Weakly separated self-affine carpets

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Abstract

In this paper, we study the Hausdorff and box-counting dimensions of diagonally aligned self-affine carpets whose projections to the x and y-axes satisfy the weak separation condition. In particular, we will see that the Hausdorff dimension equals to the limit of the Barański formula and box-counting dimension is the limit of the Feng-Wang formula taken over the nth level functions. We also prove various equivalent formulas for the box-counting dimension, and we provide an alternative limiting formula for Gatzouras-Lalley carpets.

We demonstrate our theorems through two examples that were unobtainable previously, and we calculate their Hausdorff and box-counting dimensions.

Keywords: self-affine carpets, weak separation condition, Hausdorff dimension, box-counting dimension

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1 Introduction

The dimension theory of iterated function systems gained a reasonable attention in the last decades. Let $\mathbb{G}$ be a finite collection of contractions on $\mathbb{R}^d$, which is called an iterated function system (IFS). Hutchinson [15] showed that there exists a unique non-empty compact set Λ such that $\Lambda = \bigcup_{S\in\mathbb{G}} S(\Lambda)$. We call Λ the attractor of $\mathbb{G}$. If the maps of $\mathbb{G}$ are affine mappings we call the IFS $\mathbb{G}$ and its attractor Λ self-affine, and specially, if the mappings are similarities then we call the IFS and its attractor self-similar.

One of the focus points in the theory of iterated function systems is the Hausdorff and box-counting dimension of the attractor. Let us recall here the definitions. Let $E \subseteq \mathbb{R}^d$, and denote the diameter of the set E with respect to the usual Euclidean metric by $|E|$. We define the s-dimensional (outer) Hausdorff measure $\mathcal{H}^s(.)$ by

$$\mathcal{H}^s(E) := \lim_{\delta\to 0} \inf\left\{ \sum_{i\in I} |U_i|^s \;\middle|\; |U_i| \le \delta,\ \bigcup_{i\in I} U_i \supseteq E,\ I \text{ is countable}\right\}.$$

Moreover, we define the Hausdorff dimension of the set E by $\dim_{\mathrm{H}}(E) := \inf\left\{s \ge 0 \;\middle|\; \mathcal{H}^s(E) = 0\right\}$.

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For a bounded set $E \subset \mathbb{R}^d$, define the box-counting dimension as

$$\dim_{\mathrm{B}}(E) := \lim_{\delta \to 0^+} \frac{\log N_\delta(E)}{-\log \delta}$$

if the limit exists, where $N_\delta(E) := \min\left\{m > 0 \,\middle|\, \exists x_1, x_2, \ldots, x_m : E \subseteq \bigcup_{i=1}^m B(x_i, \delta)\right\}$, and $B(x,r)$ denotes the ball of radius r centered at x. If the limit does not exists, then taking the lim inf and lim sup define the lower- ($\underline{\dim}_{\mathrm{B}}$) and the upper box-counting dimension ($\overline{\dim}_{\mathrm{B}}$). For basic properties of the Hausdorff- and box-counting dimension, we refer for Falconer's book [6].

In case of self-similar IFS on $\mathbb{R}^d$, Hutchinson [15] showed that $\overline{\dim}_{\mathrm{B}}(\Lambda) \leq \min\{d, s_0\}$, where s_0 is called the similarity dimension, and it is the unique solution of the equation $\sum_{S \in \mathbb{G}} r_S^{s_0} = 1$, where r_S denotes the contraction ratio of the similarity map $S \in \mathbb{G}$. Furthermore, Hutchinson showed that if $\mathbb{G}$ satisfies the **open set condition** (OSC), that is, there exists a non-empty, open and bounded set U such that $S(U) \subset U$ and $S(U) \cap \hat{S}(U) = \emptyset$ for every $S \neq \hat{S} \in \mathbb{G}$, then $\dim_{\mathrm{H}}(\Lambda) = \dim_{\mathrm{B}}(\Lambda) = s_0$. The Hausdorff- and box-counting dimension of self-similar sets are equal in general, regardless of its geometric structure, see Falconer [7].

Roughly speaking, the open set condition implies that the overlap between the images $S(\Lambda)$ (called cylinder sets) are negligible. The situation becomes more complicated if we allow overlaps between the cylinders. Hochman [13, 14] showed that if the exponential separation holds then the Hausdorff (and box) dimension equals to the similarity dimension. In particular, this holds for typical choice of the natural parameters in some proper sense.

Another type of separation condition is the so-called weak separation condition (WSC) introduced by Lau and Ngai [19] and Zerner [25]. We say that the IFS $\mathbb{G}$ satisfies the **weak separation condition** (WSC) if the identity map is an isolated point of the set

$$\left\{\hat{S} \circ S^{-1} \,\middle|\, \hat{S}, S \in \mathbb{G}^*\right\},$$

where $\mathbb{G}^* = \{S_1 \circ \cdots \circ S_n : S_1, \ldots, S_n \in \mathbb{G},\ n \in \mathbb{N}\}$ is the semigroup induced by the maps in $\mathbb{G}$. In particular, the WSC allows heavy overlaps between the cylinders. More precisely, it allows exact overlaps, but the non-identical cylinders are relatively well separated. This makes possible the calculation of the Hausdorff dimension of the attractor via taking the limit of the similarity dimensions of higher iterates of the IFS removing the exact overlaps.

Diagonally aligned carpets are one of the most natural not self-similar examples for self-affine sets. Their introduction is credited to Bedford [4] and McMullen [20], who simultaneously calculated the Hausdorff and box-counting dimension of self-affine sets with contractions chosen from mappings taking the unit square to a rectangle of a homogeneous rectangular grid. To contrast the result of Falconer [7], we observe that in this setup the equality of the Hausdorff and box-counting dimension is not typical, and can be characterised with a nice geometric condition.

Their construction has many regularities, which led to generalisations along various avenues. Lalley and Gatzouras [18] considered diagonally aligned self-affine carpets with a certain column structure and with strict order between the contraction rates along the coordinate axis. Later, Barański [2] generalised the rectangular grid structure to some more general net structure of rectangles. In both cases, the Hausdorff and box-counting dimension were determined. Feng and Hu [8] proved a formula for the Hausdorff dimension of self-affine ergodic measures supported on diagonally aligned self-affine sets, which establishes relations between projected entropies, Lyapunov exponents and the dimension of the projections to the coordinate axis. This result (actually, its special cases) is the base expression for the various formulas for the Hausdorff dimension of the attractor. In particular, Lalley and Gatzouras [18] and Barański [2] proved that the Hausdorff dimension of the attractor is the maximum of the Feng-Hu formula over the probability distributions of $\mathbb{G}$.

In a more general setup, namely, without any grid-like structure or order between the contraction ratios, Feng and Wang [9] determined the box-counting dimension under the rectangular open set condition for self-affine carpets.

Several articles have recently been published on the dimension theory of overlapping aligned self-affine carpets. Fraser and Shmerkin [11] modified the construction of Bedford-McMullen by considering typical translations of the columns formed by the maps, and Pardo-Simón [23] considered overlapping Barański carpets by taking random translations of the rows and columns formed by the maps. In particular, they assumed that the projection of the columns (and rows) satisfy the exponential separation condition as self-similar IFSs. Recently, Rapaport [24] and Feng [10] considered general overlapping diagonal self-affine sets under the exponential separation condition of the iterated function systems induced by the coordinate projections.

We wish to continue the study of overlapping self-affine carpets on the plane by introducing the weak separation condition for this case. Our standing assumption is that the coordinate projections (as self-similar systems) satisfy the weak separation condition. Under this assumption, we provide a formula for the Hausdorff and the box-counting dimension of the attractor.

Our results can be considered also as a generalisation of the planar case of the results of He, Lau and Rao [12], who considered systems with homogeneous linear parts being inverses of expanding integer-coefficient matrices and with translation vectors of integer coordinates. Such systems are strongly related to sofic self-affine fractals, see for example Kenyon and Peres [16, 17], and Alibabaei [1].

1.1 Setup

Now, we present in details our main assumptions and findings. First, let us introduce some notations.

(A1) Let $\mathbb{G}$ be a finite collection of maps of the form

$$S(x,y) = (r_{S,1}x + t_{S,1}, r_{S,2}y + t_{S,2})\,, \tag{1.1}$$

such that $|r_{S,i}| \in (0,1)$ for every $S \in \mathbb{G}$ and $i \in \{1,2\}$. That is, all the functions in the IFS $\mathbb{G}$ have diagonal matrix part in the standard base. Denote the attractor of $\mathbb{G}$ by Λ. In general, we call such IFS and its attractor as **diagonal** or **diagonally aligned** self-affine carpets.

Denote the orthogonal projections to the main coordinate axis by $\mathrm{p}_1(x,y) = x$ and $\mathrm{p}_2(x,y) = y$, respectively. These are called the **principal projections**. It is easy to see that the orthogonal projections of the maps in $\mathbb{G}$ are forming self-similar IFSs on $\mathbb{R}$ with attractor $\mathrm{p}_\ell(\Lambda)$. Indeed, for any map S of the form in (1.1), $\mathrm{p}_1 \circ S(x,y)$ is independent of y and so, we can define $\mathrm{p}_1 S(x) := \mathrm{p}_1 \circ S(x,y) = r_{S,1}x + t_{S,1}$ (respectively for $\mathrm{p}_2 S$). Let us denote this IFS by $\mathrm{p}_\ell\mathbb{G} := \{\mathrm{p}_\ell S : S \in \mathbb{G}\}$. Let us note that it might happen that $\mathrm{p}_\ell S \equiv \mathrm{p}_\ell \hat{S}$, although $S \neq \hat{S}$. To avoid redundancies in projections and higher iterates, we will always consider $\mathrm{p}_\ell\mathbb{G}$ as a set of maps.

(A2) We assume that $\mathrm{conv}(\mathrm{p}_1(\Lambda)) \times \mathrm{conv}(\mathrm{p}_2(\Lambda)) = [0,1]^2$, where $\mathrm{conv}(\cdot)$ refers to the convex hull of a set. If this assumption holds, we say that the attractor fills $[0,1]^2$.

Under the name of sef-affine carpets, we are considering also the following possible constructions regularities:

(B1) The IFS has **homogeneous contractions**. Under the umbrella of diagonal self-affine carpets, we mean by this that there exist positive reals r_1, r_2 such that $r_{S,1} = r_1$ and $r_{S,2} = r_2$ for every $S \in \mathbb{G}$. This can be thought of as the homogeneity of the principal projection IFSs. Without loss of generality, we will always assume in the homogeneous case that $r_1 \leq r_2$.

(B2) Both $r_{S,1}$ and $r_{S,2}$ are positive for every $S \in \mathbb{G}$. This means that the functions of the IFS are **orientation preserving**.

(C1) $\mathbb{G}$ satisfies the **rectangular open set condition** (ROSC). That is, for any different $S \neq \hat{S} \in \mathbb{G}$, we have that $S((0,1)^2) \cap \hat{S}((0,1)^2) = \emptyset$.

(C2) For $\ell = 1,2$, $\mathrm{p}_\ell\mathbb{G}$ satisfy the open set condition (OSC) with respect to the unit open interval $(0,1)$.

(C3) Half-grid structure. Meaning that $\mathrm{p}_2\mathbb{G}$ satisfies the OSC with respect to the unit open interval $(0,1)$.

(G1) There is **coordinate ordering**, meaning that for any $S \in \mathbb{G}$, we have that $|r_{S,1}| \leq |r_{S,2}|$.

In this paper, we will always work assuming (A1) and (A2). Observe that (A2) is only technical, and the only generality one might lose by assuming it is the case when the attractor is contained in a horizontal/vertical line, but that IFS is self-similar, and has well-developed, but differing theory.

Note that the previously studied cases by Bedford [4] and McMullen [20], Lalley and Gatzouras [18] and Barański [2] can be described by these regularity assumptions. For example, (B1) with r_i^{-1} being positive integer, (B2), (C1) and (C2) essentially describes Bedford-McMullen carpets; (C1), (B2), (G1) and (C3) are the principal assumptions of Lalley and Gatzouras [18]; and (B2), (C1) and (C2) corresponds to Barański [2].

In this paper, we will generalise the separation assumptions (C1) and (C2) (or (C1), (G1) and (C3)) for a much weaker version, namely,

(W1) For $\ell = 1, 2$, $\mathrm{p}_\ell\mathbb{G}$ satisfy the weak separation condition (WSC).

(W2) The IFS $\mathrm{p}_2\mathbb{G}$ satisfies the weak separation condition (WSC).

We will see that respective to the Hausdorff dimension, we need the WSC for both $\mathrm{p}_1\mathbb{G}$ and $\mathrm{p}_2\mathbb{G}$ regardless of further separation on the plane if there is no distinguished direction of contraction. The assumptions (B2) is inessential. In the case of Lalley and Gatzouras [18], their arguments would work if we would let rows of functions switch orientation simultaneously along their y-axes, while switching along the x-axes seems insignificant. For the case of Barański carpets [2], something similar would need more attention.

1.2 Main results

Here we state the elegant, well-dressed form of our theorems, with the note that in their respective sections, these theorems attain a much more overgrown form.

1.2.1 Hausdorff dimension

The first theorem asserts that, under the assumption that the projection of the IFS onto the x- and y-axes satisfies the weak separation condition, the Hausdorff dimension of diagonally self-affine carpets equals the limit of the maximums of the Barański formula applied to the functions at the nth level. To be more precise, let $p = (p_S)_{S\in\mathbb{G}}$ be a probability vector over the maps of $\mathbb{G}$. For $R \in \mathrm{p}_\ell\mathbb{G}$, let $q_R^\ell = \sum_{S\in\mathbb{G}:\mathrm{p}_\ell S=R} p_S$. For the IFS $\mathbb{G}$, defined in (A1) and probability vector $p = (p_S)_{S\in\mathbb{G}}$, let

$$D(p,\mathbb{G}) := \begin{cases} \dfrac{\sum\limits_{S\in\mathbb{G}} p_S \log p_S}{\sum\limits_{S\in\mathbb{G}} p_S \log r_{S,1}} + \dfrac{\sum\limits_{R\in\mathrm{p}_2\mathbb{G}} q_R^2 \log q_R^2}{\sum\limits_{S\in\mathbb{G}} p_S \log r_{S,2}} - \dfrac{\sum\limits_{R\in\mathrm{p}_2\mathbb{G}} q_R^2 \log q_R^2}{\sum\limits_{S\in\mathbb{G}} p_S \log r_{S,1}}, & \text{if } \sum\limits_{S\in\mathbb{G}} p_S \log \frac{r_{S,2}}{r_{S,1}} \geq 0 \\ \dfrac{\sum\limits_{S\in\mathbb{G}} p_S \log p_S}{\sum\limits_{S\in\mathbb{G}} p_S \log r_{S,2}} + \dfrac{\sum\limits_{R\in\mathrm{p}_1\mathbb{G}} q_R^1 \log q_R^1}{\sum\limits_{S\in\mathbb{G}} p_S \log r_{S,1}} - \dfrac{\sum\limits_{R\in\mathrm{p}_1\mathbb{G}} q_R^1 \log q_R^1}{\sum\limits_{S\in\mathbb{G}} p_S \log r_{S,2}}, & \text{if } \sum\limits_{S\in\mathbb{G}} p_S \log \frac{r_{S,2}}{r_{S,1}} < 0. \end{cases} \tag{1.2}$$

and let

$$\mathbb{H}_{BA}(\mathbb{G}) = \max\{D(p,\mathbb{G}) \mid p = (p_S)_{S\in\mathbb{G}} \text{ probability vector}\}.$$

Now, we are ready to state our main theorem on the Hausdorff dimension.

Theorem 1.1 (Hausdorff dimension)**.** *Let $\mathbb{G}$ be a self-affine IFS satisfying (A1), (A2) and (B2). If (W1) holds or (C1), (G1) and (W2) hold then*

$$\dim_{\mathrm{H}}(\Lambda) = \lim_{n\to\infty} \mathbb{H}_{BA}(\mathbb{G}_n), \tag{1.3}$$

where $\mathbb{G}_n$ is the set of n-fold compositions of the functions of $\mathbb{G}$.

Unfortunately, the formula given in (1.3) seems extremely difficult to calculate for general weakly separated systems. For homogeneous systems, the formula can be reasonably simplified.

Corollary 1.2 (Hausdorff dimension for homogeneous system). *Let $\mathbb{G}$ be a self-affine IFS satisfying (A1), (A2), (W2) and (B1) with contraction ratios $0 < r_1 \leq r_2 < 1$. If (W1) or (C1) holds then*

$$\dim_{\mathrm{H}}(\Lambda) = \lim_{n\to\infty} \frac{\log\left(\sum_{T\in \mathrm{p}_2\mathbb{G}_n} \#\{S \in \mathbb{G}_n \mid \mathrm{p}_\ell S = T\}^{\frac{\log r_2}{\log r_1}}\right)}{n \log r_2}.$$

In the part concerning the Hausdorff dimension, we will also show for separated systems ((C1) and (C2) or (C1), (C3) and (G1)) that measures nearly optimal in achieving the dimension are mostly supported on a confined set of cylinder-rectangles, whose height and length are close to powers of a maximising Lyapunov exponent pair. Along that lines, we will present an alternative limiting formulas for computing the Hausdorff dimension of Lalley-Gatzouras carpets, either by maximising over lim sup expressions of the Lyapunov exponents or by taking the limit of formulas with step-by-step maximisation in the Lyapunov exponents.

1.2.2 Box-counting dimension

We assert that under the assumption that the weak separation condition holds for the projections of the IFS, the box-counting dimension exists, and equals the limit of the maximums of the Feng-Wang formula applied to the functions at the nth level.

Theorem 1.3 (Box-counting dimension). *Let $\mathbb{G}$ be a self-affine IFS satisfying (A1), (A2) and (W1). For every $n \in \mathbb{N}$, let d_n^1 and d_n^2 be the unique real solutions of the equations*

$$1 = \sum_{S\in\mathbb{G}_n} \left(\frac{|r_{S,1}|}{|r_{S,2}|}\right)^{s_n^1} |r_{S,2}|^{d_n^1}, \qquad 1 = \sum_{S\in\mathbb{G}_n} \left(\frac{|r_{S,2}|}{|r_{S,1}|}\right)^{s_n^2} |r_{S,1}|^{d_n^2}, \tag{1.4}$$

where s_n^ℓ is the unique real solution of the equation $\sum_{R\in\mathrm{p}_\ell\mathbb{G}_n} |r_{R,\ell}|^{s_n^\ell} = 1$. Then

$$\dim_{\mathrm{B}}(\Lambda) = \limsup_{n\to\infty} \max\{d_n^1, d_n^2\}.$$

Let us note that the formula above remains valid by replacing (1.4) with the equations

$$1 = \sum_{S\in\mathbb{G}_n} \left(\frac{|r_{S,1}|}{|r_{S,2}|}\right)^{\dim_{\mathrm{B}}(\mathrm{p}_1(\Lambda))} |r_{S,2}|^{d_n^1}, \qquad 1 = \sum_{S\in\mathbb{G}_n} \left(\frac{|r_{S,2}|}{|r_{S,1}|}\right)^{\dim_{\mathrm{B}}(\mathrm{p}_2(\Lambda))} |r_{S,1}|^{d_n^2},$$

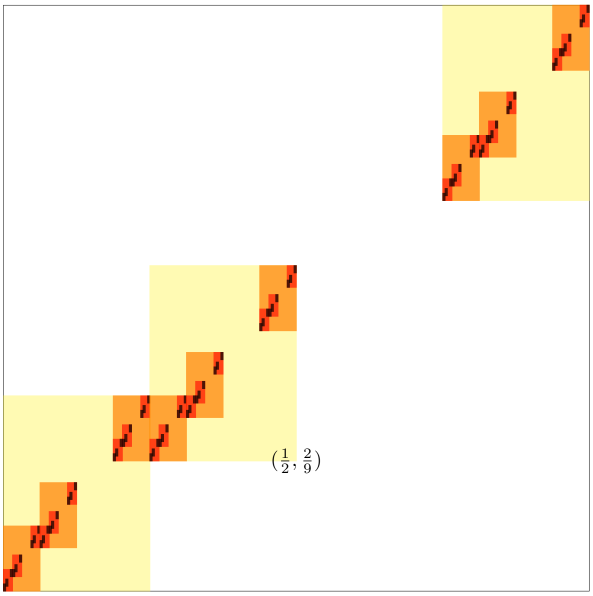
which corresponds to a limiting version of Feng and Wang's formula [9] for the box-counting dimension. Note that the case (C1) with (G1) and (W2) is already covered by the results of Zerner [25] and Feng and Wang [9], unlike the case of Hausdorff dimension.

