \rho_\lambda(x,y)r \text{ and } \rho_\lambda(x,y) \geq \delta\}) \leq K\delta^{-\varepsilon}r^{d-\varepsilon} \text{ for } \delta, r > 0 \tag{6.12}$$

hold (it exists due to assumptions (A1) and (A3)). Set $D_0 = \mathrm{diam}(X, \rho_{\lambda_0})$ and $D := LHD_0^{\frac{1}{1+\xi}}$. Note that (6.11) gives

$$\sup_{\lambda \in U'} |\Pi_\lambda(x) - \Pi_\lambda(y)| \leq LH\rho_{\lambda_0}(x,y)^{\frac{1}{1+\xi}} \leq D. \tag{6.13}$$

By Lemma 4.3, for each $n \geq 1$ take a finite cover

$$G \subset \bigcup_{x' \in F_n} B_{\lambda_0}\left(x', \frac{1}{8}D_0 2^{-n}\right) \tag{6.14}$$

such that

$$F_n \subset G \text{ and } \left\{B_{\lambda_0}\left(x', \frac{1}{16}D_0 2^{-n}\right) : x' \in F_n\right\} \text{ consists of pairwise disjoint balls.} \tag{6.15}$$

Note that for each $\mu \in V$, we have $\underline{\dim}_H \tilde{\mu} \geq \underline{\dim}_H \mu \geq q > 0$, hence $\tilde{\mu}$ has no atoms. Therefore, as Lemma 6.1 gives $A_\mu \subset G$ for $\mu \in V$, we have for each $\mu \in V$, $\lambda \in U'$ and $s > 0$

$$\begin{aligned} \mathcal{E}_s(\Pi_\lambda\tilde{\mu}) &= \int\limits_G \int\limits_G |\Pi_\lambda(x) - \Pi_\lambda(y)|^{-s} d\tilde{\mu}(x)d\tilde{\mu}(y) \\ &= \sum_{n=0}^{\infty} \int\limits_G \int\limits_G \mathbb{1}_{\{D_0 2^{-(n+1)} < \rho_{\lambda_0}(x,y) \leq D_0 2^{-n}\}} |\Pi_\lambda(x) - \Pi_\lambda(y)|^{-s} d\tilde{\mu}(x)d\tilde{\mu}(y). \end{aligned} \tag{6.16}$$

It follows from (6.13) that

$$\text{if } \rho_{\lambda_0}(x,y) \leq D_0 2^{-n}, \text{ then } |\Pi_\lambda(x) - \Pi_\lambda(y)| \leq D2^{-\frac{n}{1+\xi}}. \tag{6.17}$$

Therefore

$$\begin{aligned}(6.18)\quad &\mathbb{1}_{\left\{D_0 2^{-(n+1)}<\rho_{\lambda_0}(x,y)\le D_0 2^{-n}\right\}}|\Pi_\lambda(x)-\Pi_\lambda(y)|^{-s}\\ &\le \mathbb{1}_{\left\{D_0 2^{-(n+1)}<\rho_{\lambda_0}(x,y)\le D_0 2^{-n}\right\}} D^{-s}\sum_{m=0}^{\infty} 2^{s(m+1+\frac{n}{1+\xi})}\mathbb{1}_{\left\{|\Pi_\lambda(x)-\Pi_\lambda(y)|\le 2^{-m}D2^{-\frac{n}{1+\xi}}\right\}}.\end{aligned}$$

Indeed, if $\rho_{\lambda_0}(x,y) \le D_0 2^{-n}$ and $\Pi_\lambda(x) \ne \Pi_\lambda(y)$, then by (6.17) there exists $m \ge 0$ such that $2^{-(m+1)}D2^{-\frac{n}{1+\xi}} < |\Pi_\lambda(x)-\Pi_\lambda(y)| \le 2^{-m}D2^{-\frac{n}{1+\xi}}$, so (6.18) follows. If $\Pi_\lambda(x) = \Pi_\lambda(y)$ then both sides of the inequality in (6.18) are infinite provided $D_0 2^{-(n+1)} < \rho_{\lambda_0}(x,y) \le D_0 2^{-n}$. Applying (6.18) to (6.16) gives

$$\begin{aligned}\mathcal{E}_s(\Pi_\lambda\tilde{\mu}) \lesssim &\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} 2^{s(m+\frac{n}{1+\xi})}\\ &\times \tilde{\mu}\otimes\tilde{\mu}\left(\left\{(x,y)\in G^2 : D_0 2^{-(n+1)} < \rho_{\lambda_0}(x,y) \le D_0 2^{-n},\ |\Pi_\lambda(x)-\Pi_\lambda(y)| \le 2^{-m}D2^{-\frac{n}{1+\xi}}\right\}\right).\end{aligned}$$

To bound each term in the sum, we can cover G^2 by products of balls from (6.14) corresponding to $n+m$, obtaining

$$\begin{aligned}\mathcal{E}_s(\Pi_\lambda\tilde{\mu}) \lesssim &\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} 2^{s(m+\frac{n}{1+\xi})}\sum_{x'\in F_{n+m}}\sum_{y'\in F_{n+m}}\tilde{\mu}\left(B_{\lambda_0}\left(x',\frac{1}{8}D_0 2^{-(n+m)}\right)\right)\tilde{\mu}\left(B_{\lambda_0}\left(y',\frac{1}{8}D_0 2^{-(n+m)}\right)\right)\\ &\times \mathbb{1}_{\left\{\frac{1}{2}D_0 2^{-(n+1)}<\rho_{\lambda_0}(x',y')\le 2D_0 2^{-n}\right\}}\times \mathbb{1}_{\left\{|\Pi_\lambda(x')-\Pi_\lambda(y')|\le 2D2^{-\frac{n+m}{1+\xi}}\right\}}.\end{aligned}$$

We have used here the observation that if $x \in B_{\lambda_0}\left(x', \frac{1}{8}D_0 2^{-(n+m)}\right)$ and $y \in B_{\lambda_0}\left(y', \frac{1}{8}D_0 2^{-(n+m)}\right)$ then $D_0 2^{-(n+1)} < \rho_{\lambda_0}(x,y) \le D_0 2^{-n}$ implies $\frac{1}{2}D_0 2^{-(n+1)} < \rho_{\lambda_0}(x',y') \le 2D_0 2^{-n}$, while $|\Pi_\lambda(x) - \Pi_\lambda(y)| \le 2^{-m}D2^{-\frac{n}{1+\xi}}$ implies $|\Pi_\lambda(x')-\Pi_\lambda(y')| \le 2^{-m}D2^{-\frac{n}{1+\xi}} + 2LH8^{-\frac{1}{1+\xi}}D_0^{\frac{1}{1+\xi}}2^{-\frac{n+m}{1+\xi}} \le 2D2^{-\frac{n+m}{1+\xi}}$ by (6.13). Let $N \in \mathbb{N}$ be such that $\frac{1}{16}D_0 2^{-N} < R/2$. Then

$$\begin{aligned}&\mathcal{E}_s(\Pi_\lambda\tilde{\mu})\\ (6.19)\quad &\lesssim \sum_{m+n>N} 2^{s(m+\frac{n}{1+\xi})}\sum_{x'\in F_{n+m}}\sum_{y'\in F_{n+m}}\tilde{\mu}\left(B_{\lambda_0}\left(x',\frac{1}{8}D_0 2^{-(n+m)}\right)\right)\tilde{\mu}\left(B_{\lambda_0}\left(y',\frac{1}{8}D_0 2^{-(n+m)}\right)\right)\\ &\quad\times \mathbb{1}_{\left\{\frac{1}{2}D_0 2^{-(n+1)}<\rho_{\lambda_0}(x',y')\le 2D_0 2^{-n}\right\}}\times \mathbb{1}_{\left\{|\Pi_\lambda(x')-\Pi_\lambda(y')|\le 2D2^{-\frac{n+m}{1+\xi}}\right\}},\end{aligned}$$

as by (6.3)

$$\begin{aligned}&\sum_{n=0}^{N}\sum_{m=0}^{N} 2^{s(m+\frac{n}{1+\xi})}\sum_{x'\in F_{n+m}}\sum_{y'\in F_{n+m}}\tilde{\mu}\left(B_{\lambda_0}\left(x',\frac{1}{8}D_0 2^{-(n+m)}\right)\right)\tilde{\mu}\left(B_{\lambda_0}\left(y',\frac{1}{8}D_0 2^{-(n+m)}\right)\right)\\ &\qquad\times \mathbb{1}_{\left\{\frac{1}{2}D_0 2^{-(n+1)}<\rho_{\lambda_0}(x',y')\le 2D_0 2^{-n}\right\}}\times \mathbb{1}_{\left\{|\Pi_\lambda(x')-\Pi_\lambda(y')|\le 2D2^{-\frac{n+m}{1+\xi}}\right\}}\\ &\quad\le 2^{sN(1+\frac{1}{1+\xi})}M^2\nu(X)^2\sum_{n=0}^{N}\sum_{m=0}^{N}\#F_{n+m}^2\\ &\quad\lesssim 1.\end{aligned}$$

Continuing from (6.19), as $F_{n+m} \subset G$, we can invoke (6.2) to bound further

$$\mathcal{E}_s(\Pi_\lambda \tilde{\mu}) \lesssim \sum_{m+n>N} 2^{s(m+\frac{n}{1+\xi})+10\varepsilon(n+m)} \sum_{x' \in F_{n+m}} \sum_{y' \in F_{n+m}} \nu\left(B_{\lambda_0}\left(x', \frac{1}{8}D_0 2^{-(n+m)}\right)\right) \times \nu\left(B_{\lambda_0}\left(y', \frac{1}{8}D_0 2^{-(n+m)}\right)\right) \times \mathbb{1}_{\left\{\frac{1}{2}D_0 2^{-(n+1)} < \rho_{\lambda_0}(x',y') \leq 2D_0 2^{-n}\right\}} \times \mathbb{1}_{\left\{|\Pi_\lambda(x') - \Pi_\lambda(y')| \leq 2D 2^{-\frac{n+m}{1+\xi}}\right\}}.$$