Additionally, we provide later further explicit limiting formulas which are characterising the box-counting dimension.

1.3 Examples

Now, we illustrate our results in two examples. Their calculation is rather lengthy, and hence, the proof of these assertions will be in the last part of the paper.

Example 1.4. Let $\mathbb{G}$ be the IFS depicted in Figure 1.



$$\mathbb{G} := \{S_1, S_2, S_3\},$$
$$S_1(x,y) := \left(\frac{x}{4}, \frac{y}{3}\right),$$
$$S_2(x,y) := \left(\frac{x}{4} + \frac{1}{4}, \frac{x}{3} + \frac{2}{9}\right),$$
$$S_3(x,y) := \left(\frac{x}{4} + \frac{3}{4}, \frac{y}{3} + \frac{2}{3}\right).$$

Figure 1: Example for application of Theorem 1.1 under the assumptions (C1), (G1) and (W2).

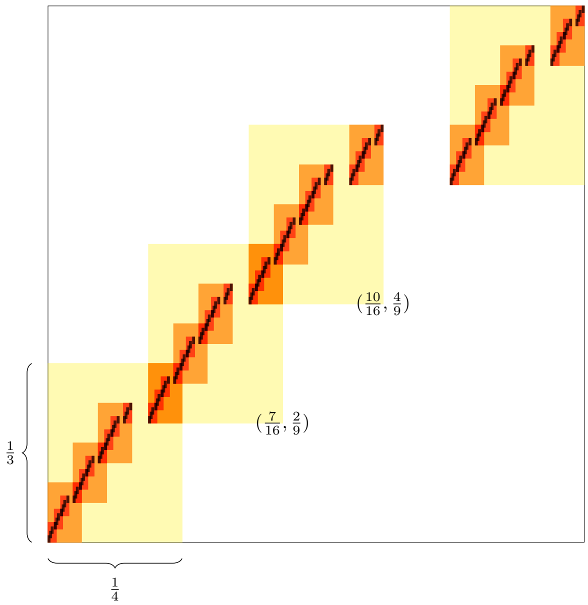
Then $\dim_{\mathrm{H}}(\Lambda) = \log_3 \lambda^*$, where λ^* the unique $\lambda > 1$ is such that

$$\lambda = \frac{1}{\lambda - 1} \sum_{k=2}^{\infty} k^{\alpha} (\lambda)^{2-k} + \frac{(\lambda)^2}{(\lambda-1)^3},$$

where $\alpha := \log_4 3$, and therefore

$$\begin{aligned} \dim_{\mathrm{H}}(\Lambda) &= \log_3 2.896001351588652942659618472486268180831798155970 1975\ldots \\ &= 0.967885533595539319438037445903385862252724017052009287837\ldots \end{aligned}$$

Example 1.5. Let $\mathbb{G}$ be the IFS depicted in Figure 2.



$$\mathbb{G} := \{S_1, S_2, S_3, S_4\},$$
$$S_1(x,y) := \left(\frac{x}{4}, \frac{y}{3}\right),$$
$$S_2(x,y) := \left(\frac{x}{4} + \frac{3}{16}, \frac{y}{3} + \frac{2}{9}\right),$$
$$S_3(x,y) := \left(\frac{x}{4} + \frac{6}{16}, \frac{y}{3} + \frac{4}{9}\right),$$
$$S_4(x,y) := \left(\frac{x}{4} + \frac{3}{4}, \frac{y}{3} + \frac{2}{3}\right).$$

Figure 2: Example for the application of Theorems 1.1 and 1.3 under the assumption (W1).

Then

$$\dim_{\mathrm{B}}(\Lambda) = \frac{\log 3}{\log 3}\Big(1 - \frac{\log 3}{\log 4}\Big) + \frac{\log(2+\sqrt{2})}{\log 4} = 1.093295401221\dots$$

while $\dim_{\mathrm{H}}(\Lambda) = \log_3 \lambda^*$, where λ^* the unique $\lambda > 1$ is such that

$$\lambda(\lambda - 1) = 2 + (\lambda + 1)\sum_{k=3}^{\infty}(k-1)^{\alpha}(\lambda)^{2-k},$$

where $\alpha := \log_4 3$, and therefore

$$\begin{aligned}\dim_{\mathrm{H}}(\Lambda) &= \log_3 3.01591034782768925227902905124997073016801934451517291\dots\\ &= 1.00481465169197886851100832276918315528286034877088758744 21\dots\end{aligned}$$

We note that the presented examples above satisfy the assumptions of He, Lau and Rao [12, Theorem 4.4] based on Kenyon and Peres [16, Theorem 2.2]. However, we present here a different method for the calculation of the dimension values.

2 Preliminaries

2.1 Some notations

Let $\mathbb{G}$ be an IFS. Let us index the maps of $\mathbb{G}$. That is, let $\Sigma = \{1, \dots, \mathfrak{m}\}$ be a finite set with $\mathfrak{m} = \#\mathbb{G}$ and $\mathbb{G} = \{S_i \mid i \in \Sigma\}$. For simplicity, we may also write for the contraction ratios $r_{S_i,\ell} = r_{i,\ell}$ and for the translations $t_{S_i,\ell} = t_{i,\ell}$.

For $k \in \mathbb{N}^+$ let Σ^k be the set of k-touples formed by the elements of Σ. Let Σ^* be the set $\bigcup_{k\in\mathbb{N}^+} \Sigma^k$. Usually, the finite words in Σ^* will be denoted by the mathfrak style: $\mathfrak{i}, \mathfrak{j}, \mathfrak{k}, \mathfrak{h}$. In particular, if we refer to an element in Σ, then we might use i, j, k, h.

For a finite word $\mathfrak{i} \in \Sigma^*$, denote $|\,\mathfrak{i}\,|$ the length of $\mathfrak{i}$. If $k < \ell \in \mathbb{N}$ and $\mathfrak{i}$ is a sufficiently large finite word with $\ell \le |\,\mathfrak{i}\,|$, we denote:

$$\begin{aligned}\mathfrak{i}\,|_{(k,\ell]} &:= (\mathfrak{i}_{k+1} \dots \mathfrak{i}_{\ell})\\ \mathfrak{i}- &:= (\mathfrak{i}_1 \dots \mathfrak{i}_{|\,\mathfrak{i}\,|-1}) = \mathfrak{i}\,|_{(0,|\,\mathfrak{i}\,|-1]}.\end{aligned}$$

For $\mathfrak{i} = (i_1, \dots, i_{|\,\mathfrak{i}\,|}) \in \Sigma^*$, write

$$S_{\mathfrak{i}} := S_{i_1} \circ \dots \circ S_{i_{|\,\mathfrak{i}\,|}}.$$

One can think of S as a function mapping the symbolic space of contractions of $\mathbb{R}^d$.

The above symbolic space describes separated sets very well, but where less is assumed for separation, it shows some shortcomings. Namely, if there are exact overlaps many finite words are redundant.

For $n \in \mathbb{N}^+$ let the nth level functions be

$$\mathbb{G}_n := \big\{S_{\mathfrak{i}} \;\big|\; \mathfrak{i} \in \Sigma^n\big\}.$$

Define $\Sigma^{\{n\}} \subseteq \Sigma^n$ iteratively. Let $\Sigma^{\{1\}} := \Sigma$. Given $\Sigma^{\{k\}}$ define $\Sigma^{\{k+1\}}$ as maximal subset of Σ^{k+1} such that

$$\mathbb{G}_{k+1} = \Big\{S_{\mathfrak{i}} \;\Big|\; \mathfrak{i} \in \Sigma^{\{k+1\}}\Big\}, S_{\mathfrak{i}} \neq S_{\mathfrak{j}} \text{ for } \mathfrak{i} \neq \mathfrak{j} \in \Sigma^{\{k+1\}} \text{ and } \Sigma^{\{k\}} \supseteq \Big\{\mathfrak{i}- \;\Big|\; \mathfrak{i} \in \Sigma^{\{k+1\}}\Big\}$$

for every $k \in \mathbb{N}$. By the construction, there is a one-to-one correspondence between $\Sigma^{\{n\}}$ and the maps $\mathbb{G}_n$. Similarly, for $\ell \in \{1, 2\}$, let $\Gamma_\ell^{\{n\}} \subseteq \Sigma^{\{n\}}$ be such that

$$\mathrm{p}_\ell\mathbb{G}_n = \Big\{\mathrm{p}_\ell S_{\mathfrak{i}} \;\Big|\; \mathfrak{i} \in \Gamma_\ell^{\{n\}}\Big\}, \mathrm{p}_\ell S_{\mathfrak{i}} \neq \mathrm{p}_\ell S_{\mathfrak{j}} \text{ for } \mathfrak{i} \neq \mathfrak{j} \in \Gamma_\ell^{\{n\}} \text{ and } \Gamma_\ell^{\{n-1\}} \supseteq \Big\{\mathfrak{i}- \;\Big|\; \mathfrak{i} \in \Gamma_\ell^{\{n\}}\Big\}.$$

Let $\Sigma^{\{*\}} = \bigcup_{n=1}^{\infty} \Sigma^{\{n\}}$. Furthermore, for $m \in \mathbb{N}^+$, let

$$\Sigma^{\{n\}m} := \left\{ \mathtt{i}^1 \mathtt{i}^2 \dots \mathtt{i}^m \;\middle|\; \mathtt{i}^j \in \Sigma^{\{n\}} \ \forall j \in \{1, \dots, m\} \right\}$$
$$\Gamma_\ell^{\{n\}m} := \left\{ \mathtt{j}^1 \mathtt{j}^2 \dots \mathtt{j}^m \;\middle|\; \mathtt{j}^k \in \Gamma_\ell^{\{n\}} \ \forall k \in \{1, \dots, m\} \right\}.$$

Now we follow with some specific notation appropriate for diagonally aligned self-affine sets. For $\ell \in \{1, 2\}$, let

$$r_{\max,\ell} := \max_{i \in \Sigma} \left\{ |r_{i,\ell}| \right\} \qquad r_{\min,\ell} := \min_{i \in \Sigma} \left\{ |r_{i,\ell}| \right\}$$
$$r_{\max} := \max \left\{ r_{\max,1}, r_{\max,2} \right\} \qquad r_{\min} := \min \left\{ r_{\min,1}, r_{\min,2} \right\}.$$

For $\mathtt{i} = (i_1, i_2, \dots, i_{|\mathtt{i}|}) \in \Sigma^*$, write

$$r_{\mathtt{i},1} := \prod_{\ell=1}^{|\mathtt{i}|} r_{i_\ell,1} \qquad r_{\mathtt{i},2} := \prod_{\ell=1}^{|\mathtt{i}|} r_{i_\ell,2}.$$

Define the δ-Moran cut-set of the self-similar IFS $\mathbb{G} = \left\{ S_i \right\}_{i \in \Sigma}$ and $\delta \in (0, 1)$ as

$$M_\delta(\mathbb{G}, \Sigma) := \left\{ \mathtt{i} \in \Sigma^* \;\middle|\; |S_\mathtt{i}(\Lambda)| \le \delta < |S_{\mathtt{i}^-}(\Lambda)| \right\}.$$

Furthere define

$$M_\delta^*(\mathbb{G}, \Sigma) := \left\{ \mathtt{i} \in \Sigma^{\{*\}} \;\middle|\; |S_\mathtt{i}(\Lambda)| \le \delta < |S_{\mathtt{i}^-}(\Lambda)| \right\}.$$

In particular, given a diagonally aligned self-affine set $\mathbb{G}_n$, for $\ell \in \{1, 2\}$ denote $M_\delta(\mathrm{p}_\ell \mathbb{G}, \Gamma_\ell)$ as M_δ^ℓ, and $M_\delta^*(\mathrm{p}_\ell \mathbb{G}, \Gamma_\ell)$ as $M_\delta^{\ell,*}$.

Let

$$\triangle_\delta := \left\{ \mathtt{i} \in \Sigma^{\{*\}} \;\middle|\; \min\{|r_{\mathtt{i},1}|, |r_{\mathtt{i},2}|\} \le \delta < \min\{|r_{\mathtt{i}^-,1}|, |r_{\mathtt{i}^-,2}|\} \right\} \text{ and } \mathbb{S}_\delta := \left\{ S_\mathtt{i} \;\middle|\; \mathtt{i} \in \triangle_\delta \right\}.$$

Let us also define

$$\triangle_\delta^1 := \left\{ \mathtt{i} \in \triangle_\delta \;\middle|\; |r_{\mathtt{i},1}| > |r_{\mathtt{i},2}| \right\} \text{ and } \triangle_\delta^2 := \left\{ \mathtt{i} \in \triangle_\delta \;\middle|\; |r_{\mathtt{i},2}| \ge |r_{\mathtt{i},1}| \right\}.$$

and

$$\mathbb{S}_\delta^1 := \left\{ S_\mathtt{i} \;\middle|\; \mathtt{i} \in \triangle_\delta^1 \right\} \text{ and } \mathbb{S}_\delta^2 := \left\{ S_\mathtt{i} \;\middle|\; \mathtt{i} \in \triangle_\delta^2 \right\}.$$

2.1.1 The Weak Separation Condition

The weak separation condition (WSC) was introduced by Lau and Ngai [19] and Zerner [25]. It ensures that the overlapping structure only affects the dimension values through the exact-overlap structure. For a general discussion on WSC, we direct the reader to [3, Chapter 4].

The WSC will be used through the following lemma, which, although is stated as a consequence, is equivalent to the WSC.

Lemma 2.1. *Suppose that the self-similar IFS $\mathbb{F} = \{f_i\}_{i \in \Sigma}$ satisfies the weak separation condition. Then*

$$\exists C \in (0, \infty) \ \forall x \in \mathbb{R}^d \ \forall \delta > 0 : \quad \# \left\{ \mathtt{i} \in M_{\delta/|\Lambda|}^*(\mathbb{F}, \Sigma) \;\middle|\; f_\mathtt{i}(\Lambda) \cap B(x, \delta) \ne \emptyset \right\} \le C.$$

In particular, given a diagonally aligned self-affine set $\mathbb{G}$ satisfying (A1), (A2) and (W1) we have that

$$\exists C \in (0, \infty) \ \forall \ell \in \{1, 2\} \ \forall x \in \mathbb{R}^d \ \forall \delta > 0 : \quad \# \left\{ \mathtt{i} \in M_\delta^{\ell,*} \;\middle|\; \mathrm{p}_\ell S_\mathtt{i}(\mathrm{p}_\ell(\Lambda)) \cap B(x, \delta) \ne \emptyset \right\} \le C.$$

The proof is due to Zerner [25, Theorem 1]. The following theorem is a typical statement concerning the WSC.

Theorem 2.2 (Zerner [25])**.** *If the self-similar IFS* $\mathbb{F}$ *satisfies the weak separation condition then*

$$\dim_{\mathrm{H}}(\Lambda) = \dim_{\mathrm{B}}(\Lambda) = \lim_{\delta\to 0} \frac{\log \# M_\delta^*(\mathbb{F},\Sigma)}{-\log\delta}.$$

In particular, given a diagonally aligned self-affine set $\mathbb{G}$ *satisfying (A1), (A2) and (W1) we have that*

$$\forall \ell \in \{1,2\}: \ \dim_{\mathrm{H}}(\mathrm{p}_\ell(\Lambda)) = \dim_{\mathrm{B}}(\mathrm{p}_\ell(\Lambda)) = \lim_{\delta\to 0} \frac{\log \# M_\delta^{\ell,*}}{-\log\delta}.$$

The proof can be found in Zerner [25, Theorem 2]. Another way of expressing these facts is to use $\mathrm{p}_\ell\mathbb{G}_n$ instead of $M_\delta^{\ell,*}$. This works, since the transition affects only at a subexponential cost.

Lemma 2.3. *Let* $\mathbb{F} = \{f_i\}_{i\in\Sigma}$ *be and IFS satisfying the weak separation condition. Let* $\mathbb{F}_n := \{f_{\mathrm{i}} \,|\, \mathrm{i} \in \Sigma^n\}$*. Then*

$$\dim_{\mathrm{H}}(\Lambda) = \dim_{\mathrm{B}}(\Lambda) = \lim_{n\to\infty} s_n,$$

where s_n*, the similarity dimension of* $\mathbb{F}_n$*, is defined by* $\sum_{f\in\mathbb{F}_n} |r_f|^{s_n} = 1$*. In particular, given a diagonally aligned self-affine set* $\mathbb{G}$ *satisfying (A1), (A2) and (W1) we have that*

$$\forall \ell \in \{1,2\}: \ \dim_{\mathrm{H}}(\mathrm{p}_\ell(\Lambda)) = \dim_{\mathrm{B}}(\mathrm{p}_\ell(\Lambda)) = \lim_{n\to\infty} s_n^\ell,$$

where s_n^ℓ*, the similarity dimension of* $\mathrm{p}_\ell\mathbb{G}_n$*, is defined by* $\sum_{\mathrm{i}\in\Gamma_\ell^{\{n\}}} |r_{\mathrm{i},\ell}|^{s_n^\ell} = 1$*.*

For a proof, we refer to Remark 4.2.17 from [3].

2.2 Dimension estimates for carpets

Now, we will state result from the dimension theory of self-affine carpets which we will rely on.

Theorem 2.4 (Feng and Wang [9])**.** *Let* $\mathbb{G}$ *be a self-affine IFS satisfying (A1), (A2), (C1). Then* $\dim_{\mathrm{B}}(\Lambda) = \max\{d_1, d_2\}$*, where*

$$1 = \sum_{S\in\mathbb{G}} \left(\frac{|r_{S,1}|}{|r_{S,2}|}\right)^{\dim_{\mathrm{B}}(\mathrm{p}_1(\Lambda))} |r_{S,2}|^{d_1} \ \textit{and}\ 1 = \sum_{S\in\mathbb{G}} \left(\frac{|r_{S,2}|}{|r_{S,1}|}\right)^{\dim_{\mathrm{B}}(\mathrm{p}_2(\Lambda))} |r_{S,1}|^{d_2}.$$

Let us note that the theorem of Feng and Wang is not stated in this way in [9]. For a proof of this formula, we refer for [3, Theorem 11.4.2].