Note further that if $\frac{1}{2}D_0 2^{-(n+1)} < \rho_{\lambda_0}(x', y')$, then (6.11) implies $\rho_\lambda(x', y') > Q_\xi 2^{-\frac{n}{1-\xi}}$, where $Q_\xi = H^{-1}\left(D_0/4\right)^{\frac{1}{1-\xi}}$, and so

$$\mathbb{1}_{\left\{\frac{1}{2}D_0 2^{-(n+1)} < \rho_{\lambda_0}(x',y') \leq 2D_0 2^{-n}\right\}} \times \mathbb{1}_{\left\{|\Pi_\lambda(x') - \Pi_\lambda(y')| \leq 2D 2^{-\frac{n+m}{1+\xi}}\right\}} \leq \mathbb{1}_{\left\{\frac{1}{2}D_0 2^{-(n+1)} < \rho_{\lambda_0}(x',y') \leq 2D_0 2^{-n}\right\}} \times \mathbb{1}_{\left\{|\Pi_\lambda(x') - \Pi_\lambda(y')| \leq C\rho_\lambda(x',y') 2^{-\frac{m}{1+\xi}+n\left(\frac{1}{1-\xi}-\frac{1}{1+\xi}\right)},\ \rho_\lambda(x',y') > Q_\xi 2^{-\frac{n}{1-\xi}}\right\}}$$

for a constant $C = C(\lambda_0, \xi)$. Combing the last two bounds, which are uniform in $\mu \in V$ and $\lambda \in U'$, we obtain

$$\sup_{\mu \in V} \mathcal{E}_s(\Pi_\lambda \tilde{\mu}) \lesssim \sum_{m+n>N} 2^{s(m+\frac{n}{1+\xi})+10\varepsilon(n+m)} \sum_{x' \in F_{n+m}} \sum_{y' \in F_{n+m}} \nu\left(B_{\lambda_0}\left(x', \frac{1}{8}D_0 2^{-(n+m)}\right)\right) \times \nu\left(B_{\lambda_0}\left(y', \frac{1}{8}D_0 2^{-(n+m)}\right)\right) \times \mathbb{1}_{\left\{\frac{1}{2}D_0 2^{-(n+1)} < \rho_{\lambda_0}(x',y') \leq 2D_0 2^{-n}\right\}} \times \mathbb{1}_{\left\{|\Pi_\lambda(x') - \Pi_\lambda(y')| \leq C\rho_\lambda(x',y') 2^{-\frac{m}{1+\xi}+n\left(\frac{1}{1-\xi}-\frac{1}{1+\xi}\right)},\ \rho_\lambda(x',y') > Q_\xi 2^{-\frac{n}{1-\xi}}\right\}}.$$

Integrating with respect to $d\eta(\lambda)$ and using the transversality condition (6.12) yields

$$\begin{aligned}
\int_{U'} \sup_{\mu \in V} \mathcal{E}_s(\Pi_\lambda \tilde{\mu})\, d\eta(\lambda) \lesssim & \sum_{m+n>N} 2^{s(m+\frac{n}{1+\xi})+10\varepsilon(n+m)} \\
& \times \sum_{x' \in F_{n+m}} \sum_{y' \in F_{n+m}} \nu\left(B_{\lambda_0}\left(x', \frac{1}{8}D_0 2^{-(n+m)}\right)\right) \nu\left(B_{\lambda_0}\left(y', \frac{1}{8}D_0 2^{-(n+m)}\right)\right) \\
& \times \mathbb{1}_{\left\{\frac{1}{2}D_0 2^{-(n+1)} < \rho_{\lambda_0}(x',y') \leq 2D_0 2^{-n}\right\}} \\
& \eta\left(\left\{\lambda \in U' : |\Pi_\lambda(x') - \Pi_\lambda(y')| \leq C\rho_\lambda(x',y') 2^{-\frac{m}{1+\xi}+n\left(\frac{1}{1-\xi}-\frac{1}{1+\xi}\right)},\ \rho_\lambda(x',y') > Q_\xi 2^{-\frac{n}{1-\xi}}\right\}\right) \\
\lesssim & \sum_{m+n>N} 2^{s(m+\frac{n}{1+\xi})+10\varepsilon(n+m)-\frac{m(d-\varepsilon)}{1+\xi}+n\left((d-\varepsilon)\left(\frac{1}{1-\xi}-\frac{1}{1+\xi}\right)+\frac{\varepsilon}{1-\xi}\right)} \\
& \times \sum_{x' \in F_{n+m}} \sum_{y' \in F_{n+m}} \nu\left(B_{\lambda_0}\left(x', \frac{1}{8}D_0 2^{-(n+m)}\right)\right) \nu\left(B_{\lambda_0}\left(y', \frac{1}{8}D_0 2^{-(n+m)}\right)\right) \\
& \times \mathbb{1}_{\left\{\frac{1}{2}D_0 2^{-(n+1)} < \rho_{\lambda_0}(x',y') \leq 2D_0 2^{-n}\right\}}.
\end{aligned} \tag{6.20}$$

To deal with the sums over x', y' we recall that $F_{n+m} \subset G$ and use the definition of G to obtain

(6.21)
$$\begin{aligned}&\sum_{x' \in F_{n+m}} \sum_{y' \in F_{n+m}} \nu\left(B_{\lambda_0}\left(x', \frac{1}{8} D_0 2^{-(n+m)}\right)\right) \nu\left(B_{\lambda_0}\left(y', \frac{1}{8} D_0 2^{-(n+m)}\right)\right) \\ &\qquad \times \mathbb{1}_{\left\{\frac{1}{2} D_0 2^{-(n+1)} < \rho_{\lambda_0}(x', y') \leq 2 D_0 2^{-n}\right\}} \\ &\lesssim 2^{6\varepsilon(n+m)} \sum_{x' \in F_{n+m}} \sum_{y' \in F_{n+m}} \nu\left(B_{\lambda_0}\left(x', \frac{1}{16} D_0 2^{-(n+m)}\right)\right) \nu\left(B_{\lambda_0}\left(y', \frac{1}{16} D_0 2^{-(n+m)}\right)\right) \\ &\qquad \times \mathbb{1}_{\left\{\frac{1}{2} D_0 2^{-(n+1)} < \rho_{\lambda_0}(x', y') \leq 2 D_0 2^{-n}\right\}}.\end{aligned}$$

By (6.15) and the definition of G we have for $n+m > N$ (recall that N was chosen so that $\frac{1}{16} D_0 2^{-N} < R/2$)

(6.22)
$$\begin{aligned}&\sum_{y' \in F_{n+m}} \nu\left(B_{\lambda_0}\left(y', \frac{1}{16} D_0 2^{-(n+m)}\right)\right) \mathbb{1}_{\left\{\frac{1}{2} D_0 2^{-(n+1)} < \rho_{\lambda_0}(x', y') \leq 2 D_0 2^{-n}\right\}} \\ &\qquad \leq \nu\left(B_{\lambda_0}(x', 4 D_0 2^{-n})\right) \\ &\qquad \lesssim 2^{-n(q-2\varepsilon)},\end{aligned}$$

where the last inequality holds for large enough n by the definition of G, while for the remaining finitely many n's it holds as $\nu(X) < \infty$. Applying (6.22) to (6.21) and invoking once more disjointness of the balls in (6.15) gives

$$\begin{aligned}&\sum_{x' \in F_{n+m}} \sum_{y' \in F_{n+m}} \nu\left(B_{\lambda_0}\left(x', \frac{1}{8} D_0 2^{-(n+m)}\right)\right) \nu\left(B_{\lambda_0}\left(y', \frac{1}{8} D_0 2^{-(n+m)}\right)\right) \\ &\qquad \times \mathbb{1}_{\left\{\frac{1}{2} D_0 2^{-(n+1)} < \rho_{\lambda_0}(x', y') \leq 2 D_0 2^{-n}\right\}} \\ &\qquad \lesssim \nu(X) 2^{-n(q-2\varepsilon)+6\varepsilon(n+m)}.\end{aligned}$$

Combining this with (6.20) gives finally

$$\int_{U'} \sup_{\mu \in V} \mathcal{E}_s(\Pi_\lambda \tilde{\mu})\, d\eta(\lambda) \lesssim \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} 2^{m\left(s+16\varepsilon-\frac{d-\varepsilon}{1+\xi}\right)+n\left(\frac{s}{1+\xi}+18\varepsilon+(d-\varepsilon)\left(\frac{1}{1-\xi}-\frac{1}{1+\xi}\right)+\frac{\varepsilon}{1-\xi}-q\right)}.$$

The last sum is finite provided that

$$s < \frac{d-\varepsilon}{1+\xi} - 16\varepsilon \quad \text{and} \quad s < (1+\xi)\left(q - 18\varepsilon - (d-\varepsilon)\left(\frac{1}{1-\xi} - \frac{1}{1+\xi} - \frac{\varepsilon}{1-\xi}\right)\right).$$

□

Proof of point (1) of Theorem 2.7. By taking a countable intersection over λ, it suffices to prove that for every $L \geq 1$, the conclusion of the theorem holds for η-a.e. $\lambda \in U_L = \{\lambda \in U : \mathrm{Lip}(\Pi_\lambda, \rho_\lambda) \leq L\}$.

It is well known that for a Borel measure μ on $\mathbb{R}^n$, if $\mathcal{E}_s(\mu) < \infty$, then $\underline{\dim}_H \mu \geq s$, see [Fal14, Theorem 4.13]. Therefore, a consequence of Proposition 6.3 is that for fixed $q, \varepsilon > 0$ and $0 < \xi \leq 1$ there exists an open neighbourhood U' of λ_0 in U_L such for every $\nu \in \mathcal{V}_{\lambda_0}$ and every $R > 0$ we have that for η-a.e. $\lambda \in U'$ inequality

(6.23)
$$\underline{\dim}_H \Pi_\lambda(\mu|_{A_{\lambda_0,q,\varepsilon,R}(\mu,\nu)}) \geq \min\left\{\frac{d-\varepsilon}{1+\xi} - 16\varepsilon,\ (1+\xi)\left(q - 18\varepsilon - (d-\varepsilon)\left(\frac{1}{1-\xi} - \frac{1}{1+\xi}\right) - \frac{\varepsilon}{1-\xi}\right)\right\}$$

holds for every $\mu \in \underline{V}_{\lambda_0,q,\varepsilon}(\nu)$.

Point (1) of Theorem 2.7 follows from (6.23) by invoking Lemma 6.2, letting $q, \varepsilon, \xi, R \to 0$ and taking countable intersections over the parameter space. Below we explain this more precisely.