Theorem 2.5 (Lalley and Gatzouras [18])**.** *Let* $\mathbb{G}$ *be a self-affine IFS satisfying (A1), (A2), (B2), (C1), (G1) and (C2). Then*

$$\dim_{\mathrm{H}}(\Lambda) = \mathbb{H}_{BA}(\mathbb{G}) = \max_{p\in P} \left\{ \frac{\sum\limits_{S\in\mathbb{G}} p_S \log p_S}{\sum\limits_{S\in\mathbb{G}} p_S \log r_{S,1}} + \frac{\sum\limits_{R\in\mathrm{p}_2\mathbb{G}} q_R^2 \log q_R^2}{\sum\limits_{S\in\mathbb{G}} p_S \log r_{S,2}} - \frac{\sum\limits_{R\in\mathrm{p}_2\mathbb{G}} q_R^2 \log q_R^2}{\sum\limits_{S\in\mathbb{G}} p_S \log r_{S,1}} \right\}, \tag{2.1}$$

where $P = \{(p_S)_{S\in\mathbb{G}} \mid p_S \geq 0,\ \sum_{S\in\mathbb{G}} p_S = 1\}$ *is the set of probability vectors and* $q_R^2 = \sum_{S\in\mathbb{G}\,:\,\mathrm{p}_2 S = R} p_S$.

The proof can be found in [18, Theorem 5.3].

Theorem 2.6 (Pardo-Simón [23])**.** *Let* $\mathbb{G}$ *be a self-affine IFS satisfying (A1), (A2). Then*

$$\dim_{\mathrm{H}}(\Lambda) \leq \mathbb{H}_{BA}(\mathbb{G}) = \max\{D(p,\mathbb{G}) \mid p = (p_S)_{S\in\mathbb{G}} \textit{ probability vector}\}. \tag{2.2}$$

For a proof, we refer for [23, Proposition 3.5].

Let us note that in both of the proofs of Theorems 2.5 and 2.6 are assuming (B2) (the positivity of $r_{i,j}$), but the proofs can be modified in a straightforward way for the more general situation.

3 Box-counting dimension

First, we will study the box-counting dimension of the attractor Λ. Let us now state a more detailed version of Theorem 1.3.

Theorem 3.1 (Box-counting dimension)**.** *Let*

$$\mathbb{G} := \{S_i(x_1, x_2) := (r_{i,1}x + t_{i,1}, r_{i,2}y + t_{i,2})\}_{i\in\Sigma}$$

be a diagonal self-affine IFS, with attractor Λ satisfying the conditions (A1), (A2) and (W1). Let $\mathrm{p}_\ell\mathbb{G}$ the principal projection IFSs with attractors $\mathrm{p}_\ell(\Lambda)$ for $\ell = 1, 2$. Then

$$\dim_{\mathrm{B}}(\Lambda) = \max_{\ell\in\{1,2\}} \limsup_{\delta\to 0} \left(\frac{\log \#\mathbb{S}_\delta^\ell}{-\log\delta} + \dim_{\mathrm{B}}(\mathrm{p}_\ell(\Lambda)) \left(1 + \frac{\log \mathcal{M}_{\dim_{\mathrm{B}}(\mathrm{p}_\ell(\Lambda))}\{|r_{S,\ell}| \mid S \in \mathbb{S}_\delta^\ell\}}{-\log\delta} \right) \right) \tag{3.1}$$

where $\mathcal{M}_p(x_1, \ldots, x_n) := \left(\frac{1}{n}\sum_{i=1}^n x_i^p\right)^{\frac{1}{p}}$ is the power mean with exponent p.

Moreover

$$\dim_{\mathrm{B}}(\Lambda) = \limsup_{\delta\to 0} \max\left\{D_\delta^1, D_\delta^2\right\} = \limsup_{n\to\infty} \max\left\{d_n^1, d_n^2\right\} = \limsup_{n\to\infty} \max\left\{\mathfrak{D}_n^1, \mathfrak{D}_n^2\right\},$$

where for $\ell = 1, 2$ we quantities D_δ^ℓ, d_n^ℓ and $\mathfrak{D}_n^\ell$ are the unique roots of the equations

$$1 = \sum_{S\in\mathbb{S}_\delta} \left(\frac{|r_{S,\ell}|}{|r_{S,3-\ell}|}\right)^{\dim_{\mathrm{B}} \mathrm{p}_\ell(\Lambda)} |r_{S,3-\ell}|^{D_\delta^\ell}, \qquad 1 = \sum_{S\in\mathbb{G}_n} \left(\frac{|r_{S,\ell}|}{|r_{S,3-\ell}|}\right)^{\dim_{\mathrm{B}} \mathrm{p}_\ell(\Lambda)} |r_{S,3-\ell}|^{d_n^\ell} \text{ and}$$

$$1 = \sum_{S\in\mathbb{G}_n} \left(\frac{|r_{S,\ell}|}{|r_{S,3-\ell}|}\right)^{s_n^\ell} |r_{S,3-\ell}|^{\mathfrak{D}_n^\ell}, \text{ where } s_n^\ell \text{ is the similarity dimension of } \mathrm{p}_\ell\mathbb{G}_n\text{: } \sum_{S\in\mathrm{p}_\ell\mathbb{G}_n} |r_{S,\ell}|^{s_n^\ell} = 1.$$

In general, the sets $\mathbb{S}_\delta^1$, $\mathbb{S}_\delta^2$ and $\mathbb{S}_\delta$ are uncomfortable to compute and it is relatively simpler to take higher and higher iterates inductively. The rest of the section is devoted to prove Theorem 3.1.

3.1 Proof of the first part of Theorem 3.1

Let $\delta > 0$ and Δ_δ^ℓ be as in Section 2.1.

Lemma 3.2. *Assuming the conditions of Theorem 3.1, for any $\varepsilon > 0$, there is a $c = c(\varepsilon) > 0$ such that for any $\mathfrak{i} \in \triangle_\delta^\ell$ we have*

$$N_\delta(S_{\mathfrak{i}}(\Lambda)) \in \left(c^{-1}\left(\frac{|r_{\mathfrak{i},\ell}|}{\delta}\right)^{\dim_{\mathrm{B}}(\mathrm{p}_\ell(\Lambda))-\varepsilon}, \; c\left(\frac{|r_{\mathfrak{i},\ell}|}{\delta}\right)^{\dim_{\mathrm{B}}(\mathrm{p}_\ell(\Lambda))+\varepsilon} \right)$$

for every $\ell \in \{1, 2\}$.

Proof: Let $\ell \in \{1, 2\}$. We start with a geometric idea. Observe that if $\mathfrak{i} \in \triangle_\delta^\ell$, then we certainly have $|r_{\mathfrak{i},3-\ell}| \le \delta$, and then

$$N_\delta\big(S_{\mathfrak{i}}(\Lambda)\big) = N_{\delta\cdot|r_{\mathfrak{i},\ell}|^{-1}}(\mathrm{p}_\ell(\Lambda)).$$

See Figure 3 for some intuition.
For self-similar sets, and in particular for $\mathrm{p}_1(\Lambda)$ and $\mathrm{p}_2(\Lambda)$, the box-counting dimension exists, and therefore for any $\varepsilon > 0$ $\exists\Gamma = \Gamma(\varepsilon) > 0$ such that $\forall\delta \le \Gamma$ we have:

$$N_\delta(\mathrm{p}_\ell(\Lambda)) \in \left(\delta^{-\dim_{\mathrm{B}}(\mathrm{p}_\ell(\Lambda))+\varepsilon}, \delta^{-\dim_{\mathrm{B}}(\mathrm{p}_\ell(\Lambda))-\varepsilon}\right).$$

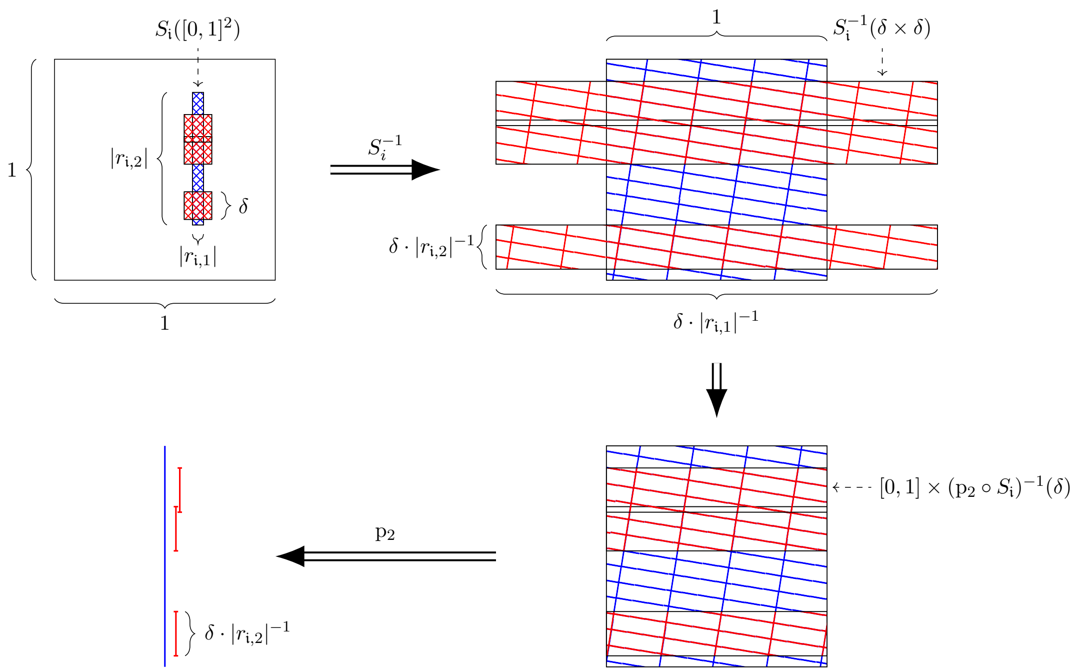


Figure 3: Visualizing the argument for a fixed cylinder and three δ by δ square transforming. Notice that for the covering of the cylinder by δ by δ squares, it is enough to consider the covering of the projection of the attractor to the y-axis by $\delta\cdot|r_{{\rm i},2}|^{-1}$ intervals.

This alone wouldn't be enough, since as δ approaches 0, the value of $\delta\cdot|r_{{\rm i},\ell}|^{-1}$ may not converge, but rather cycle back ever so often, since square-like cylinders ought to happen unless there is coordinate ordering. This can be controlled with the fact that for ${\rm i}\in\triangle_\delta^\ell$: $\delta\cdot|r_{{\rm i},\ell}|^{-1}\le r_{\min}^{-1}$, and then

$$\sup_{{\rm i}\in\triangle_\delta^\ell}\left\{\frac{N_{\delta|r_{{\rm i},\ell}|^{-1}}(\mathrm{p}_\ell(\Lambda))}{\left(\frac{\delta}{|r_{{\rm i},\ell}|}\right)^{-\dim_{\rm B}(\mathrm{p}_\ell(\Lambda))-\varepsilon}}\right\}=\max\left\{1,\sup_{\substack{{\rm i}\in\triangle_\delta^\ell\\ \delta|r_{{\rm i},\ell}|^{-1}>\Gamma}}\left\{\frac{N_{\delta|r_{{\rm i},\ell}|^{-1}}(\mathrm{p}_\ell(\Lambda))}{\left(\frac{\delta}{|r_{{\rm i},\ell}|}\right)^{-\dim_{\rm B}(\mathrm{p}_\ell(\Lambda))-\varepsilon}}\right\}\right\}$$

$$\le\max\left\{1,N_\Gamma(\mathrm{p}_\ell(\Lambda))\cdot r_{\min}^{-\dim_{\rm B}(\mathrm{p}_\ell(\Lambda))-\varepsilon}\right\}=:c_1$$

and

$$\inf_{{\rm i}\in\triangle_\delta^\ell}\left\{\frac{N_{\delta|r_{{\rm i},\ell}|^{-1}}(\mathrm{p}_\ell(\Lambda))}{\left(\frac{\delta}{|r_{{\rm i},\ell}|}\right)^{-\dim_{\rm B}(\mathrm{p}_\ell(\Lambda))+\varepsilon}}\right\}=\min\left\{1,\inf_{\substack{{\rm i}\in\triangle_\delta^\ell\\ \delta|r_{{\rm i},\ell}|^{-1}>\Gamma}}\left\{\frac{N_{\delta|r_{{\rm i},\ell}|^{-1}}(\mathrm{p}_\ell(\Lambda))}{\left(\frac{\delta}{|r_{{\rm i},\ell}|}\right)^{-\dim_{\rm B}(\mathrm{p}_\ell(\Lambda))+\varepsilon}}\right\}\right\}$$

$$\ge\min\left\{1,N_{r_{\min}^{-1}}(\mathrm{p}_\ell(\Lambda))\cdot\Gamma^{\dim_{\rm B}(\mathrm{p}_\ell(\Lambda))+\varepsilon}\right\}=:c_2.$$

provides a sufficinet constant $c:=\max\{c_1,c_2^{-1}\}$. □

Lemma 3.3.

$$\overline{\dim}_{\rm B}(\Lambda)\le\max_{\ell\in\{1,2\}}\limsup_{\delta\to0}\frac{\log\left(\sum_{S\in\mathbb{S}_\delta^\ell}\left(\frac{|r_{S,\ell}|}{\delta}\right)^{\dim_{\rm B}(\mathrm{p}_\ell(\Lambda))}\right)}{-\log\delta}. \tag{3.2}$$

Proof: Let $\delta>0$ and let Δ_δ and $\mathbb{S}_\delta$ be as in Section 2.1. It is easy to see that $\Lambda=\bigcup_{S\in\mathbb{S}_\delta}S(\Lambda)$. Also, by definition, $\mathbb{S}_\delta = \bigcup_{\ell\in\{1,2\}} \mathbb{S}_\delta^\ell$. Moreover, for every $\mathfrak{i} \in \Delta_\delta^\ell$,

$$|r_{\mathfrak{i},\ell}| \in \left[\delta r_{\min},\ \delta^{\frac{\log(r_{\max})}{\log(r_{\min})}}\right].$$

Thus, by Lemma 3.2, for every $\varepsilon > 0$ there exists $c > 0$ such that for every $\delta > 0$

$$\begin{aligned} N_\delta(\Lambda) = N_\delta\left(\bigcup_{\ell\in\{1,2\}} \bigcup_{S\in\mathbb{S}_\delta^\ell} S(\Lambda)\right) &\leq 2 \max_{\ell\in\{1,2\}} \sum_{S\in\mathbb{S}_\delta^\ell} N_\delta(S(\Lambda)) \\ \leq 2c \max_{\ell\in\{1,2\}} \sum_{S\in\mathbb{S}_\delta^\ell} \left(\frac{|r_{S,\ell}|}{\delta}\right)^{\dim_{\mathrm{B}}(\mathrm{p}_\ell(\Lambda))+\varepsilon} &\leq 2c\delta^{(\frac{\log(r_{\max})}{\log(r_{\min})}-1)\varepsilon} \max_{\ell\in\{1,2\}} \sum_{S\in\mathbb{S}_\delta^\ell} \left(\frac{|r_{S,\ell}|}{\delta}\right)^{\dim_{\mathrm{B}}(\mathrm{p}_\ell(\Lambda))}. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, equation (3.2) follows. □

Now, we continue by defining a sufficiently large subsystem that satisfies the ROSC and of which dimension is sufficiently close to the dimension of the original set. Let $\varepsilon > 0$ and $\ell \in \{1,2\}$ be arbitrary. Let us decompose $\mathbb{S}_\delta^\ell$ with respect to it's greater contraction. It is easy to see that for every $S \in \mathbb{S}_\delta^\ell$, $|r_{S,3-\ell}| \leq |r_{S,\ell}| \leq |r_{S,3-\ell}|^{\frac{\log r_{\max}}{\log r_{\min}}}$. Thus,

$$\mathbb{S}_\delta^\ell = \bigcup_{n=\left\lfloor\frac{\log r_{\max}}{\varepsilon \log r_{\min}}\right\rfloor}^{\lceil 1/\varepsilon\rceil} \mathbb{S}_{\delta,\varepsilon}^{\ell,n}, \quad \text{where} \quad \mathbb{S}_{\delta,\varepsilon}^{\ell,n} := \left\{S \in \mathbb{S}_\delta^\ell \;\middle|\; |r_{S,3-\ell}|^{(n+1)\varepsilon} < |r_{S,\ell}| \leq |r_{S,3-\ell}|^{n\varepsilon}\right\}. \tag{3.3}$$

For every $S \in \mathbb{S}_{\delta,\varepsilon}^{\ell,n}$

$$(\delta r_{\min})^{(n+1)\varepsilon} \leq |r_{S,\ell}| \leq \delta^{n\varepsilon}.$$

We now construct a separated subset $\mathbb{S}_{\delta,\varepsilon}^{\ell,n,*}$ from $\mathbb{S}_{\delta,\varepsilon}^{\ell,n}$: write

$$\mathbb{S}_{\delta,\varepsilon}^{\ell,n} = \bigcup_{T\in\mathrm{p}_\ell\mathbb{S}_{\delta,\varepsilon}^{\ell,n}} R(T) \quad \text{where} \quad R(T) := \left\{S \in \mathbb{S}_{\delta,\varepsilon}^{\ell,n} \;\middle|\; \mathrm{p}_\ell S = T\right\}.$$

For every $T \in \mathrm{p}_\ell\mathbb{S}_{\delta,\varepsilon}^{\ell,n}$, let $R^*(T) \subseteq R(T)$ be a maximal populous cylinder-disjoint subset. Now, let $\mathrm{p}_\ell\mathbb{S}_{\delta,\varepsilon}^{\ell,n,*}$ be cylinder-disjoint such that for every $T \in \mathrm{p}_\ell\mathbb{S}_{\delta,\varepsilon}^{\ell,n}$ either $T \in \mathrm{p}_\ell\mathbb{S}_{\delta,\varepsilon}^{\ell,n,*}$ or there is an $\hat{T} \in \mathrm{p}_\ell\mathbb{S}_{\delta,\varepsilon}^{\ell,n,*}$ such that $T((0,1)) \cap \hat{T}((0,1)) \neq \emptyset$, and $R^*(T) \leq R^*(\hat{T})$. Finally, let

$$\mathbb{S}_{\delta,\varepsilon}^{\ell,n,*} := \bigcup_{\mathrm{p}_\ell S\in\mathrm{p}_\ell\mathbb{S}_{\delta,\varepsilon}^{\ell,n,*}} R^*(\mathrm{p}_\ell S). \tag{3.4}$$

Intuitively we factorise the elements of, let's say $\mathbb{S}_{\delta,\varepsilon}^{2,n}$, into rows (columns if $\mathbb{S}_{\delta,\varepsilon}^{1,n}$) based on what their projection to the y-axis (x-axis for $\mathbb{S}_{\delta,\varepsilon}^{1,n}$) is. Then from each row choose as many cylinder-disjoint functions as possible, and then select the largest rows to include, considering that their defining projections have to be cylinder-disjoint.