Letting $R \to 0$ in (6.23) and taking countable intersection over λ we see by (6.6) that for η-a.e. $\lambda \in U'$ inequality

$$(6.24)\quad \underline{\dim}_H \Pi_\lambda \mu \geq \min\left\{ \frac{d-\varepsilon}{1+\xi} - 16\varepsilon,\ (1+\xi)\left(q - 18\varepsilon - (d-\varepsilon)\left(\frac{1}{1-\xi} - \frac{1}{1+\xi}\right) - \frac{\varepsilon}{1-\xi}\right)\right\}.$$

holds for every $\mu \in \underline{V}_{\lambda_0,q,\varepsilon}(\nu)$. By (6.4), as $\mathcal{V}_{\lambda_0}$ is countable, we have that for μ-a.e. $\lambda \in U'$, inequality (6.24) holds for every $\mu \in \mathcal{M}$ with $\underline{\dim}_H(\mu, \rho_{\lambda_0}) \geq q > 0$. We can assume by (A1) that U' is such that $H^{-1}\rho_\lambda(x,y)^{1+\xi} \leq \rho_{\lambda_0}(x,y) \leq H\rho_\lambda(x,y)^{1-\xi}$ holds for $\lambda \in U$ and therefore $\underline{\dim}_H(\mu, \rho_{\lambda_0}) \geq \frac{\underline{\dim}_H(\mu,\rho_\lambda)}{1+\xi}$ for all $\lambda \in U'$ and $\mu \in \mathcal{M}$ (since $B_{\lambda_0}(x,r) \subset B_\lambda(x, Hr^{\frac{1}{1+\xi}})$). We conclude that for η-a.e. $\lambda \in U'$, inequality (6.24) holds for every $\mu \in \mathcal{M}$ satisfying $\underline{\dim}_H(\mu, \rho_\lambda) \geq (1+\xi)q > 0$. As U_L is Lindelöf (since U is hereditary Lindelöf) we can find a countable cover $\{U_i'\}_{i=1}^\infty$ of U_L such that for η-a.e. $\lambda \in U_i'$ inequality (6.24) holds for every $\mu \in \mathcal{M}$ with $\underline{\dim}_H(\mu, \rho_\lambda) \geq (1+\xi)q > 0$ and hence the same is true for η-a.e $\lambda \in U_L$. Finally, taking countable intersections over $q, \varepsilon, \xi \in \mathbb{Q} \cap (0,\infty)$ we conclude that for η-a.e. $\lambda \in U_L$ we have

$$\underline{\dim}_H \Pi_\lambda \mu \geq \min\{d, \underline{\dim}_H \mu\} \text{ for every } \mu \in \mathcal{M} \text{ such that } \underline{\dim}_H(\mu, \rho_\lambda) > 0.$$

The proof of the Theorem for the lower Hausdorff dimension is finished upon noting that $\underline{\dim}_H \Pi_\lambda \mu \leq \underline{\dim}_H(\mu, \rho_\lambda)$ as Π_λ is Lipschitz in ρ_λ (so in particular $\underline{\dim}_H \Pi_\lambda = 0$ if $\underline{\dim}_H \mu = 0$) and $\underline{\dim}_H \Pi_\lambda \mu \leq d$ as $\Pi_\lambda \mu$ is a measure on $\mathbb{R}^d$.

The case of the upper Hausdorff dimension can be treated in exactly the same way, with the use of (6.5) and (6.7) instead of (6.4) and (6.6), respectively. ☐

6.3. **Theorem 2.7 - Assouad dimension and Hölder regularity.** The remainder of this section is devoted to the proof of point (2) of Theorem 2.7. We begin with a lemma which combines the transversality assumption (A3) with covering bounds coming from the Assouad dimension.

Lemma 6.4. *Fix $L > 0$ and set $U_L = \{\lambda \in U : \mathrm{Lip}(\Pi_\lambda, \rho_\lambda) \leq L\}$. For every $\lambda_0 \in U_L$, $\theta > 0$ there exists an open neighbourhood U' of $\lambda_0 \in U_L$ and a constant $D = D(\lambda_0, L, \theta)$ such that for every $x \in X, 0 < r \leq \delta \leq 1$ inequality*

$$(6.25)\qquad \eta\left(\left\{\lambda \in U' : \underset{y\in X}{\exists}\ |\Pi_\lambda(x) - \Pi_\lambda(y)| \leq r,\ \rho_{\lambda_0}(x,y) \geq \delta\right\}\right) \leq D\left(\frac{r}{\delta}\right)^{d-\dim_A(X,\rho_{\lambda_0})-\theta} \delta^{-\theta}$$

holds provided that $\dim_A(X, \rho_{\lambda_0}) < d$.

Proof. Fix $\xi > 0$ (we will specify it later in terms of θ and θ') and let $U' \subset U_L$ be a neighbourhood of λ_0 such that $\eta(U') < \infty$ and for every $x, y \in X$ inequalities

$$(6.26)\qquad H^{-1}\rho_\lambda(x,y)^{1+\xi} \leq \rho_{\lambda_0}(x,y) \leq H\rho_\lambda(x,y)^{1-\xi}$$

and

$$(6.27)\qquad \eta(\{\lambda \in U' : |\Pi_\lambda(x) - \Pi_\lambda(y)| < \rho_\lambda(x,y)r \text{ and } \rho_\lambda(x,y) \geq \delta\}) \leq K\delta^{-\xi}r^{d-\xi} \text{ for } \delta, r > 0$$

hold. Fix $x \in X$. Given $i \geq 0$ take a cover

$$\left\{y \in X : 2^i\delta \leq \rho_{\lambda_0}(x,y) < 2^{i+1}\delta\right\} \subset \bigcup_{m=1}^{N_i} B_{\lambda_0}(y_{i,m}, r)$$

such that

$$N_i \le C_\xi \left(\frac{2^{i+1}\delta}{r}\right)^{\dim_A(X,\rho_{\lambda_0})+\xi} \quad \text{and} \quad 2^i\delta \le \rho_{\lambda_0}(x,y_{i,m}) < 2^{i+1}\delta. \tag{6.28}$$

It exists due to the definition of the Assouad dimension (note that C_ξ does not depend on x). By (6.26) we have for $\lambda \in U',\ i \ge 0,\ 1 \le m \le N_i$

$$B_{\lambda_0}(y_{i,m}, r) \subset B_\lambda(y_{i,m}, Hr^{\frac{1}{1+\xi}}) \text{ and } H^{-1}2^{\frac{i}{1-\xi}}\delta^{\frac{1}{1-\xi}} \le \rho_\lambda(x,y_{i,m}) < H2^{\frac{i+1}{1+\xi}}\delta^{\frac{1}{1+\xi}}. \tag{6.29}$$

This gives

$$\begin{aligned}
&\eta\left(\left\{\lambda \in U' : \underset{y\in X}{\exists}\ |\Pi_\lambda(x) - \Pi_\lambda(y)| \le r,\ \rho_{\lambda_0}(x,y) \ge \delta\right\}\right)\\
&\le \sum_{i=0}^{\infty} \eta\left(\left\{\lambda \in U' : \underset{y\in X}{\exists}\ |\Pi_\lambda(x) - \Pi_\lambda(y)| \le r,\ 2^i\delta \le \rho_{\lambda_0}(x,y) < 2^{i+1}\delta\right\}\right)\\
&\le \sum_{i=0}^{\infty}\sum_{m=1}^{N_i} \eta\left(\left\{\lambda \in U' : \underset{y\in B_{\lambda_0}(y_{i,m},r)}{\exists}\ |\Pi_\lambda(x) - \Pi_\lambda(y)| \le r\right\}\right)\\
&\le \sum_{i=0}^{\infty}\sum_{m=1}^{N_i} \eta\left(\left\{\lambda \in U' : \underset{y\in B_\lambda(y_{i,m},Hr^{\frac{1}{1+\xi}})}{\exists}\ |\Pi_\lambda(x) - \Pi_\lambda(y)| \le r\right\}\right)
\end{aligned} \tag{6.30}$$

If $\lambda \in U'$ and $y \in B_\lambda(y_{i,m}, Hr^{\frac{1}{1+\xi}})$ is such that $|\Pi_\lambda(x) - \Pi_\lambda(y)| \le r$, then $|\Pi_\lambda(x) - \Pi_\lambda(y_{i,m})| \le |\Pi_\lambda(x) - \Pi_\lambda(y)| + |\Pi_\lambda(y) - \Pi_\lambda(y_{i,m})| \le r + LHr^{\frac{1}{1+\xi}} \le Mr^{\frac{1}{1+\xi}}$ for a constant $M = LH + 1$. Using this together with (6.29), we can continue (6.30) to obtain

$$\begin{aligned}
&\eta\left(\left\{\lambda \in U' : \underset{y\in X}{\exists}\ |\Pi_\lambda(x) - \Pi_\lambda(y)| \le r,\ \rho_{\lambda_0}(x,y) \ge \delta\right\}\right)\\
&\le \sum_{i=0}^{\infty}\sum_{m=1}^{N_i} \eta\left(\left\{\lambda \in U' : |\Pi_\lambda(x) - \Pi_\lambda(y_{i,m})| \le Mr^{\frac{1}{1+\xi}},\ \rho_\lambda(x,y_{i,m}) \ge H^{-1}2^{\frac{i}{1-\xi}}\delta^{\frac{1}{1-\xi}}\right\}\right)\\
&\le \sum_{i=0}^{\infty}\sum_{m=1}^{N_i} \eta\left(\left\{\lambda \in U' : |\Pi_\lambda(x) - \Pi_\lambda(y_{i,m})| \le \rho_\lambda(x,y_{i,m})MHr^{\frac{1}{1+\xi}}2^{\frac{-i}{1-\xi}}\delta^{\frac{-1}{1-\xi}},\ \rho_\lambda(x,y_{i,m}) \ge H^{-1}2^{\frac{i}{1-\xi}}\delta^{\frac{1}{1-\xi}}\right\}\right).
\end{aligned}$$

Applying (6.27) and (6.28) gives

$$\begin{aligned}
&\eta\left(\left\{\lambda \in U' : \underset{y\in X}{\exists}\ |\Pi_\lambda(x) - \Pi_\lambda(y)| \le r,\ \rho_{\lambda_0}(x,y) \ge \delta\right\}\right)\\
&\le K(MH)^{d-\xi} r^{\frac{d-\xi}{1+\xi}} H^{\xi} \delta^{\frac{-d}{1-\xi}} \sum_{i=0}^{\infty} N_i 2^{\frac{-id}{1-\xi}}\\
&\le Cr^{\frac{d-\xi}{1+\xi} - \dim_A(X,\rho_{\lambda_0}) - \xi}\delta^{-\frac{d}{1-\xi} + \dim_A(X,\rho_{\lambda_0}) + \xi} \sum_{i=0}^{\infty} 2^{i\left(\dim_A(X,\rho_{\lambda_0}) + \xi - \frac{d}{1-\xi}\right)},
\end{aligned}$$

for some constant $C = C(\lambda_0, L, \xi)$. If now $\xi > 0$ was chosen small enough to guarantee $\dim_A(X,\rho_{\lambda_0}) + \xi - \frac{d}{1-\xi} < 0$ (recall that we consider only the case $\dim_A(X,\rho_{\lambda_0}) < d$), then the above sum converges, giving

$$\eta\left(\left\{\lambda\in U' : \underset{y\in X}{\exists}\ |\Pi_\lambda(x)-\Pi_\lambda(y)|\le r,\ \rho_{\lambda_0}(x,y)\ge\delta\right\}\right)$$
$$\le D\left(\frac{r}{\delta}\right)^{\frac{d-\xi}{1+\xi}-\dim_A(X,\rho_{\lambda_0})-\xi}\delta^{-\left(\frac{d}{1-\xi}-\frac{d-\xi}{1+\xi}\right)},$$

for some constant $D=D(\lambda_0,L,\xi)$. Finally, if $\xi>0$ was chosen small enough to satisfy $\frac{d-\xi}{1+\xi}-\dim_A(X,\rho_{\lambda_0})-\xi\ge d-\dim_A(X,\rho_{\lambda_0})-\theta$ and $\frac{d}{1-\xi}-\frac{d-\xi}{1+\xi}\le\theta$, then (6.25) holds. □

Proof of point (2) **of Theorem 2.7.** Fix $\alpha\in(0,1)$, $L\ge1$ and set

$$U_L=\{\lambda\in U:\mathrm{Lip}(\Pi_\lambda,\rho_\lambda)\le L\ \text{ and }\ \dim_A(X,\rho_\lambda)<d\}.$$

As U is hereditary Lindelöf, by taking countable intersections in the parameter space U it suffices to prove that every $\lambda_0\in U_L$ has an open neighbourhood U' in U_L such that for η-a.e. $\lambda\in U'$, every $\mu\in\mathcal{M}$ has the property that for μ-a.e. $x\in X$ there exists C such that

$$\rho_\lambda(x,y)\le C|\Pi_\lambda(x)-\Pi_\lambda(y)|^\alpha \text{ for every } y\in X. \tag{6.31}$$

For that, by Lemma 6.2 with $q=0$ (and again taking countable intersections over λ) it suffices to show that for $\varepsilon>0$ small enough one has for every $\nu\in\mathcal{V}_{\lambda_0}$ and $R>0$

$$\lim_{C\to\infty}\int\limits_{U'}\sup_{\mu\in\underline{V}_{\lambda_0,0,\varepsilon}(\nu)}\mu|_{A_{\lambda_0,0,\varepsilon,R}(\mu,\nu)}(E_{C,\lambda})\,d\eta(\lambda)=0, \tag{6.32}$$

where

$$E_{C,\lambda}=\left\{x\in X:\underset{y\in X}{\exists}\ \rho_\lambda(x,y)>C|\Pi_\lambda(x)-\Pi_\lambda(y)|^\alpha\right\}.$$

Indeed, if (6.32) holds, then (note that $\mu|_{A_{\lambda_0,0,\varepsilon,R}(\mu,\nu)}(E_{C,\lambda})$ is decreasing in C)

$$\lim_{C\to\infty}\sup_{\mu\in\underline{V}_{\lambda_0,0,\varepsilon}(\nu)}\mu|_{A_{\lambda_0,0,\varepsilon,R}(\mu,\nu)}(E_{C,\lambda})=0 \text{ for } \eta\text{-a.e. } \lambda\in U'$$

and hence

$$\text{for } \eta\text{-a.e. } \lambda\in U',\ \lim_{C\to\infty}\mu|_{A_{\lambda_0,0,\varepsilon,R}(\mu,\nu)}(E_{C,\lambda})=0 \text{ for every } \mu\in\underline{V}_{\lambda_0,0,\varepsilon}(\nu).$$

This shows that for η-a.e. $\lambda\in U'$ and every $\mu\in\underline{V}_{\lambda_0,0,\varepsilon}(\nu)$, for μ-a.e. every $x\in A_{\lambda_0,0,\varepsilon,R}(\mu,\nu)$ there exists C such that (6.31) holds. By (6.6) the above holds then for μ-a.e. $x\in X$ and by (6.4) we have $\mathcal{M}\subset\bigcup\limits_{\nu\in\mathcal{V}_{\lambda_0}}\underline{V}_{\lambda_0,0,\varepsilon}(\nu)$ with the sum being countable, so we can extend it further to every $\mu\in\mathcal{M}$.

We shall now prove that for fixed $\lambda_0\in U_L$ there exists a neighbourhood U' of λ_0 in U_L such that (6.32) holds provided that $\varepsilon>0$ is small enough. For simplicity denote $V=\underline{V}_{\lambda_0,0,\varepsilon}(\nu)$, $A_\mu=A_{\lambda_0,0,\varepsilon,R}(\mu,\nu)$, $G=G_{\lambda_0,0,\varepsilon,R}(\nu)$ and $\tilde\mu=\mu|_{A_\mu}$. For $\xi>0$ (we will specify later how small ξ has to be) let U' be a neighbourhood of λ_0 such that for $\lambda\in U'$

$$H^{-1}\rho_\lambda(x,y)^{1+\xi}\le\rho_{\lambda_0}(x,y)\le H\rho_\lambda(x,y)^{1-\xi} \text{ for every } x,y\in X. \tag{6.33}$$

Set $D_0=\mathrm{diam}(X,\rho_{\lambda_0})$ and $D:=LHD_0^{\frac{1}{1+\xi}}$, so that $\sup\limits_{\lambda\in U'}\sup\limits_{x,y\in X}|\Pi_\lambda(x)-\Pi_\lambda(y)|\le D$. By Lemma 4.3, for $i\ge0$ take a finite cover

$$G\subset\bigcup_{x'\in F_i}B_{\lambda_0}(x',D_02^{-i}) \tag{6.34}$$

such that

$$F_i \subset G \text{ and } \left\{ B_{\lambda_0}(x', \frac{1}{2}D_0 2^{-i}) : x' \in F_i \right\} \text{ consists of pairwise disjoint balls.} \tag{6.35}$$

Let

$$R_{C,i} = \left\{ (x,\lambda) \in G \times U' : \underset{y\in X}{\exists}\ D2^{-(i+1)} < |\Pi_\lambda(x) - \Pi_\lambda(y)| \le D2^{-i},\ \rho_\lambda(x,y) > C|\Pi_\lambda(x) - \Pi_\lambda(y)|^\alpha \right\}.$$

Note that if $x \in B_{\lambda_0}(x', D_0 2^{-i})$ and $(x,\lambda) \in R_{C,i}$, then there exists $y \in X$ such that

$$|\Pi_\lambda(x') - \Pi_\lambda(y)| \le |\Pi_\lambda(x') - \Pi_\lambda(x)| + |\Pi_\lambda(x) - \Pi_\lambda(y)| \le LHD_0^{\frac{1}{1+\xi}} 2^{-\frac{i}{1+\xi}} + D2^{-i} \le 2D2^{-\frac{i}{1+\xi}}$$

and

$$\rho_{\lambda_0}(x', y) \ge \rho_{\lambda_0}(x,y) - \rho_{\lambda_0}(x,x') \ge H^{-1}C^{1+\xi}D^{1+\xi}2^{-\alpha(1+\xi)(i+1)} - D_0 2^{-i} \ge aC2^{-\alpha(1+\xi)i}$$

for some constant $a = a(\lambda_0, L, \xi) > 0$ provided that C is large enough and $\xi > 0$ is small enough to guarantee $\alpha(1+\xi) < 1$. Therefore, for such C and ξ we have by (6.34)

$$R_{C,i} \subset \bigcup_{x' \in F_i} B_{\lambda_0}(x', D_0 2^{-i}) \times Q_{C,i}(x'),$$

where

$$Q_{C,i}(x') = \{\lambda \in U' : |\Pi_\lambda(x') - \Pi_\lambda(y)| \le 2D2^{-\frac{i}{1+\xi}},\ \rho_{\lambda_0}(x', y) \ge aC2^{-\alpha(1+\xi)i}\}.$$

This gives

$$\begin{aligned} \int_{U'} \sup_{\mu\in V} \tilde{\mu}\left(E_{C,\lambda}\right) d\eta(\lambda) &\le \sum_{i=0}^{\infty} \int_{U'} \sup_{\mu\in V} \tilde{\mu}\left(\{x \in G : (x,\lambda) \in R_{C,i}\}\right) d\eta(\lambda) \\ &\le \sum_{i=0}^{\infty} \int_{U'} \sup_{\mu\in V} \sum_{x'\in F_i} \tilde{\mu}(B_{\lambda_0}(x', D_0 2^{-i})) \mathbb{1}_{Q_{C,i}(x')}(\lambda) d\eta(\lambda). \end{aligned} \tag{6.36}$$

Below we use the following notation: $A \lesssim B$ means that there exists a constant $C = C(\lambda_0, \varepsilon, \xi, R, L, \nu)$ such that $A \le CB$. Let now N be such that $\frac{1}{2}D_0 2^{-N} < R/2$ and note that applying (6.3) gives

$$\begin{aligned} &\lim_{C\to\infty} \sum_{i=0}^{N} \int_{U'} \sup_{\mu\in V} \sum_{x'\in F_i} \tilde{\mu}(B_{\lambda_0}(x', D_0 2^{-i})) \mathbb{1}_{Q_{C,i}(x')}(\lambda) d\eta(\lambda) \\ &\le \lim_{C\to\infty} \sum_{i=0}^{N} \int_{U'} \sup_{\mu\in V} \sum_{x'\in F_i} \tilde{\mu}(B_{\lambda_0}(x', D_0 2^{-i})) \mathbb{1}_{Q_{C,i}(x')}(\lambda) d\eta(\lambda) \\ &\le \lim_{C\to\infty} M\nu(X) \sum_{i=0}^{N} \sum_{x'\in F_i} \eta(Q_{C,i}(x')) \\ &= 0, \end{aligned} \tag{6.37}$$

since for each $i \in \mathbb{N}$, one has $Q_{C,i}(x') = \emptyset$ for C large enough (as $\operatorname{diam}(X, \rho_{\lambda_0}) < \infty$). Therefore, by (6.36), in order to prove (6.32) it suffices to show

$$\lim_{C\to\infty} \sum_{i=N}^{\infty} \int_{U'} \sup_{\mu\in V} \sum_{x'\in F_i} \tilde{\mu}(B_{\lambda_0}(x', D_0 2^{-i})) \mathbb{1}_{Q_{C,i}(x')}(\lambda) d\eta(\lambda) = 0. \tag{6.38}$$

As $F_i \subset G$, we can invoke (6.2) and recall the definition of G to obtain (we use here $\frac{1}{2}D_0 2^{-N} < R/2$)

$$\begin{aligned}\sum_{i=N}^{\infty} \int_{U'} \sup_{\mu\in V} \sum_{x'\in F_i} \tilde{\mu}(B_{\lambda_0}(x', D_0 2^{-i}))\mathbb{1}_{Q_{C,i}(x')}(\lambda)d\eta(\lambda) \\ \lesssim \sum_{i=N}^{\infty} 2^{5\varepsilon i} \int_{U'} \sum_{x'\in F_i} \nu(B_{\lambda_0}(x', D_0 2^{-i}))\mathbb{1}_{Q_{C,i}(x')}(\lambda)d\eta(\lambda) \\ \lesssim \sum_{i=N}^{\infty} 2^{8\varepsilon i} \int_{U'} \sum_{x'\in F_i} \nu(B_{\lambda_0}(x', \frac{1}{2}D_0 2^{-i}))\mathbb{1}_{Q_{C,i}(x')}(\lambda)d\eta(\lambda) \\ = \sum_{i=N}^{\infty} 2^{8\varepsilon i} \sum_{x'\in F_i} \nu(B_{\lambda_0}(x', \frac{1}{2}D_0 2^{-i}))\eta(Q_{C,i}(x')).\end{aligned}$$