Lemma 3.4. *There are universal (meaning that it depends only on $\mathbb{G}$) constants s, s' such that*

$$s \cdot \delta^{-\varepsilon s'} \cdot \#\mathbb{S}_{\delta,\varepsilon}^{\ell,n,*} \geq \#\mathbb{S}_{\delta,\varepsilon}^{\ell,n}.$$

Proof: For $j \in \{1,2\}$ let $\mathrm{p}_j\mathbb{S}_{\delta,\varepsilon}^{\ell,n} = \left\{\mathrm{p}_j S_{\mathfrak{i}} \middle| \mathfrak{i} \in \mathbb{S}_{\delta,\varepsilon}^{\ell,n}\right\}$. Since $\mathrm{p}_{3-\ell}\mathbb{S}_{\delta,\varepsilon}^{\ell,n} \subseteq M_\delta^{3-\ell}$, using Lemma 2.1, there is a constant s_1 (indepenedent of $\delta, \ell, n, \varepsilon$ and of T) such that for any $T \in \mathrm{p}_{3-\ell}\mathbb{S}_{\delta,\varepsilon}^{\ell,n}$

$$s_1 \cdot \#R^*(T) \geq \#R(T).$$

On the other hand using Lemma 2.1 on the ℓ projection gives that there is a constant s_2 (indepenedent of $\delta, \ell, n, \varepsilon$ and of T) such that any $T \in \mathrm{p}_\ell \mathbb{S}^{\ell,n}_{\delta,\varepsilon}$ may only cut into s_2 many cylinders from its Moran cut-set. It is possible that those s_2-many functions are not in $\mathrm{p}_\ell \mathbb{S}^{j,n}_{\delta,\varepsilon}$, but their ancestors/offsprings might. Observe that

$$\mathbb{S}^{\ell,n}_{\delta,\varepsilon} \subseteq \left\{ S_{\mathrm{i}} \;\middle|\; |r_{\mathrm{i},3-\ell}| \leq \delta < |r_{\mathrm{i}^-,3-\ell}|,\ (\delta r_{\min})^{(n+1)\varepsilon} \leq |r_{\mathrm{i},\ell}| < \delta^{n\varepsilon} \right\}, \tag{3.5}$$

therefore $\mathrm{p}_\ell \mathbb{S}^{\ell,n}_{\delta,\varepsilon} \subseteq \bigcup_{i\in I} \mathrm{p}_\ell \mathbb{G}_{n_i}$, where we take the union of at most $\left\lceil \varepsilon \frac{\log \delta}{\log r_{\max}} + (n+1)\,\varepsilon \frac{\log r_{\min}}{\log r_{\max}} \right\rceil$-many consecutive levels of $\mathrm{p}_\ell \mathbb{G}_n$. This also meas that $\mathrm{p}_j \mathbb{S}^{\ell,n}_{\delta,\varepsilon} \subseteq \bigcup_{i\in I} M^\ell_{\delta^{n\,\varepsilon}(r_{\max})^i}$ where we again take the union of at most $s_3\,\varepsilon \log \delta$-many consecutive $i \in \mathbb{N}$ for a constant s_3. Taking bigger and bigger intervals over the cylinder rectangle of $\mathrm{p}_\ell S_{\mathrm{i}}$, with Lemma 2.1 we have that any $T \in \mathrm{p}_\ell \mathbb{S}^{\ell,n}_{\delta,\varepsilon}$ may be cylinder-intersected by at most $s_2 s_3\,\varepsilon \log \delta$-many $\hat{T} \in \mathrm{p}_\ell \mathbb{S}^{\ell,n}_{\delta,\varepsilon}$ assuming $|r_{T,\ell}| \leq |r_{\hat{T},\ell}|$.

The other direction is simpler, but gives the more crude bound. As stated above, any $T \in \mathrm{p}_\ell \mathbb{S}^{\ell,n}_{\delta,\varepsilon}$ may only cut into s_2-many cylinders from its Moran cut-set, and with the bound of the number of levels of $\mathrm{p}_\ell \mathbb{G}_n$ covering $\mathrm{p}_\ell \mathbb{S}^{\ell,n}_{\delta,\varepsilon}$ we have that at most $s_2(\#\Sigma)^{s_3\,\varepsilon \log \delta}$-many $\hat{T} \in \mathrm{p}_\ell \mathbb{S}^{\ell,n}_{\delta,\varepsilon}$ assuming $|r_{T,\ell}| \geq |r_{\hat{T},\ell}|$.

Finally, using these bounds, and the construction of $\mathbb{S}^{\ell,n,*}_{\delta,\varepsilon}$, we have the claimed bound. □

Now, let $\mathbb{F} = \mathbb{F}^{n,\ell}_{\delta,\epsilon}(\mathbb{S}^{\ell,n,*}_{\delta,\varepsilon}) \subseteq \{S_{\mathrm{i}} \mid \mathrm{i} \in \Sigma^*\}$ be maximal with respect to containment, such that $\mathrm{p}_\ell \mathbb{F} \subseteq M^{\ell,*}_{\delta^{n\varepsilon}}$, such that the maps in $\mathrm{p}_\ell \mathbb{F}$ are pairwise disjoint and for every $T \in \mathbb{F}$ and $S \in \mathbb{S}^{\ell,n,*}_{\delta,\varepsilon}$, $\mathrm{p}_\ell T((0,1)) \cap \mathrm{p}_\ell S((0,1)) = \emptyset$. Such kind of set can be defined by induction. Denote $\Lambda^{\ell,n}_{\delta,\varepsilon}$ the attractor of the IFS $\mathbb{F} \cup \mathbb{S}^{n,\ell,*}_{\delta,\epsilon}$. By construction, $\mathrm{p}_\ell \mathbb{F} \cup \mathrm{p}_\ell \mathbb{S}^{n,\ell,*}_{\delta,\epsilon}$ satisfies the OSC and so,

$$\sum_{T \in \mathrm{p}_\ell \mathbb{F} \cup \mathrm{p}_\ell \mathbb{S}^{n,\ell,*}_{\delta,\epsilon}} |r_{T,\ell}|^{\dim_{\mathrm{B}}(\mathrm{p}_\ell(\Lambda^{\ell,n}_{\delta,\varepsilon}))} = 1.$$

By definition, the cover

$$\{T((0,1)) \mid T \in \mathrm{p}_\ell \mathbb{F} \cup \mathrm{p}_\ell \mathbb{S}^{n,\ell,*}_{\delta,\epsilon}\}$$

forms a $(\delta r_{\min})^{(n+1)\varepsilon}$-packing and a $3\delta^{n\varepsilon}$-cover of $\mathrm{p}_\ell(\Lambda)$. Hence,

$$\begin{aligned} 1 &\geq (\delta r_{\min})^{(n+1)\varepsilon \dim_{\mathrm{B}}(\mathrm{p}_\ell(\Lambda^{\ell,n}_{\delta,\varepsilon}))} \#(\mathrm{p}_\ell \mathbb{F} \cup \mathrm{p}_\ell \mathbb{S}^{n,\ell,*}_{\delta,\epsilon}) \\ &\geq c^{-1} (\delta r_{\min})^{(n+1)\varepsilon \dim_{\mathrm{B}}(\mathrm{p}_\ell(\Lambda^{\ell,n}_{\delta,\varepsilon}))} \delta^{-n\varepsilon(\dim_{\mathrm{B}}(\mathrm{p}_\ell(\Lambda)) - \varepsilon)} \\ &\geq c^{-1} r_{\min} \delta^{n\varepsilon(\dim_{\mathrm{B}}(\mathrm{p}_\ell(\Lambda^{\ell,n}_{\delta,\varepsilon})) - \dim_{\mathrm{B}}(\mathrm{p}_\ell(\Lambda)) + \varepsilon) + \varepsilon}. \end{aligned}$$

Thus,

$$\begin{aligned} \dim_{\mathrm{B}}(\mathrm{p}_\ell(\Lambda)) &\leq \frac{\log(c r_{\min}^{-1}) - \varepsilon \log \delta}{-n\varepsilon \log \delta} + \dim_{\mathrm{B}}(\mathrm{p}_\ell(\Lambda^{\ell,n}_{\delta,\varepsilon})) + \varepsilon \\ &\leq \frac{2 \log r_{\min} \log(c r_{\min}^{-1})}{-\log r_{\max} \log \delta} + \left(\frac{2 \log r_{\min}}{\log r_{\max}} + 1 \right) \varepsilon + \dim_{\mathrm{B}}(\mathrm{p}_\ell(\Lambda^{\ell,n}_{\delta,\varepsilon})), \end{aligned} \tag{3.6}$$

since $n \geq \frac{\log r_{\max}}{2\varepsilon \log r_{\min}}$. Moreover, $\dim_B(\Lambda^{\ell,n}_{\delta,\varepsilon}) \geq d$ by Feng and Wang's theorem Theorem 2.4, where

$$\sum_{S \in \mathbb{F} \cup \mathbb{S}^{n,\ell,*}_{\delta,\epsilon}} \left(\frac{r_{S,\ell}}{r_{S,3-\ell}} \right)^{\dim_{\mathrm{B}}(\mathrm{p}_\ell(\Lambda^{\ell,n}_{\delta,\varepsilon}))} |r_{S,3-\ell}|^d = 1, \tag{3.7}$$

since the IFS $\mathbb{F} \cup \mathbb{S}^{n,\ell,*}_{\delta,\epsilon}$ satisfies the ROSC.

Lemma 3.5. *Assuming the conditions of Theorem 3.1,*

$$\max_{\ell \in \{1,2\}} \limsup_{\delta \to 0} \frac{\log \left(\sum_{S \in \mathbb{S}^\ell_\delta} \left(\frac{|r_{S,\ell}|}{\delta} \right)^{\dim_{\mathrm{B}}(\mathrm{p}_\ell(\Lambda))} \right)}{-\log \delta} \leq \underline{\dim}_{\mathrm{B}}(\Lambda). \tag{3.8}$$

Proof: Let $\varepsilon > 0$ and $\ell \in \{1,2\}$ be arbitrary, and let $\mathbb{S}^{\ell,n}_{\delta,\varepsilon}$ as in (3.3). Then

$$\sum_{S\in\mathbb{S}^\ell_\delta}\left(\frac{|r_{S,\ell}|}{\delta}\right)^{\dim_{\mathrm{B}}(\mathrm{p}_\ell(\Lambda))} \leq \left(\lceil 1/\varepsilon\rceil - \left\lfloor \frac{\log r_{\max}}{\varepsilon\log r_{\min}}\right\rfloor\right) \max_{n\in\{\lfloor\frac{\log r_{\max}}{\varepsilon\log r_{\min}}\rfloor,\dots,\lceil 1/\varepsilon\rceil\}} \#\mathbb{S}^{\ell,n}_{\delta,\epsilon}\cdot\delta^{(n\varepsilon-1)\dim_{\mathrm{B}}(\mathrm{p}_\ell(\Lambda))}$$

by Lemma 3.4

$$\leq s\left(\lceil 1/\varepsilon\rceil - \left\lfloor \frac{\log r_{\max}}{\varepsilon\log r_{\min}}\right\rfloor\right)\delta^{-s'\varepsilon} \max_{n\in\{\lfloor\frac{\log r_{\max}}{\varepsilon\log r_{\min}}\rfloor,\dots\lceil 1/\varepsilon\rceil\}} \#\mathbb{S}^{\ell,n,*}_{\delta,\epsilon}\cdot\delta^{(n\varepsilon-1)\dim_{\mathrm{B}}(\mathrm{p}_\ell(\Lambda))}$$

by (3.6) there exist constants $c_3 = r_{\min}^{(\frac{\log r_{\max}}{\varepsilon\log r_{\min}}\varepsilon+\varepsilon-1)\frac{2\log(cr_{\min}^{-1})}{\log r_{\max}}} s\left(\lceil 1/\varepsilon\rceil - \left\lfloor \frac{\log r_{\max}}{\varepsilon\log r_{\min}}\right\rfloor\right) > 0$ independent of $\delta > 0$ and $s_3 = s + \left(\frac{2\log r_{\min}}{\log r_{\max}}+1\right)\left(2-\frac{\log r_{\max}}{\log r_{\min}}\right) > 0$ independent of $\varepsilon > 0$ such that

$$\leq c_3\delta^{-s_3\varepsilon} \max_{n\in\{\lfloor\frac{\log r_{\max}}{\varepsilon\log r_{\min}}\rfloor,\dots,\lceil 1/\varepsilon\rceil\}} \#\mathbb{S}^{\ell,n,*}_{\delta,\epsilon}\cdot\delta^{(n\varepsilon-1)\dim_{\mathrm{B}}(\mathrm{p}_\ell(\Lambda^{\ell,n}_{\delta,\varepsilon}))}$$

by (3.5)

$$\leq c_3 r_{\min}^{-1}\delta^{-(s_3+1)\varepsilon} \max_{n\in\{\lfloor\frac{\log r_{\max}}{\varepsilon\log r_{\min}}\rfloor,\dots,\lceil 1/\varepsilon\rceil\}} \sum_{S\in\mathbb{S}^{\ell,n,*}_{\delta,\epsilon}}\left(\frac{|r_{S,\ell}|}{|r_{S,3-\ell}|}\right)^{\dim_{\mathrm{B}}(\mathrm{p}_\ell(\Lambda^{\ell,n}_{\delta,\varepsilon}))}$$

$$\leq c_3 r_{\min}^{-1}\delta^{-(s_3+1)\varepsilon} \max_{n\in\{\lfloor\frac{\log r_{\max}}{\varepsilon\log r_{\min}}\rfloor,\dots,\lceil 1/\varepsilon\rceil\}} \delta^{-d}\sum_{S\in\mathbb{S}^{\ell,n,*}_{\delta,\epsilon}}\left(\frac{|r_{S,\ell}|}{|r_{S,3-\ell}|}\right)^{\dim_{\mathrm{B}}(\mathrm{p}_\ell(\Lambda^{\ell,n}_{\delta,\varepsilon}))}|r_{S,3-\ell}|^d$$

by (3.7)

$$\leq c_3 r_{\min}^{-1}\delta^{-(s_3+1)\varepsilon}\delta^{-d} \leq c_3 r_{\min}^{-1}\delta^{-(s_3+1)\varepsilon-\underline{\dim}_{\mathrm{B}}(\Lambda)},$$

where in the last inequality we used $d \leq \dim_{\mathrm{B}}(\Lambda^{\ell,n}_{\delta,\varepsilon}) \leq \underline{\dim}_{\mathrm{B}}(\Lambda)$. Hence,

$$\limsup_{\delta\to 0}\frac{\log\left(\sum_{S\in\mathbb{S}^\ell_\delta}\left(\frac{|r_{S,\ell}|}{\delta}\right)^{\dim_{\mathrm{B}}(\mathrm{p}_\ell(\Lambda))}\right)}{-\log\delta} \leq \underline{\dim}_{\mathrm{B}}(\Lambda) + (s_3+1)\varepsilon.$$

Since $s_3 > 0$ is independent of $\varepsilon > 0$ and $\varepsilon > 0$ was arbitrary, the claim follows. □

Corollary 3.6. *Assuming the conditions of Theorem 3.1, the box-counting dimension of the attractor exists. Moreover,*

$$\dim_{\mathrm{B}}(\Lambda) = \max_{\ell\in\{1,2\}}\limsup_{\delta\to 0}\frac{\log\left(\sum_{S\in\mathbb{S}^\ell_\delta}\left(\frac{|r_{S,\ell}|}{\delta}\right)^{\dim_{\mathrm{B}}(\mathrm{p}_\ell(\Lambda))}\right)}{-\log\delta}. \tag{3.9}$$

Proof: It is an immediate corollary of Lemma 3.3 and Lemma 3.5. □

From here, it is natural to prove Theorem 3.1. Simple calculations show that

$$\frac{\log\left(\sum_{S\in\mathbb{S}^\ell_\delta}\left(\frac{|r_{S,\ell}|}{\delta}\right)^{\dim_{\mathrm{B}}(\mathrm{p}_\ell(\Lambda))}\right)}{-\log\delta} = \frac{\log\#\mathbb{S}^\ell_\delta}{-\log\delta} + \dim_{\mathrm{B}}(\mathrm{p}_\ell(\Lambda))\left(1+\frac{\log\mathcal{M}_p\left(\{|r_{S,\ell}|\,\big|\,S\in\mathbb{S}^\ell_\delta\}\right)}{-\log\delta}\right),$$

where we have the power mean

$$\mathcal{M}_p(x_1,\dots,x_n) = \left(\frac{1}{n}\sum_{i=1}^n x_i^p\right)^{\frac{1}{p}}$$

with exponent $p = \dim_{\mathrm{B}}(\mathrm{p}_\ell(\Lambda)) \in [0,1]$ and with $n = \#\mathbb{S}^\ell_\delta$. This completes the proof of (3.1).

3.2 Proof of the second part of Theorem 3.1

For $\ell \in \{1,2\}$, let us define D_δ^ℓ with the equation:

$$\sum_{S\in\mathbb{S}_\delta} \left(\frac{|r_{S,\ell}|}{|r_{S,3-\ell}|}\right)^{\dim_B(p_\ell(\Lambda))} |r_{S,3-\ell}|^{D_\delta^\ell} = 1$$

Denote $D_*^\ell := \limsup_{\delta\to 0} D_\delta^\ell$. Notice that

$$\begin{aligned}
\dim_B(\Lambda) &= \max_{\ell\in\{1,2\}} \limsup_{\delta\to 0} \frac{\log\left(\sum_{S\in\mathbb{S}_\delta^\ell}\left(\frac{|r_{S,\ell}|}{|r_{S,3-\ell}|}\right)^{\dim_B(p_\ell(\Lambda))}\right)}{-\log\delta} \\
&= \limsup_{\delta\to 0} \frac{\log\left(\sum_{\ell\in\{1,2\}}\delta^{-D_\delta^\ell}\sum_{S\in\mathbb{S}_\delta^\ell}\left(\frac{|r_{S,\ell}|}{|r_{S,3-\ell}|}\right)^{\dim_B(p_\ell(\Lambda))}|r_{S,3-\ell}|^{D_\delta^\ell}\right)}{-\log\delta} \\
&\le \limsup_{\delta\to 0} \frac{\log\left(\sum_{\ell\in\{1,2\}}\delta^{-D_\delta^\ell}\sum_{S\in\mathbb{S}_\delta}\left(\frac{|r_{S,\ell}|}{|r_{S,3-\ell}|}\right)^{\dim_B(p_\ell(\Lambda))}|r_{S,3-\ell}|^{D_\delta^\ell}\right)}{-\log\delta} \\
&= \limsup_{\delta\to 0} \frac{\log\left(\sum_{\ell\in\{1,2\}}\delta^{-D_\delta^\ell}\right)}{-\log\delta} = \max_{\ell\in\{1,2\}}\left\{\limsup_{\delta\to 0} D_\delta^\ell\right\}. \qquad (3.10)
\end{aligned}$$

For the other direction, let ℓ_δ^* be such that $D_\delta^{\ell_\delta^*} = \max_{\ell\in\{1,2\}}\left\{D_\delta^\ell\right\}$. The following two lemmas are similar to the lemmas from page 335 in [3].

Lemma 3.7. $\dim_B(p_1(\Lambda)) + \dim_B(p_2(\Lambda)) \ge \limsup_{\delta\to 0} D_\delta^{\ell_\delta^*}$.