Applying Lemma 6.4 with $\theta = \theta(\lambda_0, \varepsilon, \xi, \alpha)$ to be specified later gives

$$\begin{aligned}\sum_{i=N}^{\infty} \int_{U'} \sup_{\mu\in V} \sum_{x'\in F_i} \tilde{\mu}(B_{\lambda_0}(x', D_0 2^{-i}))\mathbb{1}_{Q_{C,i}(x')}(\lambda)d\eta(\lambda) \\ \lesssim C^{\dim_A(X,\rho_{\lambda_0})-d} \sum_{i=0}^{\infty} 2^{8\varepsilon i} \sum_{x'\in F_i} \nu(B_{\lambda_0}(x', \frac{1}{2}D_0 2^{-i}))2^{i(\alpha(1+\xi)-\frac{1}{1+\xi})(d-\dim_A(X,\rho_{\lambda_0})-\theta)}2^{\alpha(1+\xi)\theta i} \\ = C^{\dim_A(X,\rho_{\lambda_0})-d} \sum_{i=0}^{\infty} 2^{i\left(8\varepsilon+(\alpha(1+\xi)-\frac{1}{1+\xi})(d-\dim_A(X,\rho_{\lambda_0})-\theta)+\alpha\theta(1+\xi)\right)} \sum_{x'\in F_i} \nu(B_{\lambda_0}(x', \frac{1}{2}D_0 2^{-i})) \\ \lesssim C^{\dim_A(X,\rho_{\lambda_0})-d} \sum_{i=0}^{\infty} 2^{i\left(8\varepsilon+(\alpha(1+\xi)-\frac{1}{1+\xi})(d-\dim_A(X,\rho_{\lambda_0})-\theta)+\alpha\theta(1+\xi)\right)},\end{aligned}$$

where the last step uses the disjointness in (6.35). Since $\alpha < 1$ and $\dim_A(X, \rho_{\lambda_0}) < d$, the last sum converges provided that $\xi, \theta, \varepsilon > 0$ are chosen small enough. This choice establishes a neighbourhood U' of λ_0 such that

$$\sum_{i=N}^{\infty} \int_{U'} \sup_{\mu\in V} \sum_{x'\in F_i} \tilde{\mu}(B_{\lambda_0}(x', D_0 2^{-i}))\mathbb{1}_{Q_{C,i}(x')}(\lambda)d\eta(\lambda) \lesssim C^{\dim_A(X,\rho_{\lambda_0})-d}$$

and hence (6.38) holds as $\dim_A(X, \rho_{\lambda_0}) < d$. Together with (6.37), this establishes (6.32) and concludes the proof. □

7. Iterated function systems and measures on symbolic spaces

In this section we develop tools needed for applying Theorem 2.7 in the setting of symbolic dynamics related to IFS. In particular we prove Propositions 3.4 and 3.10.

7.1. **Relative dimension separability in symbolic spaces.** The goal of this subsection is to prove Proposition 3.4. The proof is based on approximating ergodic measures with Markov measures in relative entropy, a technique well-known in information theory. Let us introduce some notation and recall useful results. Most of this exposition follows [Gra11], but note that we use a different notation.

Let $\mathcal{A}$ be a finite set and let $\Sigma = \mathcal{A}^{\mathbb{N}}$ be the corresponding symbolic space. We endow Σ with the product topology. We will denote by $\mathcal{M}_\sigma(\Sigma)$ the set of all shift-invariant Borel probability measures on Σ and by $\mathcal{E}_\sigma(\Sigma) \subset \mathcal{M}_\sigma(\Sigma)$ the set of all ergodic measures. For $\mu \in \mathcal{M}_\sigma(\Sigma)$, we will denote by

$\mu|_n$ the distribution of μ on words of length n, i.e. $\mu|_n \in \mathcal{M}(\mathcal{A}^n)$ is given by $\mu|_n(\{\omega\}) = \mu([\omega])$ for $\omega \in \mathcal{A}^n$.

Definition 7.1. Measure $\nu \in \mathcal{M}_\sigma(\Sigma)$ is called a k**-step Markov measure** for $k \in \mathbb{N}$ if for every $n \geq k$ and every $\omega = (\omega_1, \ldots, \omega_n) \in \mathcal{A}^n$ it satisfies

$$\nu([\omega_1, \ldots, \omega_n]) = \nu([\omega_1, \ldots, \omega_k]) \prod_{j=1}^{n-k} P_\nu(\omega_{j+k}|\omega_j, \ldots, \omega_{j+k-1}), \tag{7.1}$$

where P_ν is the transition kernel given by

$$P_\nu(\tau|\tau_1, \ldots, \tau_k) = \begin{cases} \frac{\nu([\tau_1,\ldots,\tau_k,\tau])}{\nu([\tau_1,\ldots,\tau_k])} & \text{if } \nu([\tau_1, \ldots, \tau_k]) > 0 \\ 0 & \text{if } \nu([\tau_1, \ldots, \tau_k]) = 0 \end{cases} \quad \text{for } \tau, \tau_1, \ldots \tau_k \in \mathcal{A}.$$

Note that a k-step Markov measure is uniquely determined by its stationary distribution $\nu|_k$ and its transition kernel P_ν (so it is in fact uniquely determined by its $k+1$-dimensional distribution $\nu|_{k+1}$). Entropy of a k-step Markov measure is given by the following formula (see e.g. [Gra11, Lemma 3.16]):

$$h(\nu) = -\sum_{\omega \in \mathcal{A}^{k+1}} \nu([\omega]) \log P_\nu(\omega_{k+1}|\omega_1, \ldots, \omega_k). \tag{7.2}$$

There is also the following criterion for ergodicity of a k-step Markov measure:

Lemma 7.2. *A k-step Markov measure ν is ergodic if and only if for every $\omega, \tau \in \mathcal{A}^k$ with $\nu([\omega]) > 0, \nu([\tau]) > 0$, there exists $u \in \Sigma_*$ with $\nu([u]) > 0$ such that ω is a prefix of u and τ is a suffix of u.*

Proof. For 1-step Markov measures this can be found e.g. in [Wal82, Theorem 1.13]. Statement for a k-step Markov measure follows by noting that it is isomorphic to a 1-step Markov measure over alphabet $\mathcal{A}^k$. Also, statement in [Wal82] requires the stationary distribution $\nu|_k$ to be strictly positive, hence in general case one has to consider only the states with positive measure. □

Definition 7.3. Let $\mu, \nu \in \mathcal{M}_\sigma(\Sigma)$ be such that $\mu|_n \ll \nu|_n$ for every $n \in \mathbb{N}$. The **relative entropy** of μ with respect to ν is defined as

$$h(\mu||\nu) = \lim_{n \to \infty} \frac{1}{n} \sum_{\omega \in \mathcal{A}^n} \mu([\omega]) \log \frac{\mu([\omega])}{\nu([\omega])},$$

whenever the limit exists (we use the standard convention $0 \log \frac{0}{0} = 0 \log 0 = 0$).

Whenever the relative entropy exists, it satisfies $h(\mu||\nu) \geq 0$, see [Gra11, Lemma 3.1]. It may fail to exists for general shift-invariant measures, but it is guaranteed to exists if the reference measure ν is a k-step Markov measure:

Lemma 7.4 ([Gra11, Lemma 3.10])**.** *Let $\mu, \nu \in \mathcal{M}_\sigma(\Sigma)$ be such that $\mu|_n \ll \nu|_n$ for every $n \in \mathbb{N}$ and assume that ν is a k-step Markov measure. Then $h(\mu||\nu)$ is well defined and satisfies*

$$h(\mu||\nu) = -h(\mu) - \sum_{\omega \in \mathcal{A}^{k+1}} \mu([\omega]) \log P_\nu(\omega_{k+1}|\omega_1, \ldots, \omega_k).$$

We will make use of the following ergodic theorem for relative entropy.

Theorem 7.5 ([Gra11, Theorem 11.1]). *Let $\mu \in \mathcal{E}_\sigma(\Sigma)$ and $\nu \in \mathcal{M}_\sigma(\Sigma)$ be such that $\mu|_n \ll \nu|_n$ for every $n \in \mathbb{N}$ and assume that ν is a k-step Markov measure. Then*

$$\lim_{n\to\infty} \frac{1}{n} \log \frac{\mu([\omega|_n])}{\nu([\omega|_n])} = h(\mu||\nu) \text{ for } \mu\text{-a.e. } \omega \in \Sigma.$$

We will be interested in approximating ergodic measures by Markov measures.

Definition 7.6. For $\mu \in \mathcal{M}_\sigma(\Sigma)$ and $k \in \mathbb{N}$, we define the ***k*-th Markov approximation of** μ to be the k-step Markov measure $\mu^{(k)}$ with initial distribution $\mu^{(k)}|_k = \mu|_k$ and transition kernel $P_{\mu^{(k)}}$ given by

$$P_{\mu^{(k)}}(\tau|\tau_1,\ldots,\tau_k) = \begin{cases} \frac{\mu([\tau_1,\ldots,\tau_k,\tau])}{\mu([\tau_1,\ldots,\tau_k])} & \text{if } \mu([\tau_1,\ldots,\tau_k]) > 0 \\ 0 & \text{if } \mu([\tau_1,\ldots,\tau_k]) = 0 \end{cases} \quad \text{for } \tau, \tau_1, \ldots \tau_k \in \mathcal{A}.$$

It is straightforward to check that $\mu^{(k)}$ is well defined and it is a shift-invariant measure (as we have assumed μ to be shift-invariant). Moreover, Markov approximations have the following properties.

Lemma 7.7. *Let $\mu \in \mathcal{M}_\sigma(\Sigma)$. The following hold:*

(1) for every $k, n \in \mathbb{N}$, we have $\mu|_n \ll \mu^{(k)}|_n$,
(2) if $\mu \in \mathcal{E}_\sigma(\Sigma)$, then $\mu^{(k)} \in \mathcal{E}_\sigma(\Sigma)$ for every $k \in \mathbb{N}$,
(3) $h(\mu||\mu^{(k)}) = h(\mu^{(k)}) - h(\mu)$ for every $k \in \mathbb{N}$,
(4) $\lim\limits_{k\to\infty} h(\mu||\mu^{(k)}) = 0$.

Proof. (1) It is clear that $\mu|_n \ll \mu^{(k)}|_n$ for $n \le k$ (the two distributions are equal in this case). For $n > k$, it follows from (7.1) for $\nu = \mu^{(k)}$ that $\mu^{(k)}([\omega_1,\ldots,\omega_n]) = 0$ implies that either $\mu([\omega_1,\ldots,\omega_k]) = 0$ or $P_{\mu^{(k)}}(\omega_{j+k}|\omega_j,\ldots,\omega_{j+k-1}) = 0$ for some $j \in \{1,\ldots,n-k\}$. As the latter case gives $\mu([\omega_j,\ldots,\omega_{j+k-1}]) = 0$ or $\mu([\omega_j,\ldots,\omega_{j+k}]) = 0$, we obtain $\mu([\omega_1,\ldots,\omega_n]) = 0$ in both cases, so $\mu|_n \ll \mu^{(k)}|_n$.

(2) We will apply Lemma 7.2 to $\mu^{(k)}$. Let $\omega, \tau \in \mathcal{A}^k$ be such that $\mu^{(k)}([\omega]) > 0, \mu^{(k)}([\tau]) > 0$. Then also $\mu([\omega]) > 0, \mu([\tau]) > 0$ and by ergodicity of μ, there exists $u \in \Sigma_*$ with $\mu([u]) > 0$ such that ω is a prefix of u and τ is a suffix of u. By point (1) we have $\mu^{(k)}([u]) > 0$.

(3) This follows by combining Lemma 7.4 with (7.2), as $\mu^{(k)}|_{k+1} = \mu|_{k+1}$.

(4) This is [Gra11, Theorem 3.4].

□

Proposition 7.8. *There exists a countable set $\mathcal{V} \subset \mathcal{E}_\sigma(\Sigma)$ such that every $\nu \in \mathcal{V}$ is a k-step Markov measure for some $k \in \mathbb{N}$ and for every $\mu \in \mathcal{E}_\sigma(\Sigma)$ and $\varepsilon > 0$ there exists $\nu \in \mathcal{V}$ such that $0 \le h(\mu||\nu) < \varepsilon$, and $\mu|_n \ll \nu|_n$ for every $n \in \mathbb{N}$.*

Proof. We define $\mathcal{V}$ to consist of all ergodic k-step Markov measures ν for which the transition kernel P_ν takes only rational values. This is clearly a countable set. Fix $\mu \in \mathcal{E}_\sigma(\Sigma)$ and $\varepsilon > 0$. By Lemma 7.7 there exists k_0 such that for all $k \ge k_0$

$$0 \le h(\mu||\mu^{(k)}) = h(\mu^{(k)}) - h(\mu) < \varepsilon/2. \tag{7.3}$$

Consider ν which is a k-step Markov measure for which the transition kernel P_ν has exactly the same zeros as $P_{\mu^{(k)}}$. As $\mu^{(k)}$ is ergodic by Lemma 7.7, every such ν is also ergodic (by Lemma 7.2) and hence stationary distribution $\nu|_k$ depends continuously on the transition kernel P_ν (by the uniqueness of the stationary distribution, as long as we preserve the zeros of the transition kernel, see [Wal82, Theorem 1.19]). Therefore $\nu|_{k+1}$ depends continuously on the transition kernel

P_ν and hence we can find $\nu \in \mathcal{V}$ such that $\nu|_{k+1}$ and $\mu^{(k)}|_{k+1} = \mu|_{k+1}$ are arbitrarily close and have the same zeros. Consequently, by the formula in Lemma 7.4, we can choose $\nu \in \mathcal{V}$ such that $|h(\mu||\nu) - h(\mu||\mu^{(k)})| < \varepsilon/2$. Combining this with (7.3) and noting that $\mu|_n \ll \mu^{(k)}|_n \ll \nu|_n$ for every $n \in \mathbb{N}$ (by Lemma 7.7 and the fact that $P_{\mu^{(k)}}$ and P_ν have the same zeros) finishes the proof. □

It remains to connect the relative entropy with relative dimension. We do so for a class of metrics ρ on Σ satisfying the following assumptions:

(i) there exists a function $\psi : \Sigma^* \to (0,\infty)$ so that $\rho(\omega,\tau) = \psi(\omega \wedge \tau)$ for every $\omega \neq \tau \in \Sigma$,

(ii) there exists $\gamma \in (0,1)$ such that $\psi(\omega|_{n+1}) \leq \gamma\psi(\omega|_n)$ for each $n \geq 1$ and $\omega \in \Sigma$.

Lemma 7.9. *Let ρ be a metric on Σ satisfying (i) and (ii). Fix $N \in \mathbb{N}$ so that $\gamma^N \leq 1/2$. Then for every $r > 0$ small enough the following holds: for every $\omega \in \Sigma$ there exists $n \geq 1$ such that for*

$$[\omega|_{n+1}] \subset B(\omega, r) \subset [\omega|_n] \tag{7.4}$$

and

$$B(\omega, 2r) \subset [\omega|_{n-N}] \tag{7.5}$$

where $B(\omega,r)$ denotes the ball in the metric ρ. Moreover, n can be chosen so that $n \leq \frac{\log r}{\log\gamma} + B$ for some constant B (depending only on ψ).

Proof. Take $0 < r < \min\{\psi(i) : i \in \mathcal{A}\}$ and given $\omega \in \Omega$, let $n \geq 1$ to be the unique integer such that

$$\psi(\omega|_{n+1}) < r \leq \psi(\omega|_n). \tag{7.6}$$

If $\tau \in B(\omega,r)$, then $\rho(\omega,\tau) = \psi(\omega\wedge\tau) < r \leq \psi(\omega|_n)$, and hence, by (ii), $\omega|_n$ is a prefix of $\omega \wedge \tau$. This implies $\tau \in [\omega|_n]$, so $B(\omega,r) \subset [\omega|_n]$. On the other hand, if $\tau \in [\omega|_{n+1}]$, then $\omega|_{n+1}$ is a prefix of $\omega\wedge\tau$, so (ii) gives $\rho(\omega,\tau) = \psi(\omega\wedge\tau) \leq \psi(\omega|_{n+1}) < r$. Therefore $[\omega|_{n+1}] \subset B(\omega,r)$. This proves (7.4). Set $A = \max\{\psi(i) : i \in \mathcal{A}\}$ and note that (ii) implies

$$\psi(\omega|_n) \leq A\gamma^{n-1},$$

hence if (7.6) holds, then $r \leq A\gamma^{n-1}$, so $n \leq \frac{\log r}{\log\gamma} - \frac{\log A}{\log\gamma} + 1$. Finally, if (7.6) holds, then by (ii)

$$2r \leq 2\psi(\omega|_n) \leq 2\gamma^N\psi(\omega|_{n-N}) \leq \psi(\omega|_{n-N}),$$

since N is chosen so that $\gamma^N \leq 1/2$. Then the same argument as before shows $B(\omega,2r) \subset [\omega|_{n-N}]$, proving (7.5). □

Now we are ready to establish relative dimension separability of the set of ergodic measures on Σ with respect to a large class of metrics.

Proposition 7.10. *Let ρ be a metric on Σ satisfying (i) and (ii). Then the set $\mathcal{E}_\sigma(\Sigma)$ of ergodic shift-invariant probability measures on Σ is relative dimension separable with respect to ρ.*

Proof. Let $\mathcal{V} \subset \mathcal{E}_\sigma(\Sigma)$ be as in Proposition 7.8. Given $\mu \in \mathcal{E}_\sigma(\Sigma)$ and $\varepsilon > 0$, let $\nu \in \mathcal{V}$ be such that $h(\mu||\nu) < \varepsilon$. By Theorem 7.5, for μ-a.e. $\omega \in \Sigma$ there exists $n_0 = n_0(\omega,\varepsilon) > 0$ such that for all $n \geq n_0$

$$2^{-\varepsilon n}\nu([\omega|_n]) \leq \mu([\omega|_n]) \leq 2^{\varepsilon n}\nu([\omega|_n]). \tag{7.7}$$

Moreover, as ν is a k-step Markov measure we have

$$K_\nu := \max\left\{\frac{\nu([\omega|_n])}{\nu([\omega|_{n+1}])} : \omega \in \Sigma, n \geq k \text{ with } \nu([\omega|_{n+1}]) > 0\right\} < \infty,$$

as $K_\nu = \max\{P_\nu(\omega_{k+1}|\omega_1,\dots,\omega_k)^{-1} : \omega_1,\dots,\omega_{k+1} \in \mathcal{A} \text{ and } P_\nu(\omega_{k+1}|\omega_1,\dots,\omega_k) \neq 0\}$. Fix now $\omega \in \operatorname{supp}(\mu)$ such that (7.7) holds. Note that as $\omega \in \operatorname{supp}(\mu)$, we have $\mu([\omega|_n]) > 0$ for every $n \geq 1$ and hence (7.7) implies $\nu([\omega|_n]) > 0$ for all $n \geq 1$. Let $A = \frac{1}{-\log\gamma}$. By Lemma 7.9, we have for all $r > 0$ small enough and n satisfying $n_0 \leq n \leq A \log\frac{1}{r} + B$

$$\mu(B(\omega,r)) \leq \mu([\omega|_n]) \leq 2^{\varepsilon n}\nu([\omega|_n]) \leq 2^{B\varepsilon}K_\nu r^{-A\varepsilon}\nu([\omega|_{n+1}]) \leq 2^{B\varepsilon}K_\nu r^{-A\varepsilon}\nu(B(\omega,r)),$$

and similarly

$$\begin{aligned}\mu(B(\omega,r)) &\geq \mu([\omega|_{n+1}]) \geq 2^{-\varepsilon(n+1)}\nu([\omega|_{n+1}]) \geq (2^\varepsilon K_\nu)^{-1}2^{-\varepsilon n}\nu([\omega|_n]) \\ &\geq (2^{(B+1)\varepsilon}K_\nu)^{-1}r^{A\varepsilon}\nu(B(\omega,r)).\end{aligned}$$

The last two inequalities give

$$-A\varepsilon \leq \liminf_{r\to0}\frac{\log\frac{\mu(B(\omega,r))}{\nu(B(\omega,r))}}{\log r} \leq \limsup_{r\to0}\frac{\log\frac{\mu(B(\omega,r))}{\nu(B(\omega,r))}}{\log r} \leq A\varepsilon.$$

As this holds for μ-a.e. $\omega \in \Sigma$, we have

$$\dim(\mu||\nu,\rho) \leq A\varepsilon.$$

Since A is a constant depending only on ρ, we see that $\mathcal{E}_\sigma(\Sigma)$ is relative dimension separable with respect to ρ. □

If $\mathcal{F}$ is a conformal $C^{1+\theta}$ IFS, then setting $\psi(\omega) = \|f'_\omega\|$ one obtains Proposition 3.4 from Proposition 7.10.

7.2. **Gibbs measures.** Let us recall definition of a Gibbs measure on a symbolic space $\Sigma = \mathcal{A}^{\mathbb{N}}$.

Definition 7.11. Let $\phi : \Sigma \to \mathbb{R}$ be a continuous function on Σ. A shift-invariant ergodic probability measure μ on Ω is called a **Gibbs measure of the potential** ϕ if there exists $P \in \mathbb{R}$ and $C \geq 1$ such that for every $\omega \in \Sigma$ and $n \in \mathbb{N}$, holds the inequality

$$C^{-1} \leq \frac{\mu([\omega|_n])}{\exp(-Pn + \sum\limits_{k=0}^{n-1}\phi(\sigma^k\omega))} \leq C. \tag{7.8}$$

It is known that if ϕ is Hölder continuous, then there exists a unique Gibbs measure of ϕ (see [Bow08]). Here, Hölder continuity means Hölder continuity with respect to any metric of the form $\rho(\omega,\tau) = \gamma^{|\omega\wedge\tau|}$ on Σ for $\gamma \in (0,1)$. We shall prove that the set of Gibbs measures is uniform relative dimension separable (recall Definition 5.3) with respect to metrics on Σ satisfying (i) and (ii).