Proof: Firstly, by the definition of $D_\delta^{\ell_\delta^*}$

$$1 = \sum_{S\in\mathbb{S}_\delta} |r_{S,\ell_\delta^*}|^{\dim_B(p_{\ell_\delta^*}(\Lambda))}|r_{S,3-\ell_\delta^*}|^{-\dim_B(p_{\ell_\delta^*}(\Lambda))+D_\delta^{\ell_\delta^*}},$$

Since $|r_{S,\ell}| \in \left[r_{\min}\delta, \delta^{\frac{\log r_{\max}}{\log r_{\min}}}\right]$ for every $\ell = 1,2$ and $S\in\mathbb{S}_\delta$, it is enough to see that

$$0 \ge \limsup_{\delta\to 0} \frac{\log\left(\sum_{S\in\mathbb{S}_\delta}|r_{S,1}|^{\dim_B(p_1(\Lambda))}|r_{S,2}|^{\dim_B(p_2(\Lambda))}\right)}{-\log\delta}. \qquad (3.11)$$

Let us argue by contradiction. Suppose the opposite of (3.11), namely that that

$$0 < \limsup_{\delta\to 0} \frac{\log\left(\sum_{S\in\mathbb{S}_\delta}|r_{S,1}|^{\dim_B(p_1(\Lambda))}|r_{S,2}|^{\dim_B(p_2(\Lambda))}\right)}{-\log\delta}. \qquad (3.12)$$

Now

$$\begin{aligned}
\dim_B(\Lambda) &= \limsup_{\delta\to 0} \frac{\log\left(\sum_{\ell\in\{1,2\}}\sum_{S\in\mathbb{S}_\delta^\ell}\left(\frac{|r_{S,\ell}|}{|r_{S,3-\ell}|}\right)^{\dim_B(p_\ell(\Lambda))}\right)}{-\log\delta} \\
&= \limsup_{\delta\to 0} \frac{\log\left(\delta^{-\dim_B(p_1(\Lambda))-\dim_B(p_2(\Lambda))}\sum_{S\in\mathbb{S}_\delta}\left(|r_{S,1}|^{\dim_B(p_1(\Lambda))}|r_{S,2}|^{\dim_B(p_2(\Lambda))}\right)\right)}{-\log\delta} \\
&= \dim_B(p_1(\Lambda)) + \dim_B(p_2(\Lambda)) + \limsup_{\delta\to 0} \frac{\log\left(\sum_{S\in\mathbb{S}_\delta}|r_{S,1}|^{\dim_B(p_1(\Lambda))}|r_{Si,2}|^{\dim_B(p_2(\Lambda))}\right)}{-\log\delta}.
\end{aligned}$$

Therefore, assuming (3.12) implies that

$$\dim_{\mathrm{B}}(\Lambda) > \dim_{\mathrm{B}}(\mathrm{p}_1\Lambda) + \dim_{\mathrm{B}}(\mathrm{p}_2\Lambda),$$

what cannot occur, since $\Lambda \subseteq \mathrm{p}_1\Lambda \times \mathrm{p}_2\Lambda$, and for any two set, A, B we have that

$$\dim_{\mathrm{B}}(A) + \dim_{\mathrm{B}}(B) \geq \overline{\dim}_{\mathrm{B}}(A \times B),$$

see Falconer's book [5]. □

Lemma 3.8. *There is a constant $c > 0$ such that*

$$\limsup_{\delta\to 0} \max_{S\in\mathbb{S}_\delta^{3-\ell_\delta^*}} \frac{\left(\frac{|r_{S,3-\ell_\delta^*}|}{|r_{S,\ell_\delta^*}|}\right)^{\dim_{\mathrm{B}}(\mathrm{p}_{3-\ell_\delta^*}(\Lambda))} |r_{S,\ell_\delta^*}|^{D_\delta^{3-\ell_\delta^*}}}{\left(\frac{|r_{S,\ell_\delta^*}|}{|r_{S,3-\ell_\delta^*}|}\right)^{\dim_{\mathrm{B}}(\mathrm{p}_{\ell_\delta^*}(\Lambda))} |r_{S,3-\ell_\delta^*}|^{D_\delta^{\ell_\delta^*}} \delta^{D_\delta^{3-\ell_\delta^*} - D_\delta^{\ell_\delta^*}}} \geq c.$$

Proof: Let $S \in \mathbb{S}_\delta^{3-\ell_\delta^*}$, observe that

$$\frac{\left(\frac{|r_{S,3-\ell_\delta^*}|}{|r_{S,\ell_\delta^*}|}\right)^{\dim_{\mathrm{B}}(\mathrm{p}_{3-\ell_\delta^*}(\Lambda))} |r_{S,\ell_\delta^*}|^{D_\delta^{3-\ell_\delta^*}}}{\left(\frac{|r_{S,\ell_\delta^*}|}{|r_{S,3-\ell_\delta^*}|}\right)^{\dim_{\mathrm{B}}(\mathrm{p}_{\ell_\delta^*}(\Lambda))} |r_{S,3-\ell_\delta^*}|^{D_\delta^{\ell_\delta^*}}} = \left(\frac{|r_{S,3-\ell_\delta^*}|}{|r_{S,\ell_\delta^*}|}\right)^{\dim_{\mathrm{B}}(\mathrm{p}_{3-\ell_\delta^*}(\Lambda)) + \dim_{\mathrm{B}}(\mathrm{p}_{\ell_\delta^*}(\Lambda)) - D_\delta^{\ell_\delta^*}}$$

$$\cdot |r_{S,\ell_\delta^*}|^{D_\delta^{3-\ell_\delta^*} - D_\delta^{\ell_\delta^*}}.$$

Since $\frac{|r_{S,3-\ell_\delta^*}|}{|r_{S,\ell_\delta^*}|} \geq 1$ and $\delta \geq |r_{S,\ell_\delta^*}| > \delta \cdot r_{\min}$, the claim follows by Lemma 3.7. □

Finally, use the Lemma 3.8 as

$$\begin{aligned}
\dim_{\mathrm{B}}(\Lambda) &= \limsup_{\delta\to 0} \frac{\log\left(\sum_{\ell\in\{1,2\}} \sum_{S\in\mathbb{S}_\delta^\ell} \left(\frac{|r_{S,\ell}|}{|r_{S,3-\ell}|}\right)^{\dim_{\mathrm{B}}(\mathrm{p}_\ell(\Lambda))}\right)}{-\log\delta} \\
&\geq \limsup_{\delta\to 0} \frac{1}{-\log\delta}\left(\log\left(\delta^{-D_\delta^{\ell_\delta^*}} \sum_{S\in\mathbb{S}_\delta^{\ell_\delta^*}} \left(\frac{|r_{S,\ell_\delta^*}|}{|r_{S,3-\ell_\delta^*}|}\right)^{\dim_{\mathrm{B}}(\mathrm{p}_{\ell_\delta^*}(\Lambda))} |r_{S,3-\ell_\delta^*}|^{-D_\delta^{\ell_\delta^*}}\right.\right. \\
&\qquad \left.\left. + \delta^{-D_\delta^{3-\ell_\delta^*}} \sum_{S\in\mathbb{S}_\delta^{3-\ell_\delta^*}} c\delta^{D_\delta^{3-\ell_\delta^*} - D_\delta^{\ell_\delta^*}} \left(\frac{|r_{S,\ell_\delta^*}|}{|r_{S,3-\ell_\delta^*}|}\right)^{\dim_{\mathrm{B}}(\mathrm{p}_{\ell_\delta^*}(\Lambda))} |r_{S,3-\ell_\delta^*}|^{D_\delta^{\ell_\delta^*}}\right)\right) \\
&= \limsup_{\delta\to 0} \frac{\log\left(\delta^{-D_\delta^{\ell_\delta^*}} \sum_{S\in\mathbb{S}_\delta} \left(\frac{|r_{S,\ell_\delta^*}|}{|r_{S,3-\ell_\delta^*}|}\right)^{\dim_{\mathrm{B}}(\mathrm{p}_{\ell_\delta^*}(\Lambda))} |r_{S,3-\ell_\delta^*}|^{-D_\delta^{\ell_\delta^*}}\right)}{-\log\delta} \\
&= \limsup_{\delta\to 0} D_\delta^{\ell_\delta^*}.
\end{aligned}$$

The inequality above together with (3.10) imply that

$$\dim_{\mathrm{B}}(\Lambda) = \limsup_{\delta\to 0} \max\{D_\delta^1, D_\delta^2\}. \tag{3.13}$$

3.3 Proof of the third part of Theorem 3.1

Recall that
$$\mathbb{G}_n := \{S_{\mathtt{i}} \mid \mathtt{i} \in \Sigma^n\}.$$
For $\ell \in \{1,2\}$, define d_n^ℓ with the equation
$$\sum_{S\in\mathbb{G}_n} \left(\frac{|r_{S,\ell}|}{|r_{S,3-\ell}|}\right)^{\dim_{\mathrm{B}}(\mathrm{p}_\ell(\Lambda))} |r_{S,3-\ell}|^{d_n^\ell} = 1$$
Let $d_*^\ell := \limsup_{n\to\infty} d_n^\ell$. Similarly, let us define $D_*^\ell := \limsup_{\delta\to 0} D_\delta^\ell$.

Lemma 3.9. *For any $\eta \in [r_{\max}, 1)$, we have that*
$$D_*^\ell = D^\ell := \inf\left\{\alpha > 0 \;\middle|\; \limsup_{k\to\infty}\left(\sum_{S\in\mathbb{S}_{\eta^k}} \left(\frac{|r_{S,\ell}|}{|r_{S,3-\ell}|}\right)^{\dim_{\mathrm{B}}(\mathrm{p}_\ell(\Lambda))} |r_{S,3-\ell}|^{\alpha}\right)^{1/k} < 1\right\},$$
moreover
$$d_*^\ell = d^\ell := \inf\left\{\alpha > 0 \;\middle|\; \limsup_{n\to\infty}\left(\sum_{S\in\mathbb{G}_n} \left(\frac{|r_{S,\ell}|}{|r_{S,3-\ell}|}\right)^{\dim_{\mathrm{B}}(\mathrm{p}_\ell(\Lambda))} |r_{S,3-\ell}|^{\alpha}\right)^{1/n} < 1\right\}.$$

Proof: For any $\alpha > D_*^\ell$ there exists $\varepsilon > 0$, $k_0 > 0$ such that for any $k > k_0$ satisfies $\alpha - \varepsilon > D_{\eta^k}^\ell$ (we will have the hierarchy: $\alpha > \alpha - \varepsilon > D_{\eta^k}^\ell$), and therefore by the definition of $D_{\eta^k}^\ell$, and using that $|r_{S,3-\ell}| \le \eta^k$ for $S \in \mathbb{S}_{\eta^k}$, we will have that either
$$0 = \limsup_{k\to\infty}\left(\sum_{S\in\mathbb{S}_{\eta^k}} \left(\frac{|r_{S,\ell}|}{|r_{S,3-\ell}|}\right)^{\dim_{\mathrm{B}}(\mathrm{p}_\ell(\Lambda))} |r_{S,3-\ell}|^{\alpha}\right)^{1/k} < 1$$
or
$$\begin{aligned}
0 &< \limsup_{k\to\infty}\left(\sum_{S\in\mathbb{S}_{\eta^k}} \left(\frac{|r_{S,\ell}|}{|r_{S,3-\ell}|}\right)^{\dim_{\mathrm{B}}(\mathrm{p}_\ell(\Lambda))} |r_{S,3-\ell}|^{\alpha}\right)^{1/k} \\
&< \eta^{-\varepsilon} \cdot \limsup_{k\to\infty}\left(\sum_{S\in\mathbb{S}_{\eta^k}} \left(\frac{|r_{S,\ell}|}{|r_{S,3-\ell}|}\right)^{\dim_{\mathrm{B}}(\mathrm{p}_\ell(\Lambda)} |r_{S,3-\ell}|^{\alpha}\right)^{1/k} \\
&= \limsup_{k\to\infty}\left(\sum_{S\in\mathbb{S}_{\eta^k}} \left(\frac{|r_{S,\ell}|}{|r_{S,3-\ell}|}\right)^{\dim_{\mathrm{B}}(\mathrm{p}_\ell(\Lambda))} |r_{S,3-\ell}|^{\alpha}\eta^{-\varepsilon k}\right)^{1/k} \\
&\le \limsup_{k\to\infty}\left(\sum_{S\in\mathbb{S}_{\eta^k}} \left(\frac{|r_{S,\ell}|}{|r_{S,3-\ell}|}\right)^{\dim_{\mathrm{B}}(\mathrm{p}_\ell\Lambda))} |r_{S,3-\ell}|^{\alpha-\varepsilon}\right)^{1/k} \\
&\le \limsup_{k\to\infty}\left(\sum_{S\in\mathbb{S}_{\eta^k}} \left(\frac{|r_{S,\ell}|}{|r_{S,3-\ell}|}\right)^{\dim_{\mathrm{B}}(\mathrm{p}_\ell(\Lambda))} |r_{S,3-\ell}|^{D_{\eta^k}^1}\right)^{1/k} = 1.
\end{aligned}$$
Meaning that $D^\ell \le D_*^\ell$.

For the other inequality, let $\alpha < D_*^\ell$, then for any $k_0 > 0$ there exists $k > k_0$ such that $\alpha < D_{\eta^k}^\ell$ and now
$$\left(\sum_{S\in\mathbb{S}_{\eta^k}} \left(\frac{|r_{S,\ell}|}{|r_{S,3-\ell}|}\right)^{\dim_{\mathrm{B}}(\mathrm{p}_\ell(\Lambda))} |r_{S,3-\ell}|^{\alpha}\right)^{1/k} \ge \left(\sum_{S\in\mathbb{S}_{\eta^k}} \left(\frac{|r_{S,\ell}|}{|r_{S,3-\ell}|}\right)^{\dim_{\mathrm{B}}(\mathrm{p}_\ell(\Lambda))} |r_{S,3-\ell}|^{D_{\eta^k}^1}\right)^{1/k} = 1.$$
This proves that $D^\ell \ge D_*^\ell$. The proof of $d_*^\ell = d^\ell$, apply the same procedure. □

Lemma 3.10. *For $\ell \in \{1,2\}$, $d^\ell = D^\ell$.*

Proof: By Lemma 3.9 it is enough to see that the two sums in their definitions differ only by a subexponential amount, and that is what we show. For the first direction, let $\eta \in [r_{\max}, 1)$ and $\mathfrak{i} \in \Delta_{\eta^k}$, then it is easy to see that

$$k \cdot \frac{\log \eta}{\log r_{\min}} \le |\,\mathfrak{i}\,| < k \cdot \frac{\log \eta}{\log r_{\max}} + 1. \tag{3.14}$$

Hence, taking $\alpha > d^\ell$ we observe that

$$\begin{aligned}
&\limsup_{k\to\infty} \left(\sum_{S \in \mathbb{S}_{\eta^k}} \left(\frac{|r_{S,\ell}|}{|r_{S,3-\ell}|} \right)^{\dim_{\mathrm{B}}(\mathrm{p}_\ell(\Lambda))} |r_{S,3-\ell}|^\alpha \right)^{1/k} \\
&\le \limsup_{k\to\infty} \left(\sum_{n=\left\lceil k \frac{\log \eta}{\log r_{\min}} \right\rceil}^{\left\lceil k \frac{\log \eta}{\log r_{\max}} \right\rceil} \sum_{S \in \mathbb{G}_n} \left(\frac{|r_{S,\ell}|}{|r_{S,3-\ell}|} \right)^{\dim_{\mathrm{B}}(\mathrm{p}_\ell(\Lambda))} |r_{S,3-\ell}|^\alpha \right)^{1/k} \\
&= \limsup_{k\to\infty} \left(\max_{n \in \left\{ \left\lceil k \frac{\log \eta}{\log r_{\min}} \right\rceil, \dots, \left\lceil k \frac{\log \eta}{\log r_{\max}} \right\rceil \right\}} \sum_{S \in \mathbb{G}_n} \left(\frac{|r_{S,\ell}|}{|r_{S,3-\ell}|} \right)^{\dim_{\mathrm{B}}(\mathrm{p}_\ell(\Lambda))} |r_{S,3-\ell}|^\alpha \right)^{1/k} < 1,
\end{aligned}$$

meaning that $d^\ell \ge D^\ell$.

Absorbing (3.14) once again, we also have that for any $\mathfrak{i} \in \Sigma^{\{*\}}$, there exists at least one $k \in \mathbb{N}$ that $S_{\mathfrak{i}} \in \mathbb{S}_{\eta^k}$, where

$$(|\,\mathfrak{i}\,| - 1) \frac{\log r_{\max}}{\log \eta} < k \le |\,\mathfrak{i}\,| \frac{\log r_{\min}}{\log \eta}.$$

Then proceed with the prequel argument to get $d^\ell \le D^\ell$, ending the proof of Lemma 3.10. □

Lemma 3.10 with (3.13) implies that

$$\dim_{\mathrm{B}}(\Lambda) = \limsup_{n\to\infty} \max\{d_n^1, d_n^2\}. \tag{3.15}$$

3.4 Proof of the fourth part of Theorem 3.1

In view of the theorem of Zerner (Theorem 2.2) and it's variants, we assumed in the previous section that the box-counting dimension of the principal projections are known exactly. This is not necessary, as in the next setup we show that for computation, only one limit is sufficient.

Let s_n^ℓ be the similarity dimension of $\mathrm{p}_\ell \mathbb{G}_n$ $\sum_{S \in \mathrm{p}_\ell \mathbb{G}_n} |r_{S,\ell}|^{s_n^\ell} = 1$, and let $\mathfrak{D}_n^\ell$ be the unique solution of $1 = \sum_{S \in \mathbb{G}_n} \left(\frac{|r_{S,\ell}|}{|r_{S,3-\ell}|} \right)^{s_n^\ell} |r_{S,3-\ell}|^{\mathfrak{D}_n^\ell}$. By Lemma 2.3, for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \ge N$ we have that $s_n^\ell \in [\dim_{\mathrm{B}}(\mathrm{p}_\ell \Lambda), \dim_{\mathrm{B}}(\mathrm{p}_\ell \Lambda) + \varepsilon]$. Now, for $n \ge N$

$$\begin{aligned}
1 &= \sum_{S \in \mathbb{G}_n} \left(\frac{|r_{S,\ell}|}{|r_{S,3-\ell}|} \right)^{\dim_{\mathrm{B}}(\mathrm{p}_\ell(\Lambda))} |r_{S,3-\ell}|^{d_n^\ell} \\
&\ge \sum_{S \in \mathbb{G}_n} \left(\frac{|r_{S,\ell}|}{|r_{S,3-\ell}|} \right)^{s_n^\ell} \min\left\{ 1, \min_{\substack{S \in \mathbb{G}_n \\ |r_{S,\ell}| \ge |r_{S,3-\ell}|}} \left\{ \left(\frac{|r_{S,\ell}|}{|r_{S,3-\ell}|} \right)^{-\varepsilon} \right\} \right\} |r_{S,3-\ell}|^{d_n^\ell} \\
&\ge \sum_{S \in \mathbb{G}_n} \left(\frac{|r_{S,\ell}|}{|r_{S,3-\ell}|} \right)^{s_n^\ell} |r_{S,3-\ell}|^{d_n^\ell + \varepsilon z},
\end{aligned}$$

where

$$z := \max\left\{ 0, \left(1 - \frac{\log r_{\max}}{\log r_{\min}} \right) \right\} \in [0,1].$$

Since ε can be chosen arbitrary small as n tends to ∞, we can conclude that $\limsup_{n\to\infty} d_n^\ell \geq \limsup_{n\to\infty} \mathfrak{D}_n^\ell$. The proof is finished with a mirror argument for the other direction.