Proposition 7.12. *Let ρ be a metric on Σ satisfying (i) and (ii). Then the set $G_\sigma(\Sigma)$ consisting of all Gibbs measures corresponding to Hölder continuous potentials on Σ is uniform relative dimension separable with respect to ρ. Moreover, each $\mu \in G_\sigma(\Sigma)$ is uniformly diametrically regular with respect to ρ.*

Proof. Let μ_ϕ denote the Gibbs measure corresponding to a Hölder continuous potential $\phi : \Sigma \to \mathbb{R}$. It well known that the constant $P = P(\phi)$ for which μ_ϕ satisfies (7.8) can be expressed via the following pressure formula

$$P(\phi) = \lim_{n\to\infty}\frac{1}{n}\log\sum_{|u|=n}\exp(\sum_{k=0}^{n-1}\phi(\sigma^k(u\omega)))$$

for every $\omega \in \Sigma$ (see e.g. [Bow08, Proof of Theorem 1.16]). Am immediate consequence is the following: for a pair of Hölder continuous potentials $\phi_1, \phi_2 : \Sigma \to \mathbb{R}$

$$\|P(\phi_1) - P(\phi_2)\| \leq \|\phi_1 - \phi_2\|,$$

where $\|\cdot\|$ denotes the supremum norm on Σ (see also [Wal82, Theorem 9.7.(iv)]). Combining this with (7.8) gives that for every ϕ_1, ϕ_2 there exists $M = M(\phi_1, \phi_2)$ such that for every $\omega \in \Sigma$ and $n \in \mathbb{N}$

$$M^{-1} \exp(-2\|\phi_1 - \phi_2\|n) \leq \frac{\mu_{\phi_1}([\omega|_{n-1}])}{\mu_{\phi_2}([\omega|_n])} \text{ and } \frac{\mu_{\phi_1}([\omega|_n])}{\mu_{\phi_2}([\omega|_{n-1}])} \leq M \exp(2\|\phi_1 - \phi_2\|n), \tag{7.9}$$

To bound $\dim_u(\mu_{\phi_1}||\mu_{\phi_2}, \rho)$ we can use Lemma 7.9. Combined with (7.9), it gives that for every $r > 0$ small enough and every $\omega \in \Sigma$ there exists $n \leq \frac{\log r}{\log \gamma} + B$ such that

$$\begin{aligned}
\mu_{\phi_1}(B(x,r)) &\leq \mu_{\phi_1}([\omega|_n]) \leq M \exp(2\|\phi_1 - \phi_2\|n)\mu_{\phi_2}([\omega|_{n-1}]) \\
&\leq M \exp(2\|\phi_1 - \phi_2\|B) \exp\left(2\|\phi_1 - \phi_2\|\frac{\log r}{\log \gamma}\right) \mu_{\phi_2}(B(x,r)) \\
&= M' r^{-c\|\phi_1 - \phi_2\|} \mu_{\phi_2}(B(x,r)),
\end{aligned}$$

for a constant $M' = M'(\phi_1, \phi_2)$ and $c = c(\gamma) > 0$. Repeating the above calculation using lower bounds, one obtains similarly

$$\mu_{\phi_1}(B(x,r)) \geq M'^{-1} r^{c\|\phi_1 - \phi_2\|} \mu_{\phi_2}(B(x,r))$$

(possibly increasing constants M', c). Those two bounds together give

$$\dim_u(\mu_{\phi_1}||\mu_{\phi_2}, \rho) \leq c\|\phi_1 - \phi_2\|.$$

Therefore to prove that $G_\sigma(\Sigma)$ is uniform relative dimension separable it suffices to observe that the set of Hölder continuous functions $\phi : \Sigma \to \mathbb{R}$ is separable in the supremum norm. For example, the following collection is a countable dense set: all functions $\phi : \Sigma \to \mathbb{R}$ for which there exists n so that for every $\omega \in \Sigma^*$ with $|\omega| = n$ one has $\phi|_{[\omega]} \equiv \text{const} \in \mathbb{Q}$.

To prove that each $\mu \in G_\sigma(\Sigma)$ is uniformly diametrically regular, we invoke Lemma 7.9 once more. Fix $N \in \mathbb{N}$ so that $\gamma^N \leq 1/2$ and take $r > 0$ small enough so that for every $\omega \in \Sigma$ there exists $n \in \mathbb{N}$ so that

$$[\omega|_{n+1}] \subset B(\omega, r) \subset [\omega|_n] \text{ and } B(\omega, 2r) \subset [\omega|_{n-N}].$$

Then by (7.8)

$$\begin{aligned}
\mu(B(\omega, 2r)) &\leq \mu([\omega|_{n-N}]) \leq C \exp(-P(n-N) + \sum_{k=0}^{n-N-1} \phi(\sigma^k\omega)) \\
&= C^2 \exp(P(N+1) - \sum_{k=n-N}^{n} \phi(\sigma^k\omega)) C^{-1} \exp(-P(n+1) + \sum_{k=0}^{n} \phi(\sigma^k\omega)) \\
&\leq C^2 \exp((N+1)(P + \|\phi\|))\mu([\omega|_{n+1}]) \\
&\leq M\mu(B(\omega, r)),
\end{aligned}$$

for a constant M depending only on ρ and ϕ. Given $\varepsilon > 0$, this implies

$$\mu(B(\omega, 2r)) \leq r^{-\varepsilon}\mu(B(\omega, r))$$

for all $r > 0$ small enough to guarantee $r^{-\varepsilon} \geq M$. $\square$

7.3. **Exponential distance from the enemy.**

Proof of Proposition 3.10. First, note that we can assume that the attractor Λ of the IFS $\mathcal{F}$ is not a singleton (or equivalently $\Pi_{\mathcal{F}}$ is not constant), as otherwise Proposition 3.10 holds trivially.

We will make use of the bounded distortion property of $C^{1+\theta}$ conformal IFS: there exists a constant $C_D > 0$ such that

$$|f'_\omega(x)| \leq C_D |f'_\omega(y)| \text{ for every } x, y \in V \text{ and } \omega \in \Sigma^*.$$

For the proof see e.g. [MU03, Section 4.2]. An easy consequence (we use here that Λ is not a singleton) is that there exist a constant $A > 0$ such that

$$A^{-1}\|f'_\omega\| \leq \operatorname{diam}(f_\omega(\Lambda)) \leq A\|f'_\omega\| \text{ for all } \omega \in \Sigma^*. \tag{7.10}$$

Assume that the EDE condition (3.7) is satisfied at $\omega \in \Sigma$. Fix arbitrary $\alpha \in (0,1)$ and let $\varepsilon > 0$ be such that $\alpha = \frac{1}{1+\varepsilon}$. For $\tau \in \Sigma$ with $\tau \neq \omega$, let $n = |\omega \wedge \tau|$. Set $\gamma = \min\limits_{i \in \mathcal{A}} \inf\limits_{x \in V} \|f'_i(x)\|$ and note that by assumptions $\gamma > 0$. By (7.10), the EDE condition (3.7) gives

$$\begin{aligned}
\rho_{\mathcal{F}}(\omega,\tau) = \|f'_{\omega\wedge\tau}\| = \|f'_{\omega|_n}\| \leq \gamma^{-1}\|f'_{\omega|_{n+1}}\| \leq \frac{A}{\gamma}\operatorname{diam}(f_{\omega|_{n+1}}(\Lambda)) = \frac{A}{\gamma}\operatorname{diam}(\Pi_{\mathcal{F}}([\omega|_{n+1}]))\\
\leq \frac{AC^{-\frac{1}{1+\varepsilon}}}{\gamma}\operatorname{dist}\left(\Pi_{\mathcal{F}}(\omega), \bigcup_{\substack{|v|=n+1\\ v\neq\omega|_{n+1}}} \Pi_{\mathcal{F}}([v])\right)^{\frac{1}{1+\varepsilon}}\\
\leq \frac{AC^{-\frac{1}{1+\varepsilon}}}{\gamma}|\Pi_{\mathcal{F}}(\omega) - \Pi_{\mathcal{F}}(\tau)|^\alpha,
\end{aligned}$$

hence (3.7) implies (3.8).

In the other direction, assume that (3.8) holds. Given $\varepsilon > 0$, set $\alpha = \frac{1}{1+\varepsilon}$. For given $v \in \Sigma_*$ such that $|v| = n$ and $v \neq \omega|_n$ take any $\tau \in [v]$. Then $\tau \wedge \omega$ is a prefix of $\omega|_n$, so by (7.10)

$$\begin{aligned}
|\Pi_{\mathcal{F}}(\omega) - \Pi_{\mathcal{F}}(\tau)| &\geq C^{-\frac{1}{\alpha}}\rho_{\mathcal{F}}(\omega,\tau)^{\frac{1}{\alpha}} = C^{-(1+\varepsilon)}\|f'_{\omega\wedge\tau}\|^{1+\varepsilon}\\
&\geq C^{-(1+\varepsilon)}\|f'_{\omega|_n}\|^{1+\varepsilon} \geq (AC)^{-(1+\varepsilon)}\operatorname{diam}(\Pi_{\mathcal{F}}([\omega|_n]))^{1+\varepsilon}.
\end{aligned}$$

As $\tau \in [v]$ is arbitrary we have

$$\operatorname{dist}\left(\Pi_{\mathcal{F}}(\omega), \bigcup_{\substack{|v|=n\\ v\neq\omega|_n}} \Pi_{\mathcal{F}}([v])\right) > \frac{(AC)^{-(1+\varepsilon)}}{2}\operatorname{diam}(\Pi_{\mathcal{F}}([\omega|_n]))^{1+\varepsilon}$$

and hence (3.8) implies (3.7). □

8. Multifractal spectrum of self-similar measures

Finally, we prove Theorem 3.14 as an application of the above results. It relies on two lemmas

Lemma 8.1. *Let $\mathcal{F}^\lambda, \lambda \in U$ be a continuous family of $C^{1+\theta}$ conformal IFS on $V \subset \mathbb{R}^d$. Assume that η is a measure on U such that the transversality condition (A3) is satisfied (for the family of corresponding metrics ρ_λ as defined in (3.3)). Then for η-a.e. $\lambda \in U$ such that $s(\mathcal{F}_\lambda) < d$ the following holds: for every $\mu, \nu \in \mathcal{E}_\sigma(\Sigma)$ such that ν is a k-step Markov measure*

$$d(\Pi_\lambda\nu, x) = \frac{h(\mu) - h(\mu\|\nu)}{\chi(\mu,\lambda)} \text{ for } \Pi_\lambda\mu\text{-almost every } x. \tag{8.1}$$

Proof. It is clear that for every $\lambda \in U$

$$\overline{d}(\Pi_\lambda \nu, x) = \limsup_{n\to\infty} \frac{\log \Pi_\lambda \nu \left(B(\Pi_\lambda(u), \|(f^\lambda_{u|_n})'\|) \right)}{\log \|(f^\lambda_{u|_n})'\|} \leq \limsup_{n\to\infty} \frac{\log \nu\left([u|_n]\right)}{\log \|(f^\lambda_{u|_n})'\|} = \frac{h(\mu) - h(\mu\|\nu)}{\chi(\mu,\lambda)},$$

where $f^\lambda_{u|_n} = f^\lambda_{u_1} \circ \cdots \circ f^\lambda_{u_n}$, and so the upper bound in (8.1) follows by Theorem 7.5, the Shannon-McMillan-Breiman Theorem and Kingman's subadditive ergodic theorem (applied to the sequence $\omega \mapsto g_n(\omega) := -\log \|(f^\lambda_{\omega|_n})'\|$).