4 Hausdorff dimension

4.1 Feng and Hu formula

First, we need some preliminary calculations on the properties of (1.2). As usual, let Σ be a finite set. Now, let Γ_1, Γ_2 be finite sets such that $\#\Gamma_\ell \leq \#\Sigma$ for $\ell = 1, 2$, and let $\mathrm{p}_\ell \colon \Sigma \mapsto \Gamma_\ell$ be surjective maps. Let $(r_{i,\ell})_{i\in\Sigma,\ell\in\{1,2\}}$ be reals such that $r_{i,\ell} \in (0, 1)$, moreover, $r_{i,\ell} = r_{j,\ell}$ for every $i, j \in \Sigma$ for which $\mathrm{p}_\ell(i) = \mathrm{p}_\ell(j)$.

Let P be the set of probability distributions over the finite set Σ. For $p \in P$, the measure of $i \in \Sigma$ is denoted by p_i. Let $\chi_\ell(p) := -\sum_{i\in\Sigma} p_i \log r_{i,\ell}$. For $j \in \Gamma$, let $q_j^\ell := \sum_{i:\mathrm{p}_\ell(i)=j} p_i$. Define

$$D(p) := \frac{-\sum\limits_{i\in\Sigma} p_i \log p_i}{\chi_{3-\ell}(p)} + \frac{-\sum\limits_{j\in\Gamma_\ell} q_j^\ell \log q_j^\ell}{\chi_\ell(p)} - \frac{-\sum\limits_{j\in\Gamma_\ell} q_j^\ell \log q_j^\ell}{\chi_{3-\ell}(p)}, \text{ if } \quad \chi_\ell(p) \leq \chi_{3-\ell}(p). \tag{4.1}$$

Observe that the above quantity is well defined, since if $\chi_\ell(p) = \chi_{3-\ell}(p)$ then $D(p)$ is independent of ℓ.

Lemma 4.1. *Let $\chi_1, \chi_2 > 0$ be given. Then for every $\ell = 1, 2$*

$$\max_{p\in P} \left\{ \frac{-\sum\limits_{i\in\Sigma} p_i \log p_i}{\chi_{3-\ell}} + \frac{-\sum\limits_{j\in\Gamma_\ell} q_j^\ell \log q_j^\ell}{\chi_\ell} - \frac{-\sum\limits_{j\in\Gamma_\ell} q_j^\ell \log q_j^\ell}{\chi_{3-\ell}} \right\} = \frac{\log\left(\sum\limits_{j\in\Gamma_\ell} \#\{i \in \Sigma \mid \mathrm{p}_\ell(i) = j\}^{\frac{\chi_\ell}{\chi_{3-\ell}}} \right)}{\chi_\ell},$$

where ℓ is such that $\chi_\ell \leq \chi_{3-\ell}$.

The proof of Lemma 4.1 follows by [18, Proposition 3.4] and simple algebraic manipulations by choosing $r_{i,3-\ell} = e^{-\chi_{3-\ell}}$ and $r_{i,\ell} = e^{-\chi_\ell}$ for every $i \in \Sigma$.

Lemma 4.2. *Let $\chi_1, \chi_2 > 0$ and $\varepsilon_0 > 0$ be given, and suppose that $e^{-\chi_\ell-\varepsilon_0} \leq r_{i,\ell} \leq e^{-\chi_\ell+\varepsilon_0}$ for every $i \in \Sigma$ and $\ell \in \{1, 2\}$. Let $\ell \in \{1, 2\}$ be such that $\chi_\ell \leq \chi_{3-\ell}$. Then*

$$\max_{p\in P} \left\{ \frac{\sum\limits_{i\in\Sigma} p_i \log p_i}{\sum\limits_{i\in\Sigma} p_i \log r_{i,3-\ell}} + \frac{\sum\limits_{j\in\Gamma_\ell} q_j^\ell \log q_j^\ell}{\sum\limits_{i\in\Sigma} p_i \log r_{i,\ell}} - \frac{\sum\limits_{j\in\Gamma_\ell} q_j^\ell \log q_j^\ell}{\sum\limits_{i\in\Sigma} p_i \log r_{i,3-\ell}} \right\}$$
$$\geq \frac{\log\left(\sum\limits_{j\in\Gamma_\ell} \#\{i \in \Sigma \mid \mathrm{p}_\ell(i) = j\}^{\frac{\chi_\ell}{\chi_{3-\ell}}} \right)}{\chi_\ell} - \frac{2\varepsilon_0 \log \#\Sigma}{\log r_{\max}(\log r_{\max} - \varepsilon_0)}.$$

Proof: Recall that $q_j^\ell = \sum_{i:\mathrm{p}_\ell(i)=j} p_i$. Moreover, clearly $\log \#\Sigma \geq -\sum\limits_{i\in\Sigma} p_i \log p_i \geq -\sum\limits_{j\in\Gamma_\ell} q_j^\ell \log q_j^\ell$. So

$$\frac{\sum\limits_{i\in\Sigma} p_i \log p_i}{\sum\limits_{i\in\Sigma} p_i \log r_{i,3-\ell}} + \frac{\sum\limits_{j\in\Gamma_\ell} q_j^\ell \log q_j^\ell}{\sum\limits_{i\in\Sigma} p_i \log r_{i,\ell}} - \frac{\sum\limits_{j\in\Gamma_\ell} q_j^\ell \log q_j^\ell}{\sum\limits_{i\in\Sigma} p_i \log r_{i,3-\ell}}$$
$$\geq \frac{-\sum\limits_{i\in\Sigma} p_i \log p_i}{\chi_{3-\ell} + \varepsilon_0} + \frac{-\sum\limits_{j\in\Gamma_\ell} q_j^\ell \log q_j^\ell}{\chi_\ell + \varepsilon_0} - \frac{-\sum\limits_{j\in\Gamma_\ell} q_j^\ell \log q_j^\ell}{\chi_{3-\ell} + \varepsilon_0}$$
$$\geq \frac{-\sum\limits_{i\in\Sigma} p_i \log p_i}{\chi_{3-\ell}} + \frac{-\sum\limits_{j\in\Gamma_\ell} q_j^\ell \log q_j^\ell}{\chi_\ell} - \frac{-\sum\limits_{j\in\Gamma_\ell} q_j^\ell \log q_j^\ell}{\chi_{3-\ell}} - \frac{2\varepsilon_0 \log \#\Sigma}{\chi_\ell(\chi_\ell + \varepsilon_0)}.$$

The proof can be finished by Lemma 4.1. □

The map $\mathrm{p}_\ell \colon \Sigma \to \Gamma_\ell$ can be naturally extended to $\mathrm{p}_\ell \colon \Sigma^* \cup \Sigma^{\mathbb{N}} \to \Gamma_\ell^* \cup \Gamma_\ell^{\mathbb{N}}$, by defining $\mathrm{p}_\ell(i_1 i_2 \dots) := \mathrm{p}_\ell(i_1)\mathrm{p}_\ell(i_2)\dots$.

The distribution $p \in P$ induces naturally a distribution on Σ^n for every $n \in \mathbb{N}$ by $p_{\mathtt{i}} := p_{i_1} \cdots p_{i_n}$ for every $\mathtt{i} \in \Sigma^n$. Let us denote this distribution by p^{*n}. Also, let $q_{\mathtt{j}}^\ell := \sum_{\mathtt{i}:\mathrm{p}_\ell(\mathtt{i})=\mathtt{j}} p_{\mathtt{i}} = q_{j_1}^\ell \cdots q_{j_n}^\ell$ for $\mathtt{j} \in \Gamma_\ell^n$.

Simple algebraic manipulations show that $\chi_\ell(p^{*n}) = n\chi_\ell(p)$ for every $n \in \mathbb{N}$, and so,

$$D(p) = \frac{-\sum_{\mathtt{i}\in\Sigma^n} p_{\mathtt{i}} \log p_{\mathtt{i}}}{n\chi_{3-\ell}(p)} + \frac{-\sum_{\mathtt{j}\in\Gamma_\ell^n} q_{\mathtt{j}}^\ell \log q_{\mathtt{j}}^\ell}{n\chi_\ell(p)} - \frac{-\sum_{\mathtt{j}\in\Gamma_\ell^n} q_{\mathtt{j}}^\ell \log q_{\mathtt{j}}^\ell}{n\chi_{3-\ell}(p)}, \text{ if } \quad \chi_\ell(p) \leq \chi_{3-\ell}(p). \tag{4.2}$$

For χ_1, χ_2 positive reals, let

$$\begin{aligned}
\Sigma^n\big(\varepsilon_0, \chi_1, \chi_2\big) &:= \left\{\mathtt{i} \in \Sigma^n \;\middle|\; -\frac{1}{n}\log r_{\mathtt{i},\ell} \in (\chi_\ell - \varepsilon_0, \chi_\ell + \varepsilon_0) \text{ for } \ell \in \{1,2\}\right\},\\
\Gamma_\ell^n\big(\varepsilon_0, \chi_1, \chi_2\big) &:= \left\{\mathrm{p}_\ell(\mathtt{i}) \;\middle|\; \mathtt{i} \in \Sigma^n(\varepsilon_0, \chi_1, \chi_2)\right\} \text{ and}\\
\Sigma_{\mathtt{j}}^{n,\ell}\big(\varepsilon_0, \chi_1, \chi_2\big) &:= \left\{\mathtt{i} \in \Sigma^n(\varepsilon_0, \chi_1, \chi_2) \;\middle|\; \mathrm{p}_\ell(\mathtt{i}) = \mathtt{j}\right\} \text{ for } \mathtt{j} \in \Gamma_\ell^n.
\end{aligned}$$

Lemma 4.3. *For every probability distribution p on Σ, $n \in \mathbb{N}$ and $\varepsilon_0 > 0$*

$$D(p) \leq \frac{\log\left(\sum_{\mathtt{j}\in\Gamma_\ell^n(\varepsilon_0,\chi_1(p),\chi_2(p))} \#\Sigma_{\mathtt{j}}^{n,\ell}\big(\varepsilon_0, \chi_1(p), \chi_2(p)\big)^{\frac{\chi_\ell(p)}{\chi_{3-\ell}(p)}}\right)}{n\chi_\ell(p)} + b \cdot e^{-\frac{n\varepsilon_0^2}{\log^2 \frac{r_{\max}}{r_{\min}}}}$$

if $\chi_\ell(p) \leq \chi_{3-\ell}(p)$, where $b = \dfrac{8(\log\#\Sigma + \log 4 + \log^{-2}\frac{r_{\max}}{r_{\min}} + 1)}{-\log r_{\min}}$.

Proof: Given $p \in P$, let $\{(X_k^1, X_k^2)\}_{k\in\mathbb{N}}$ be a sequence of independent and identically distributed random variables such that

$$\mathbb{P}\left((X_k^1, X_k^2) = (\log r_1, \log r_2)\right) = \sum_{i\in\{\Sigma: r_{i,\ell}=r_\ell, \forall \ell=1,2\}} p_{\mathtt{i}}.$$

Then by Hoeffding's inequality, for every $n \in \mathbb{N}$ and $\ell \in \{1,2\}$

$$\sum_{\mathtt{i}\in(\Sigma^n(\varepsilon_0,\chi_1(p),\chi_2(p)))^c} p_{\mathtt{i}} \leq \sum_{\ell=1}^2 \mathbb{P}\left(\left|\sum_{k=1}^n X_k^\ell - n\chi_\ell(p)\right| > \varepsilon_0 n\right) \leq 4e^{-\frac{n\varepsilon_0^2}{\log^2\frac{r_{\max}}{r_{\min}}}}. \tag{4.3}$$

Let $P_n := \sum_{\mathtt{i}\in\Sigma^n(\varepsilon_0,\chi_1(p),\chi_2(p))} p_{\mathtt{i}} = \sum_{\mathtt{i}\in\Gamma_\ell^n(\varepsilon_0,\chi_1(p),\chi_2(p))} q_{\mathtt{i}}^\ell$. Then

$$\begin{aligned}
-\sum_{\mathtt{i}\in\Sigma^n} p_{\mathtt{i}}\log p_{\mathtt{i}} = &-P_n \sum_{\mathtt{i}\in\Sigma^n(\varepsilon_0,\chi_1(p),\chi_2(p))} \frac{p_{\mathtt{i}}}{P_n}\log\frac{p_{\mathtt{i}}}{P_n} - (1-P_n)\sum_{\mathtt{i}\in(\Sigma^n(\varepsilon_0,\chi_1(p),\chi_2(p)))^c} \frac{p_{\mathtt{i}}}{1-P_n}\log\frac{p_{\mathtt{i}}}{1-P_n}\\
&- P_n\log P_n - (1-P_n)\log(1-P_n)
\end{aligned}$$

by (4.3), the series expansion of the logarithm and the basic properties of the entropy

$$\leq -\sum_{\mathtt{i}\in\Sigma^n(\varepsilon_0,\chi_1(p),\chi_2(p))} \frac{p_{\mathtt{i}}}{P_n}\log\frac{p_{\mathtt{i}}}{P_n} + 4e^{-\frac{n\varepsilon_0^2}{\log^2\frac{r_{\max}}{r_{\min}}}}\left(n\log\#\Sigma + \log 4 + \frac{2n\varepsilon_0^2}{\log^2\frac{r_{\max}}{r_{\min}}} + 1\right).$$

Similarly,

$$-\sum_{\mathtt{i}\in\Gamma_\ell^n} q_{\mathtt{i}}^\ell\log q_{\mathtt{i}}^\ell \leq -\sum_{\mathtt{i}\in\Gamma_\ell^n(\varepsilon_0,\chi_1(p),\chi_2(p))} \frac{q_{\mathtt{i}}^\ell}{P_n}\log\frac{q_{\mathtt{i}}^\ell}{P_n} + 4e^{-\frac{n\varepsilon_0^2}{\log^2\frac{r_{\max}}{r_{\min}}}}\left(n\log\#\Sigma + \log 4 + \frac{2n\varepsilon_0^2}{\log^2\frac{r_{\max}}{r_{\min}}} + 1\right).$$

Let $b = 8(\log \#\Sigma + \log 4 + \log^{-2} \frac{r_{\max}}{r_{\min}} + 1)/(-\log r_{\min})$. By (4.2),

$$D(p) \leq \frac{-\sum_{\mathrm{i}\in\Sigma^n(\varepsilon_0,\chi_1(p),\chi_2(p))} \frac{p_\mathrm{i}}{P_n}\log\frac{p_\mathrm{i}}{P_n}}{n\chi_{3-\ell}(p)} + \left(-\sum_{\mathrm{i}\in\Gamma_\ell^n(\varepsilon_0,\chi_1(p),\chi_2(p))} \frac{q_\mathrm{i}^\ell}{P_n}\log\frac{q_\mathrm{i}^\ell}{P_n}\right)\left(\frac{1}{n\chi_\ell(p)} - \frac{1}{n\chi_{3-\ell}(p)}\right) + be^{-\frac{n\varepsilon_0^2}{\log^2\frac{r_{\max}}{r_{\min}}}}.$$

Then, by applying Lemma 4.1 for the right-hand side, the claim follows. □

As a corollary of the above, we can provide an alternative, limiting formula for the dimension of Lalley-Gatzouras carpets. This shows that the dimension of the attractor can be well approximated by close to almost-homogeneous systems.

Theorem 4.4. *Let $\mathbb{G}$ be a self-affine IFS satisfying the assumptions (A1), (A2) (C1), (C3) and (G1) with attractor Λ. Let $\varepsilon_0(n) > 0$ be a non-negative sequence such that $\varepsilon_0(n) < -\frac{1}{2}\log r_{\max}$ and $\lim_{n\to\infty}\big(n\cdot\varepsilon_0^2(n)\big) = \infty$. Then*

$$\begin{aligned}\dim_\mathrm{H}(\Lambda) &= \lim_{n\to\infty}\max_{(\chi_1,\chi_2)\in\mathbb{X}} \frac{\log\Big(\sum_{\mathrm{i}\in\Gamma_2^n(\varepsilon_0(n),\chi_1,\chi_2)} \#\Sigma_\mathrm{i}^{n,2}\big(\varepsilon_0(n),\chi_1,\chi_2\big)^{\frac{\chi_2}{\chi_1}}\Big)}{-n\chi_2} \\ &= \sup_{(\chi_1,\chi_2)\in\mathbb{X}}\limsup_{n\to\infty} \frac{\log\Big(\sum_{\mathrm{i}\in\Gamma_2^n(\varepsilon_0(n),\chi_1,\chi_2)} \#\Sigma_\mathrm{i}^{n,2}\big(\varepsilon_0(n),\chi_1,\chi_2\big)^{\frac{\chi_2}{\chi_1}}\Big)}{-n\chi_2}.\end{aligned}$$

Proof: It is clear that

$$\begin{aligned}&\sup_{(\chi_1,\chi_2)\in\mathbb{X}}\liminf_{n\to\infty} \frac{\log\Big(\sum_{\mathrm{i}\in\Gamma_2^n(\varepsilon_0(n),\chi_1,\chi_2)} \#\Sigma_\mathrm{i}^{n,2}\big(\varepsilon_0(n),\chi_1,\chi_2\big)^{\frac{\chi_2}{\chi_1}}\Big)}{-n\chi_2} \\ &\qquad\leq \liminf_{n\to\infty}\max_{(\chi_1,\chi_2)\in\mathbb{X}} \frac{\log\Big(\sum_{\mathrm{i}\in\Gamma_2^n(\varepsilon_0(n),\chi_1,\chi_2)} \#\Sigma_\mathrm{i}^{n,2}\big(\varepsilon_0(n),\chi_1,\chi_2\big)^{\frac{\chi_2}{\chi_1}}\Big)}{-n\chi_2}\end{aligned} \tag{4.4}$$

By Theorem 2.5, $\dim_\mathrm{H}(\Lambda) = \max_{p\in P} D(p)$. Let $p^* \in P$ be the measure where the maximum is attained. Then by Lemma 4.3

$$D(p^*) \leq \frac{\log\Big(\sum_{\mathrm{j}\in\Gamma_2^n(\varepsilon_0(n),\chi_1(p^*),\chi_2(p^*))} \#\Sigma_\mathrm{j}^{n,2}\big(\varepsilon_0(n),\chi_1(p^*),\chi_2(p^*)\big)^{\frac{\chi_2(p^*)}{\chi_1(p^*)}}\Big)}{n\chi_2(p^*)} + b\cdot e^{-\frac{n\varepsilon_0(n)^2}{\log^2\frac{r_{\max}}{r_{\min}}}}.$$

Thus,

$$\dim_\mathrm{H}(\Lambda) = D(p^*) \leq \sup_{(\chi_1,\chi_2)\in\mathbb{X}}\liminf_{n\to\infty} \frac{\log\Big(\sum_{\mathrm{j}\in\Gamma_2^n(\varepsilon_0(n),\chi_1,\chi_2)} \#\Sigma_\mathrm{j}^{n,2}\big(\varepsilon_0(n),\chi_1,\chi_2\big)^{\frac{\chi_2}{\chi_1}}\Big)}{n\chi_2}. \tag{4.5}$$

On the other hand, by Lemma 4.2 for every $(\chi_1,\chi_2)\in\mathbb{X}$

$$\begin{aligned}&\frac{\log\Big(\sum_{\mathrm{j}\in\Gamma_2^n(\varepsilon_0(n),\chi_1,\chi_2)} \#\Sigma_\mathrm{j}^{n,2}(\varepsilon_0(n),\chi_1,\chi_2)^{\frac{\chi_2}{\chi_1}}\Big)}{n\chi_2} \\ &\leq \max_{p\in P}\left\{\frac{\sum_{\mathrm{i}\in\Sigma^n(\varepsilon_0(n),\chi_1,\chi_2)} p_\mathrm{i}\log p_\mathrm{i}}{\sum_{\mathrm{i}\in\Sigma^n(\varepsilon_0(n),\chi_1,\chi_2)} p_\mathrm{i}\log r_{\mathrm{i},1}} + \frac{\sum_{\mathrm{j}\in\Gamma_2^n(\varepsilon_0(n),\chi_1,\chi_2)} q_\mathrm{j}^\ell\log q_\mathrm{j}^\ell}{\sum_{\mathrm{i}\in\Sigma^n(\varepsilon_0(n),\chi_1,\chi_2)} p_\mathrm{i}\log r_{\mathrm{i},2}} - \frac{\sum_{\mathrm{j}\in\Gamma_2^n(\varepsilon_0(n),\chi_1,\chi_2)} q_\mathrm{j}^\ell\log q_\mathrm{j}^\ell}{\sum_{\mathrm{i}\in\Sigma^n(\varepsilon_0(n),\chi_1,\chi_2)} p_\mathrm{i}\log r_{\mathrm{i},1}}\right\}\end{aligned}$$

$$+\frac{2\varepsilon_0(n)\log\#\Sigma^n(\varepsilon_0(n),\chi_1,\chi_2)}{n^2\cdot\chi_\ell(\chi_\ell-\varepsilon_0(n))}$$
$$\leq \dim_{\mathrm{H}}(\Lambda)+\frac{\varepsilon_0(n)}{n}\frac{2\log\#\Sigma}{\chi_\ell(\chi_\ell-\varepsilon_0(n)))},$$

where the last inequality follows by the Theorem 2.5 and the fact that the attractor of the IFS $\{S_{\mathfrak{i}}\}_{\mathfrak{i}\in\Sigma^n(\varepsilon_0(n),\chi_1,\chi_2)}$ is clearly contained in Λ. Hence,

$$\limsup_{n\to\infty}\max_{(\chi_1,\chi_2)\in\mathbb{X}}\frac{\log\Big(\sum_{\mathfrak{i}\in\Gamma_2^n(\varepsilon_0(n),\chi_1,\chi_2)}\#\Sigma_{\mathfrak{i}}^{n,2}\big(\varepsilon_0(n),\chi_1,\chi_2\big)^{\frac{\chi_2}{\chi_1}}\Big)}{-n\chi_2}\leq\dim_{\mathrm{H}}(\Lambda).$$

Then the claim follows by observing that the equations (4.4) and (4.5) hold with lim sup. □

Remark 4.5. We note that in case of constant $\varepsilon(n)=\varepsilon$, the limit

$$\lim_{n\to\infty}\frac{\log\Big(\sum_{\mathfrak{i}\in\Gamma_2^n(\varepsilon_0,\chi_1,\chi_2)}\#\Sigma_{\mathfrak{i}}^{n,2}\big(\varepsilon_0,\chi_1,\chi_2\big)^{\frac{\chi_2}{\chi_1}}\Big)}{-n\chi_2}$$

exists and equals to the supremum by Fekete's lemma. Indeed, we have that $\Big\{\mathfrak{i}\,\mathfrak{j}\ \Big|\ \mathfrak{i}\in\Gamma_2^n(\varepsilon_0,\chi_1,\chi_2),\mathfrak{j}\in\Gamma_2^m(\varepsilon_0,\chi_1,\chi_2)\Big\}\subseteq\Gamma_2^{nm}(\varepsilon_0,\chi_1,\chi_2)$, and similarly for $\Sigma_{\mathfrak{i}}^{n,2}\big(\varepsilon_0,\chi_1,\chi_2\big)$.

4.2 Weak separation for carpets

Let $\mathbb{G}$ be an IFS as in (A1). Let $\Sigma^{\{n\}}$ and $\Gamma_\ell^{\{n\}}$ be as in Section 2.1. With a slight abuse of notation, we define a map $\mathrm{p}_\ell^{\{n\}}:\Sigma^{\{n\}}\to\Gamma_\ell^{\{n\}}$ as follows: for every $\mathfrak{i}\in\Sigma^{\{n\}}$, $p_\ell^{\{n\}}(\mathfrak{i})$ is the unique element of $\mathfrak{j}\in\Gamma_\ell^{\{n\}}$ such that $\mathrm{p}_\ell S_{\mathfrak{i}}=\mathrm{p}_\ell S_{\mathfrak{j}}$.

Recall

$$\Sigma^{\{n\}m}=\Big\{\mathfrak{i}^1\,\mathfrak{i}^2\dots\mathfrak{i}^m\ \Big|\ \mathfrak{i}^k\in\Sigma^{\{n\}}\ \forall k\in\{1,\dots,m\}\Big\},\text{ and similarly,}$$
$$\Gamma_\ell^{\{n\}m}:=\Big\{\mathfrak{j}^1\,\mathfrak{j}^2\dots\mathfrak{j}^m\ \Big|\ \mathfrak{j}^k\in\Gamma_\ell^{\{n\}}\ \forall k\in\{1,\dots,m\}\Big\}$$

Lemma 4.6. *There is a constant $C>1$ depending only on $\mathbb{G}$ such that for any $n,m\in\mathbb{N}^+$ we have that*

$$\forall x\in[0,1]^2:\quad\#\Big\{\mathfrak{i}\in\Sigma^{\{n\}m}\ \Big|\ x\in S_{\mathfrak{i}}([0,1]^2)\Big\}\leq n^{2m}\cdot C^m$$
$$\forall x\in[0,1]:\quad\#\big\{\,\mathfrak{i}\in\Gamma_\ell^{\{n\}m}\ \big|\ x\in(\mathrm{p}_\ell S_{\mathfrak{i}})([0,1])\big\}\leq n^mC^m.$$

Lemma 4.6 follows from applying the following lemma inductively.

Lemma 4.7. *There is a constant $C'>1$ depending only on $\mathbb{G}$ such that for any $n\in\mathbb{N}^+$ we have that*

$$\forall x\in[0,1]^2:\quad\#\big\{S\in\mathbb{G}_n\ \big|\ x\in S([0,1]^2)\big\}\leq n^2\cdot C',$$
$$\forall x\in[0,1]:\quad\#\big\{S\in\mathrm{p}_\ell\mathbb{G}_n\ \big|\ x\in\mathrm{p}_\ell S([0,1])\big\}\leq n\cdot C'.$$

Proof: Notice that

$$\Sigma^{\{n\}}=\bigcup_{k=0}^{Y}\bigcup_{h=0}^{W}\Big\{\mathfrak{i}\in\Sigma^{\{n\}}\ \Big|\ r_{\mathfrak{i},1}\in[r_{\max}^{n+h+1},r_{\max}^{n+h}],r_{\mathfrak{i},2}\in[r_{\max}^{n+k+1},r_{\max}^{n+k}]\Big\},$$

where $Y,W\leq\Big\lfloor n\cdot\log_{r_{\max}}\Big(\frac{r_{\min}}{r_{\max}}\Big)\Big\rfloor$. Now for $\ell\in\{1,2\}$ we have that $r_{\mathfrak{i},\ell}\in[r_{\max}^{n+h+1},r_{\max}^{n+h}]$ implies that $r_{\mathfrak{i},\ell}\leq r_{\max}^{n+h}\leq r_{\mathfrak{i}-,\ell}$, thus $\big\{\mathrm{p}_\ell^{\{n\}}(\mathfrak{i})\ \big|\ r_{\mathfrak{i},\ell}\in[r_{\max}^{n+h+1},r_{\max}^{n+h}]\big\}\subseteq M^{\ell,*}_{r_{\max}^{n+h}}$. Then by Lemma 2.1, for any given $x\in\mathbb{R}$:

$$\#\Big\{\, \mathfrak{i} \in \Sigma^{\{n\}} \;\Big|\; r_{\mathfrak{i},\ell} \in [r_{\max}^{n+h+1}, r_{\max}^{n+h}],\; x \in \mathrm{p}_\ell S_{\mathfrak{i}}([0,1]) \Big\}$$
$$\subseteq \#\Big\{\, \mathfrak{i} \in \Sigma^{\{n\}} \;\Big|\; \mathrm{p}_\ell^{\{n\}}(\mathfrak{i}) \in M^{\ell,*}_{r_{\max}^{n+h}},\; x \in \mathrm{p}_\ell S_{\mathfrak{i}}([0,1]) \Big\} \le C",$$

where $C"$ is a constant depending on the IFS. Finally

$$\#\{S \in \mathbb{G}_n \mid x \in S([0,1]^2)\}$$

is upper bounded by the product of the upper bounds of the projections, and those upper bounds are obtained by noticing that from an order of n Moran cover, each has at most constant-many functions which has cylinder rectangle containing the projection of x. Since $\mathbb{G}_n = \{S_{\mathfrak{i}} \mid \mathfrak{i} \in \Sigma^{\{n\}}\}$, the first claim follows. The proof of the second claim is similar, so we omit it. □

Similarly to the previous section, we define subsets of $\Sigma^{\{n\}m}$ and $\Gamma_\ell^{\{n\}m}$ as follows:

$$\Sigma^{\{n\}m}(\varepsilon_0, \chi_1, \chi_2) := \left\{ \mathfrak{i} \in \Sigma^{\{n\}m} \;\middle|\; -\frac{1}{nm}\log r_{\mathfrak{i},\ell} \in (\chi_\ell - \varepsilon_0, \chi_\ell + \varepsilon_0) \text{ for } \ell \in \{1,2\} \right\},$$
$$\Gamma_\ell^{\{n\}m}(\varepsilon_0, \chi_1, \chi_2) := \left\{ \mathrm{p}_\ell(\mathfrak{i}) \;\middle|\; \mathfrak{i} \in \Sigma^{\{n\}m}(\varepsilon_0, \chi_1, \chi_2) \right\}.$$

For $\mathfrak{j} \in \Gamma_\ell^{\{n\}m}$ let

$$\Sigma_{\mathfrak{j}}^{\{n\}m,\ell}(\varepsilon_0, \chi_1, \chi_2) := \left\{ \mathfrak{i} \in \Sigma^{\{n\}m}(\varepsilon_0, \chi_1, \chi_2) \;\middle|\; p_\ell(\mathfrak{i}|_{(n(k-1),nk]}) = \mathfrak{j}|_{(n(k-1),nk]} \forall k \in \{1,\dots,m\} \right\}.$$

Lemma 4.8. *There are constants $\hat{C}, C > 0$ such that for $\ell \in \{1,2\}$ and for every $\mathfrak{i} \in \Gamma_\ell^{\{n\}m}(\varepsilon_0, \chi_1, \chi_2)$*

$$\#\Big\{\, \mathfrak{j} \in \Gamma_\ell^{\{n\}m}(\varepsilon_0, \chi_1, \chi_2) \;\Big|\; \mathrm{p}_\ell S_{\mathfrak{i}}([0,1]) \cap \mathrm{p}_\ell S_{\mathfrak{j}}([0,1]) \neq \emptyset \Big\} \le (nC\hat{C}^{n\,\varepsilon_0})^m.$$

Proof: For any $\mathfrak{i} \in \Gamma_\ell^{\{n\}m}(\varepsilon_0, \chi_1, \chi_2)$, we put in the interval $\mathrm{p}_\ell S_{\mathfrak{i}}([0,1])$ uniformly $\left\lceil \frac{e^{-nm(\chi_2-\varepsilon_0)}}{e^{-nm(\chi_2+\varepsilon_0)}} \right\rceil$ many points. Hence, the distance between any two consecutive points is at most $e^{-nm(\chi_\ell+\varepsilon_0)}$. Let us denote this collection of points by D. For some illustration, see Figure 4.

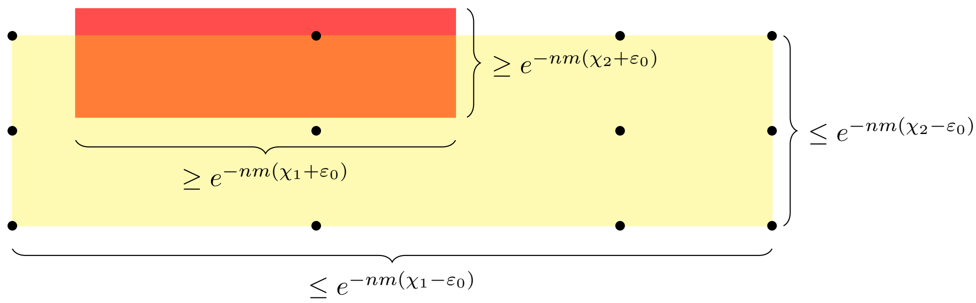


Figure 4: Illustration of a similar argument as in the proof of Lemma 4.8, exept insead of intervals, we illustrate the cylinder rectangles.

So, for any $\mathfrak{j} \in \Gamma_\ell^{\{n\}m}(\varepsilon_0, \chi_1, \chi_2)$ with $\mathrm{p}_\ell S_{\mathfrak{i}}([0,1]) \cap \mathrm{p}_\ell S_{\mathfrak{j}}([0,1]) \neq \emptyset$ we see that there exists $x \in D$ such that $x \in \mathrm{p}_\ell S_{\mathfrak{j}}([0,1])$. Then, using Lemma 4.6, we have that

$$\#\Big\{\, \mathfrak{j} \in \Gamma_\ell^{\{n\}m}(\varepsilon_0, \chi_1, \chi_2) \;\Big|\; \mathrm{p}_\ell S_{\mathfrak{i}}([0,1]) \cap \mathrm{p}_\ell S_{\mathfrak{j}}([0,1]) \neq \emptyset \Big\} \le \#D n^m C^m \le 2n^m C^m \cdot e^{-2\chi_\ell nm\,\varepsilon_0}.$$

Thus, the claim follows. □

4.3 Proof of Theorem 1.1

Let P^n be the set of probability distributions on $\Sigma^{\{n\}}$. For $p \in P^n$, let $D_n(p)$ be as in (4.1) with the symbolic spaces $\Sigma^{\{n\}}$, $\Gamma_\ell^{\{n\}}$. Let p^n be the probability vector, where the maximum of $D_n(p)$ is attained. By Theorem 2.6, for every $n \in \mathbb{N}_+$

$$\dim_{\mathrm{H}}(\Lambda) \leq \max_{p\in P^n} D_n(p) = \mathbb{H}_{BA}(\mathbb{G}_n).$$

Denote the normalised Lyapunov exponents of p^n by $\chi_{\ell,n} = \frac{1}{n}\sum_{\mathrm{i}\in\Sigma^{\{n\}}} p_{\mathrm{i}} \log r_{\mathrm{i},\ell}$, and let ℓ_n be such that $\chi_{\ell_n,n} \leq \chi_{3-\ell_n,n}$. Hence, by Lemma 4.3 there exists $b > 0$ (which is independent of n and m) such that

$$\max_{p\in P^n} D_n(p) \leq \frac{\log\left(\sum_{\mathrm{j}\in\Gamma_{\ell_n}^{\{n\}m}(\frac{1}{n}\varepsilon_0,\chi_{1,n},\chi_{2,n})} \#\Sigma_{\mathrm{j}}^{\{n\}m,\ell_n}\left(\frac{1}{n}\varepsilon_0,\chi_{1,n},\chi_{2,n}\right)^{\frac{\chi_{\ell_n,n}}{\chi_{3-\ell_n,n}}}\right)}{nm\chi_{\ell_n,n}} + b\cdot e^{-\frac{m\varepsilon_0^2}{n^4\log^2\frac{r_{\max}}{r_{\min}}}} \qquad (4.6)$$

for every $n, m \in \mathbb{N}_+$ and $\varepsilon_0 > 0$.

Now, we construct a separated subset $\Sigma^{\{n\}m,*}(\varepsilon_0,\chi_{1,n},\chi_{2,n}) \subset \Sigma^{\{n\}m}(\varepsilon_0,\chi_{1,n},\chi_{2,n})$ similarly to the construction in (3.4). For every $\mathrm{j} \in \Gamma_{\ell_n}^{\{n\}m}(\varepsilon_0,\chi_{1,n},\chi_{2,n})$, let $\Sigma_{\mathrm{j}}^{\{n\}m,\ell_n,*}(\varepsilon_0,\chi_{1,n},\chi_{2,n}) \subseteq \Sigma_{\mathrm{j}}^{\{n\}m,\ell_n}(\varepsilon_0,\chi_{1,n},\chi_{2,n})$ be a maximal populous such that $\{\mathrm{p}_{3-\ell_n}S_{\mathrm{i}}((0,1)) \mid \mathrm{i} \in \Sigma_{\mathrm{j}}^{\{n\}m,\ell_n,*}(\varepsilon_0,\chi_{1,n},\chi_{2,n})\}$ are disjoint. Since $\Sigma_{\mathrm{j}}^{\{n\}m,\ell_n}(\varepsilon_0,\chi_{1,n},\chi_{2,n})$ can be embedded into $\Gamma_{3-\ell_n}^{\{n\}m}(\varepsilon_0,\chi_{1,n},\chi_{2,n})$, we get by Lemma 4.8

$$\#\Sigma_{\mathrm{j}}^{\{n\}m,\ell_n}(\varepsilon_0,\chi_{1,n},\chi_{2,n}) \leq (nC\hat{C}^{n\varepsilon_0})^m \cdot \#\Sigma_{\mathrm{j}}^{\{n\}m,\ell_n,*}(\varepsilon_0,\chi_{1,n},\chi_{2,n})$$

for every $\mathrm{j} \in \Gamma_{\ell_n}^{\{n\}m}(\varepsilon_0,\chi_{1,n},\chi_{2,n})$. Now, let $\Gamma_{\ell_n}^{\{n\}m,*}(\varepsilon_0,\chi_{1,n},\chi_{2,n}) \subset \Gamma_{\ell_n}^{\{n\}m}(\varepsilon_0,\chi_{1,n},\chi_{2,n})$ be such that $\{\mathrm{p}_{\ell_n}S_{\mathrm{i}}((0,1)) \mid \mathrm{i} \in \Gamma_{\ell_n}^{\{n\}m,*}(\varepsilon_0,\chi_{1,n},\chi_{2,n})\}$ are disjoint and for every $\mathrm{i} \in \Gamma_{\ell_n}^{\{n\}m}(\varepsilon_0,\chi_{1,n},\chi_{2,n})$ there exists $\mathrm{j} \in \Gamma_{\ell_n}^{\{n\}m,*}(\varepsilon_0,\chi_{1,n},\chi_{2,n})$ such that $\#\Sigma_{\mathrm{i}}^{\{n\}m,\ell_n,*}(\varepsilon_0,\chi_{1,n},\chi_{2,n}) \leq \#\Sigma_{\mathrm{j}}^{\{n\}m,\ell_n,*}(\varepsilon_0,\chi_{1,n},\chi_{2,n})$. Hence, again by Lemma 4.8

$$\sum_{\mathrm{j}\in\Gamma_{\ell_n}^{\{n\}m}(\varepsilon_0,\chi_{1,n},\chi_{2,n})} \#\Sigma_{\mathrm{j}}^{\{n\}m,\ell_n}(\varepsilon_0,\chi_{1,n},\chi_{2,n})^{\frac{\chi_{3-\ell_n,n}}{\chi_{\ell,n}}}$$
$$\leq (nC\hat{C}^{n\varepsilon_0})^{m\left(1+\frac{\chi_{3-\ell_n,n}}{\chi_{\ell_n,n}}\right)} \cdot \sum_{\mathrm{j}\in\Gamma_{\ell_n}^{\{n\}m,*}(\varepsilon_0,\chi_{1,n},\chi_{2,n})} \#\Sigma_{\mathrm{j}}^{\{n\}m,\ell_n,*}(\varepsilon_0,\chi_{1,n},\chi_{2,n})^{\frac{\chi_{3-\ell_n,n}}{\chi_{\ell_n,n}}}. \qquad (4.7)$$

Finally, let $\Sigma^{\{n\}m,*}(\varepsilon_0,\chi_{1,n},\chi_{2,n}) := \bigcup_{\mathrm{j}\in\Gamma_{\ell_n}^{\{n\}m,*}(\varepsilon_0,\chi_{1,n},\chi_{2,n})} \Sigma_{\mathrm{j}}^{\{n\}m,\ell_n,*}(\varepsilon_0,\chi_{1,n},\chi_{2,n})$.