To show the lower bound, let us apply Theorem 3.11. That is, for η-almost every $\lambda \in U$ such that $s(\mathcal{F}^\lambda) < d$ we get that for every ergodic measure, almost every point has exponential distance from the enemy. Namely, for μ-almost every u and for every $\varepsilon > 0$ there exists $C > 0$ such that

$$B(\Pi_\lambda(u), \|(f^\lambda_{u|_n})'\|^{1+\varepsilon}) \cap \bigcup_{\substack{v\in\mathcal{A}^n \\ v\neq u|_n}} \Pi_\lambda([v]) = \emptyset.$$

Hence,

$$\Pi_\lambda \nu \left(B(\Pi_\lambda(u), \|(f^\lambda_{u|_n})'\|^{1+\varepsilon}) \right) \leq \nu([u|_n]),$$

and so, similarly to the upper bound,

$$\underline{d}(\Pi_\lambda \nu, x) = \liminf_{n\to\infty} \frac{\log (\Pi_\lambda)_* \nu \left(B(\Pi_\lambda(u), \|(f^\lambda_{u|_n})'\|^{1+\varepsilon}) \right)}{\log \|(f^\lambda_{u|_n})'\|^{1+\varepsilon}} \geq \liminf_{n\to\infty} \frac{\log \nu\left([u|_n]\right)}{\log \|(f^\lambda_{u|_n})'\|^{1+\varepsilon}} = \frac{h(\mu) - h(\mu\|\nu)}{(1+\varepsilon)\chi(\mu,\lambda)}.$$

Since $\varepsilon > 0$ was arbitrary, the claim follows. □

Lemma 8.2. *Let $\mathcal{F} = \{f_i(x) = \lambda_i O_i x + t_i\}_{i\in\mathcal{A}}$ be a self-similar IFS on $\mathbb{R}^d$ with similarity dimension $s_0 = s(\mathcal{F})$, let $\underline{p} = (p_i)_{i\in\mathcal{A}}$ be a probability vector and let $\mu = \underline{p}^{\mathbb{N}}$ be the Bernoulli measure corresponding to $\underline{p}$. If $(p_i)_{i\in\mathcal{A}} \neq (\lambda_i^{s_0})_{i\in\mathcal{A}}$ and $s_0 < d$ then for every $\alpha \in \left(\frac{\sum_i \lambda_i^{s_0} \log p_i}{\sum_i \lambda_i^{s_0} \log \lambda_i}, \max_i \frac{\log p_i}{\log \lambda_i} \right]$*

$$\dim_H \{x : d(\Pi\mu, x) = \alpha\} \leq \inf_{q\leq 0} (\alpha q + T(q)) = T^*(\alpha),$$

where $T(q)$ is the unique solution of $\sum_{i\in\mathcal{A}} p_i^q \lambda_i^{T(q)} = 1$.

The proof of the lemma is standard, but for completeness, we give here a proof of this lemma.

Proof. By definition,

$$\{x : d(\Pi_*\mu, x) = \alpha\} = \bigcap_{p=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{x : 2^{-n(\alpha+1/p)} \leq \Pi_*\mu(B(x, 2^{-n})) \leq 2^{-n(\alpha-1/p)}\}.$$

Hence, it is enough to show that for every $p \geq 1$, $N \geq 1$, $q \leq 0$ and $\varepsilon > 0$

$$\dim_H \bigcap_{n=N}^{\infty} \{x : 2^{-n(\alpha+1/p)} \leq \Pi_*\mu(B(x, 2^{-n})) \leq 2^{-n(\alpha-1/p)}\} \leq q(\alpha - 1/p) + T(q) + \varepsilon. \tag{8.2}$$

Let $X_{N,p} = \bigcap_{n=N}^{\infty} \{x : 2^{-n(\alpha+1/p)} \leq \Pi_*\mu(B(x, 2^{-n})) \leq 2^{-n(\alpha-1/p)}\}$ and $\mathcal{B}^N = \{B(x, 2^{-n}) : n \geq N$ and $x \in X_{N,p}\}$. Then by Besicovitch's covering theorem (see for example [BSS23, Theorem B.3.2]), there exists $Q = Q(d)$ such that for every $i = 1, \ldots, Q$ there exists a countable family

$\mathcal{B}_i^N \subseteq \mathcal{B}^N$ such that for every $B, B' \in \mathcal{B}_i^N$, $B \cap B' = \emptyset$ and $X_{N,p} \subseteq \bigcup_{i=1}^Q \bigcup_{B \in \mathcal{B}_i^N} B$. Clearly, $X_{N,p} \subseteq X_{N+1,p}$, and so $\bigcup_{i=1}^Q \mathcal{B}_i^M$ is a cover for $X_{N,p}$ for every $M \geq N$. Hence,

$$\mathcal{H}_{2^{-M}}^{q(\alpha-1/p)+T(q)+\varepsilon}(X_{N,p}) \leq \sum_{i=1}^Q \sum_{B \in \mathcal{B}_i^M} |B|^{q(\alpha-1/p)+T(q)+\varepsilon} \leq \sum_{i=1}^Q \sum_{B \in \mathcal{B}_i^M} \Pi_*\mu(B)^q |B|^{T(q)+\varepsilon} = \star.$$

For every $B \in \mathcal{B}^M$ there exists $u \in \Sigma$ such that $\Pi(u)$ is the center of B, furthermore, there exists a minimal $n = n(B) \geq 1$ such that $\Pi([u|_n]) \subseteq B$. Since $q \leq 0$ we get

$$\begin{aligned} \star &\leq \sum_{i=1}^Q \sum_{B \in \mathcal{B}_i^M} \mu([u(B)|_{n(B)}])^q |B|^{T(q)+\varepsilon} \lesssim 2^{-M\varepsilon} \sum_{i=1}^Q \sum_{B \in \mathcal{B}_i^M} \mu([u(B)|_{n(B)}])^q \lambda_{u(B)|_{n(B)}}^{T(q)} \\ &= 2^{-M\varepsilon} \sum_{i=1}^Q \sum_{B \in \mathcal{B}_i^M} p_{u(B)|_{n(B)}}^q \lambda_{u(B)|_{n(B)}}^{T(q)}. \end{aligned}$$

Since $\mathcal{B}_i^M$ is formed by disjoint balls, the cylinders $\{[u(B)|_{n(B)}]\}_{B \in \mathcal{B}_i^M}$ are disjoint too, and by the definition of $T(q)$, we get $\mathcal{H}_{2^{-M}}^{q(\alpha-1/p)+T(q)+\varepsilon}(X_{N,p}) \lesssim 2^{-M\varepsilon}$, which implies (8.2). □

Proof of Theorem 3.14. By the result of Barral and Feng [BF21], it is enough to show that (3.11) holds for $\alpha \in \left(\frac{\sum_i \lambda_i^{s_0} \log p_i}{\sum_i \lambda_i^{s_0} \log \lambda_i}, \max_i \frac{\log p_i}{\log \lambda_i}\right]$. The upper bound follows by Lemma 8.2.

If $\alpha \in \left(\frac{\sum_i \lambda_i^{s_0} \log p_i}{\sum_i \lambda_i^{s_0} \log \lambda_i}, \max_i \frac{\log p_i}{\log \lambda_i}\right)$ then there exists unique $q \leq 0$ such that $T'(q) = -\alpha$. Moreover,

$$T'(q) = -\frac{\sum_i p_i^q |\lambda_i|^{T(q)} \log p_i}{\sum_i p_i^q |\lambda_i|^{T(q)} \log |\lambda_i|} \text{ and } q\alpha + T(q) = -qT'(q) + T(q) = T^*(\alpha).$$

see for example [Fal97, Chapter 11] or [BSS23, Chapter 5].

To show the lower bound, let us apply Lemma 8.1 with ν being the Bernoulli measure with the probabilities $(p_i)_{i \in \mathcal{A}}$ and μ being the Bernoulli measure with the probabilities $(p_i^q |\lambda_i|^{T(q)})_{i \in \mathcal{A}}$. That is, let $Z \subset \mathbb{R}^N$ be a full measure subset such that (8.1) holds for every ergodic measure. So, for μ-almost every u

$$d(\Pi_{\underline{t}}\nu, \Pi_{\underline{t}}(u)) = \frac{h(\mu) - h(\mu\|\nu)}{\chi(\mu)} = \frac{\sum_i p_i^q |\lambda_i|^{T(q)} \log p_i}{\sum_i p_i^q |\lambda_i|^{T(q)} \log |\lambda_i|} = \alpha.$$

Hence,

$$\dim_H\{x : d(\Pi_{\underline{t}}\nu, x) = \alpha\} \geq \dim_H(\Pi_{\underline{t}})_*\mu = \frac{h(\mu)}{\chi(\mu)} = -q\frac{\sum_i p_i^q |\lambda_i|^{T(q)} \log p_i}{\sum_i p_i^q |\lambda_i|^{T(q)} \log |\lambda_i|} + T(q) = q\alpha + T(q) = T^*(\alpha).$$

To complete the proof, if $\alpha = \max_i \frac{\log p_i}{\log |\lambda_i|}$ then let $\mathcal{A}_{\max} = \left\{i \in \mathcal{A} : \frac{\log p_i}{\log |\lambda_i|} = \alpha\right\}$ and one can repeat the argument above with μ being the uniformly distributed Bernoulli measure on $\mathcal{A}_{\max}^{\mathbb{N}}$. □

Let us note that for higher dimensional systems, we can also extend the previous result of Barral and Feng [BF21, Theorem 1.2].

Theorem 8.3. *Let $\mathcal{A}$ be a finite set and for each $i \in \mathcal{A}$ fix $\lambda_i \in (0,1)$ and a $d \times d$ orthogonal matrix O_i. For $t = (t_i)_{i \in \mathcal{A}} \in (\mathbb{R}^d)^{\mathcal{A}}$ let $\mathcal{F}_t = \{f_i(x) = \lambda_i O_i x + t_i\}_{i \in \mathcal{A}}$ be a self-similar IFS on the line. Let $s_0 = s(\mathcal{F}_t)$ be the similarity dimension of $\mathcal{F}_t$ and assume that $s_0 < d$ and $\max_i \lambda_i < 1/2$. Then the following holds for Lebesgue almost every $(t_i)_{i \in \mathcal{A}} \in (\mathbb{R}^d)^{\mathcal{A}}$. For every probability vector $\underline{p} = (p_i)_{i \in \mathcal{A}}$*

such that $(p_i)_{i\in\mathcal{A}} \neq (\lambda_i^{s_0})_{i\in\mathcal{A}}$*, the multifractal formalism* (3.11) *holds for the self-similar measure* $\Pi_{\mathcal{F}_t}(\underline{p}^{\mathbb{N}})$ *and every* $\alpha \in \left[\frac{\sum_i p_i \log p_i}{\sum_i p_i \log \lambda_i}, \max\limits_{i\in\mathcal{A}} \frac{\log p_i}{\log \lambda_i}\right]$ *simultaneously.*

The theorem follows by the combination of [BF21, Theorem 1.2(2)], Lemma 8.1, Lemma 8.2 and Example 3.8, and left for the reader.

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