Consider the attractor $\Lambda_{n,m,\varepsilon_0}$ of the IFS $\Phi_{n,m,\varepsilon_0} := \{S_{\mathrm{i}} \mid \mathrm{i} \in \Sigma^{\{n\}m,*}(\varepsilon_0,\chi_{1,n},\chi_{2,n})\}$. By the construction, Φ_{n,m,ε_0} satisfies the ROSC. If $\chi_{3-\ell_n} < \chi_{\ell_n}$ then choosing $\varepsilon_0 := \varepsilon_0(n) > 0$ such that $\chi_{3-\ell_n} + \frac{1}{n}\varepsilon_0(n) < \chi_{\ell_n} - \frac{1}{n}\varepsilon_0(n)$ makes Φ_{n,m,ε_0} a Lalley-Gatzouras carpet, and so

$$\begin{aligned}
\dim_{\mathrm{H}}\Lambda &\geq \dim_{\mathrm{H}}\Lambda_{n,m,\varepsilon_0(n)} \\
&= \max\left\{\frac{\sum_{\mathrm{i}\in\Sigma^{\{n\}m,*}(\frac{1}{n}\varepsilon_0(n),\chi_{1,n},\chi_{2,n})} p_{\mathrm{i}}\log p_{\mathrm{i}}}{\sum_{\mathrm{i}\in\Sigma^{\{n\}m,*}(\frac{1}{n}\varepsilon_0(n),\chi_{1,n},\chi_{2,n})} p_{\mathrm{i}}\log r_{\mathrm{i},3-\ell_n}} + \frac{\sum_{\mathrm{j}\in\Gamma_{\ell_n}^{\{n\}m,*}(\frac{1}{n}\varepsilon_0(n),\chi_{1,n},\chi_{2,n})} q_{\mathrm{j}}^{\ell_n}\log q_{\mathrm{j}}^{\ell_n}}{\sum_{\mathrm{i}\in\Sigma^{\{n\}m,*}(\frac{1}{n}\varepsilon_0(n),\chi_{1,n},\chi_{2,n})} p_{\mathrm{i}}\log r_{\mathrm{i},\ell_n}}\right. \\
&\qquad\left. - \frac{\sum_{\mathrm{j}\in\Gamma_{\ell_n}^{\{n\}m,*}(\frac{1}{n}\varepsilon_0(n),\chi_{1,n},\chi_{2,n})} q_{\mathrm{j}}^{\ell_n}\log q_{\mathrm{j}}^{\ell_n}}{\sum_{\mathrm{i}\in\Sigma^{\{n\}m,*}(\frac{1}{n}\varepsilon_0(n),\chi_{1,n},\chi_{2,n})} p_{\mathrm{i}}\log r_{\mathrm{i},3-\ell_n}}\right\}
\end{aligned}$$

by Lemma 4.2

$$\geq \frac{\log\left(\sum_{\mathrm{j}\in\Gamma_{\ell_n}^{\{n\}m,*}(\frac{1}{n}\varepsilon_0(n),\chi_{1,n},\chi_{2,n})} \#\Sigma_{\mathrm{j}}^{\{n\}m,\ell_n,*}\left(\frac{1}{n}\varepsilon_0(n),\chi_{1,n},\chi_{2,n}\right)^{\frac{\chi_{\ell_n,n}}{\chi_{3-\ell_n,n}}}\right)}{nm\chi_{\ell_n,n}}$$

$$- \frac{2\varepsilon_0(n)\log\left(\#\Sigma^{\{n\}m,*}(\frac{1}{n}\,\varepsilon_0(n),\chi_{1,n},\chi_{2,n})\right)}{n^3m^2\log r_{\max}(\log r_{\max}-\frac{1}{n}\varepsilon_0(n))}$$

by (4.7)

$$\geq \frac{\log\left(\sum_{\mathrm{j}\in\Gamma_{\ell_n}^{\{n\}m}\left(\frac{1}{n}\varepsilon_0(n),\chi_{1,n},\chi_{2,n}\right)} \#\Sigma_{\mathrm{j}}^{\{n\}m,\ell_n}\left(\frac{1}{n}\varepsilon_0(n),\chi_{1,n},\chi_{2,n}\right)^{\frac{\chi_{\ell_n,n}}{\chi_{3-\ell_n,n}}}\right)}{nm\chi_{3-\ell_n,n}}$$

$$- \frac{\log\left(\left(nC\hat{C}^{\varepsilon_0(n)}\right)^{m\left(1+\frac{\chi_{3-\ell_n,n}}{\chi_{\ell_n,n}}\right)}\right)}{nm\chi_{3-\ell_n,n}} - \frac{2\varepsilon_0(n)\log\#\Sigma}{n^2m\log r_{\max}(\log r_{\max}-\frac{1}{n}\varepsilon_0(n))}$$

by (4.6)

$$\geq \max_{p\in P^n} D_n(p) - \frac{\left(1+\frac{\chi_{3-\ell_n,n}}{\chi_{\ell_n,n}}\right)\log(nC\hat{C}^{\varepsilon_0(n)})}{-n\log r_{\max}} - \frac{2\varepsilon_0(n)\log\#\Sigma}{n^2m\log r_{\max}(\log r_{\max}-\frac{1}{n}\varepsilon_0(n))} - b\cdot e^{-\frac{m\varepsilon_0^2(n)}{n^4\log^2\frac{r_{\max}}{r_{\min}}}}$$

$$\geq \dim_{\mathrm{H}}\Lambda - \frac{\left(1+\frac{\chi_{3-\ell_n,n}}{\chi_{\ell_n,n}}\right)\log(nC\hat{C}^{\varepsilon_0(n)})}{-n\log r_{\max}} - \frac{2\varepsilon_0(n)\log\#\Sigma}{n^2m\log r_{\max}(\log r_{\max}-\frac{1}{n}\varepsilon_0(n))} - b\cdot e^{-\frac{m\varepsilon_0^2(n)}{n^4\log^2\frac{r_{\max}}{r_{\min}}}}.$$

If $\chi_{3-\ell,n}=\chi_{\ell,n}=:\chi_n$ then choose $-\log r_{\max}\varepsilon_0>0$ to be arbitrary, and so,

$$\begin{aligned}
\dim_{\mathrm{H}}\Lambda \geq \dim_{\mathrm{H}}\Lambda_{n,m,\varepsilon_0} &\geq \frac{\log\#\Sigma^{\{n\}m,*}\left(\frac{1}{n}\varepsilon_0,\chi_n,\chi_n\right)}{nm(\chi_n+\frac{1}{n}\varepsilon_0)}\\
&\geq \frac{\log\#\Sigma^{\{n\}m,*}\left(\frac{1}{n}\varepsilon_0,\chi_{1,n},\chi_{2,n}\right)}{nm\chi_n} - \frac{\varepsilon_0\log\#\Sigma}{n^2m\chi_n(\chi_n+\frac{1}{n}\,\varepsilon_0)}\\
&\geq \max_{p\in P^n} D_n(p) - \frac{3\varepsilon_0\log\#\Sigma}{n^2m\log r_{\max}(\log r_{\max}-\frac{1}{n}\,\varepsilon_0)} - \frac{2\log(nC\hat{C}^{\varepsilon_0})}{-n\log r_{\max}} - b\cdot e^{-\frac{m\varepsilon_0^2}{n^4\log^2\frac{r_{\max}}{r_{\min}}}}\\
&\geq \dim_{\mathrm{H}}\Lambda - \frac{3\varepsilon_0\log\#\Sigma}{n^2m\log r_{\max}(\log r_{\max}-\frac{1}{n}\,\varepsilon_0)} - \frac{2\log(nC\hat{C}^{\varepsilon_0})}{-n\log r_{\max}} - b\cdot e^{-\frac{m\varepsilon_0^2}{n^4\log^2\frac{r_{\max}}{r_{\min}}}}
\end{aligned}$$

as before. But

$$\lim_{n\to\infty}\lim_{m\to\infty}\left\{\frac{3\varepsilon_0(n)\log\#\Sigma}{n^2m\log r_{\max}(\log r_{\max}-\frac{1}{n}\,\varepsilon_0(n))} + \frac{2\log(nC\hat{C}^{\varepsilon_0(n)})}{-n\log r_{\max}} + b\cdot e^{-\frac{m\varepsilon_0^2(n)}{n^4\log^2\frac{r_{\max}}{r_{\min}}}}\right\} = 0$$

which completes the proof of Theorem 1.1. Corollary 1.2 follows by Theorem 1.1 and Lemma 4.1.

5 On the examples

Here we derive the claimed dimension values for Example 1.4 and 1.5. By Ngai and Wang [21, Theorem 2.9] and Nguyen [22], the coordinate projections satisfy the weak separation condition.

5.1 Derivation of Example 1.4

By Equation (4.62) from [3], we have the bijection

$$\mathrm{p}_2\mathbb{G}_n \longleftrightarrow \Gamma_2^{\{n\}} := \left\{\, \mathrm{i}=(i_1i_2\dots i_n)\in\{1,2,3\}^n \;\middle|\; \forall k=1,\dots,n-1: i_ki_{k+1}\neq 13\right\}.$$

For $\mathtt{i} \in \Gamma_2^{\{n\}}$, denote $(\#\Sigma_{\mathtt{i}}^{\{n\}})^{\log_4 3}$ by $R_{\mathtt{i}}$. Then

$$\sum_{\mathtt{i}\in\Gamma_2^{\{n\}}} (\#\Sigma_{\mathtt{i}}^{\{n\}})^{\log_4 3} = \sum_{\mathtt{i}\in\mathtt{i}\in\Gamma_2^{\{n\}}} R_{\mathtt{i}} = \sum_{\substack{\mathtt{i}\in\mathtt{i}\in\Gamma_2^{\{n\}}\\ i_n=1}} R_{\mathtt{i}} + \sum_{\substack{\mathtt{i}\in\mathtt{i}\in\Gamma_2^{\{n\}}\\ i_n=3}} R_{\mathtt{i}} + \sum_{k=1}^{n-1} \sum_{\substack{\mathtt{i}\in\mathtt{i}\in\Gamma_2^{\{n\}}\\ i_n=2\\ i_{n-1}=2\\ \dots\\ i_{n-k+2}=2\\ i_{n-k+1}\neq 2}} R_{\mathtt{i}}$$

$$=: a_0^{(n)} + a_1^{(n)} + \sum_{k=1}^{n-1} a_{k+1}^{(n)}$$

where the last line defined $a_i^{(n)}$ $i \in \{0,1,\dots,n\}$ in order. Let $a_i^{(n)} := 0$ for $i > n$, and denote $a^{(n)} := \big(a_0^{(n)}, a_1^{(n)}, a_2^{(n)}, \dots\big) \in \mathbb{R}^{\mathbb{N}}$. Let $\alpha = \log_4 3$. Notice that

$$\dim_{\mathrm{H}}(\Lambda) = \lim_{n\to\infty} \left\{ \frac{1}{n} \log_3 \|a^{(n)}\|_1 \right\},$$

where $\|.\|_1$ is the usual 1-norm of real sequences. The decomposition of $a^{(n)}$ may seem ad hoc, now we show what it represents: if $\mathtt{i} \in \Gamma_2^{\{n\}}$ ends with 1 or 3, then the restriction, that 13 cannot occur, means that for any $j \in \{1,2,3\}$ we have $R_{\mathtt{i}\,j} = R_{\mathtt{i}}$. On the other hand, if $\mathtt{i}$ ends with exactly ℓ 2s, then $R_{\mathtt{i}\,2} = R_{\mathtt{i}\,3} = R_{\mathtt{i}}$, but $R_{\mathtt{i}\,1} = (\ell+1)^{\alpha} \cdot R_{\mathtt{i}}$, since

$$\mathrm{p}_2 S_{\mathtt{i}\,22\dots221} = \mathrm{p}_2 S_{\mathtt{i}\,22\dots213} = \mathrm{p}_2 S_{\mathtt{i}\,22\dots133} = \dots = \mathrm{p}_2 S_{\mathtt{i}\,13\dots333}$$

while $S_{\mathtt{i}\,22\dots221}, S_{\mathtt{i}\,22\dots213}, S_{\mathtt{i}\,22\dots133}, \dots, S_{\mathtt{i}\,13\dots333}$ are $\ell+1$ different functions, if in $\mathtt{i}\,22\dots221$ the 1 in the end was preceded by exactly ℓ 2s. Therefore

$$a_0^{(n)} = \sum_{\substack{\mathtt{i}\in\Gamma_2^{\{n\}}\\ i_n=1}} R_{\mathtt{i}} = \sum_{\substack{\mathtt{i}\in\Gamma_2^{\{n\}}\\ i_n=1\\ i_{n-1}=1}} R_{\mathtt{i}} + \sum_{\substack{\mathtt{i}\in\Gamma_n\\ i_n=1\\ i_{n-1}=3}} R_{\mathtt{i}} + \sum_{k=1}^{n-1} \sum_{\substack{\mathtt{i}\in\Gamma_2^{\{n\}}\\ i_n=1\\ i_{n-1}=2\\ \dots\\ i_{n-k+1}=2\\ i_{n-k}\neq 2}} R_{\mathtt{i}}$$

$$= a_0^{(n-1)} + a_1^{(n-1)} + \sum_{k=1}^{n-1} (k+1)^{\alpha} a_{k+1}^{(n-1)}$$

$$\Big(+ 0 \text{ disguised as } \sum_{k=n}^{\infty} (k+1)^{\alpha} a_{k+1}^{(n-1)} \Big).$$

Similarly,

$$a_1^{(n)} = \sum_{\substack{\mathtt{i}\in\Gamma_2^{\{n\}}\\ i_n=3}} R_{\mathtt{i}} = \sum_{\substack{\mathtt{i}\in\Gamma_2^{\{n\}}\\ i_n=3\\ i_{n-1}=1}} R_{\mathtt{i}} + \sum_{\substack{\mathtt{i}\in\Gamma_2^{\{n\}}\\ i_n=3\\ i_{n-1}=3}} R_{\mathtt{i}} + \sum_{k=1}^{n-1} \sum_{\substack{\mathtt{i}\in\Gamma_2^{\{n\}}\\ i_n=3\\ i_{n-1}=2\\ \dots\\ i_{n-k+1}=2\\ i_{n-k}\neq 2}} R_{\mathtt{i}}$$

$$= 0 + a_1^{(n-1)} + \sum_{k=1}^{n-1} a_{k+1}^{(n-1)} \Big(+ 0 \text{ disguised as } \sum_{k=n}^{\infty} a_{k+1}^{(n-1)} \Big).$$

Next

$$a_2^{(n)} = \sum_{\substack{\mathtt{i}\in\Gamma_2^{\{n\}}\\ i_n=2\\ i_{n-1}\neq 2}} R_{\mathtt{i}} = a_0^{(n-1)} + a_1^{(n-1)},$$

while for $j \in \{3, \dots, n\}$:

$$a_j^{(n)} = \sum_{\substack{\mathfrak{i} \in \Gamma_2^{\{n\}} \\ i_n = 2 \\ i_{n-1} = 2 \\ \cdots \\ i_{n-j+2} = 2 \\ i_{n-j+1} \neq 2}} R_{\mathfrak{i}} = \sum_{\substack{\mathfrak{i} \in \Gamma_2^{\{n-1\}} \\ i_{n-1} = 2 \\ i_{n-2} = 2 \\ \cdots \\ i_{n-j+2} = 2 \\ i_{n-j+1} \neq 2}} R_{\mathfrak{i}} = a_{j-1}^{(n-1)}.$$

From these, we conclude that

$$a^{(n)} = \mathbf{L}a^{(n-1)} = \dots = \mathbf{L}^n(1, 1, 1, 0, 0, \dots) = \mathbf{L}^{n+1}(0, 1, 0, \dots)$$

where we define the operator $\mathbf{L} : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ as

$$\mathbf{L} = \begin{bmatrix} 1 & 1 & 2^\alpha & 3^\alpha & 4^\alpha & 5^\alpha & \cdots \\ 0 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Lemma 5.1. *There exists a unique $\lambda^* \in (1, \infty)$ such that there is $a \in \mathbb{R}^{\mathbb{N}}$ with positive entries, $a_0, a_1, a_2 \geq 1$ and with $\mathbf{L}a = \lambda^* a$. Furthermore*

$$\lambda^* = \frac{1}{\lambda^* - 1} \sum_{k=2}^{\infty} k^\alpha (\lambda^*)^{2-k} + \frac{(\lambda^*)^2}{(\lambda^* - 1)^3}.$$

Proof: Let $a \in \mathbb{R}^{\mathbb{N}}$ be such that

$$\sum_{k=2}^{\infty} k^\alpha a_k < \infty.$$

Suppose $\mathbf{L}a = \lambda a$, $\lambda \in (1, \infty)$, then

$$\forall k \geq 3: \quad \lambda a_k = \mathbf{L}a_k = a_{k-1} \implies a_k = \lambda^{2-k} a_2 \tag{5.1}$$

and

$$\lambda a_1 = \mathbf{L}a_1 = a_1 + \sum_{k=2}^{\infty} a_k = a_1 + \sum_{k=2}^{\infty} \lambda^{2-k} a_2 = a_1 + a_2 \frac{\lambda}{\lambda - 1}$$
$$\implies a_1 = a_2 \frac{\lambda}{(\lambda - 1)^2}.$$

Finally

$$\lambda a_0 = \mathbf{L}a_0 = a_0 + a_1 + \sum_{k=2}^{\infty} k^\alpha a_k = a_0 + a_2 \frac{\lambda}{(\lambda - 1)^2} + \sum_{k=2}^{\infty} k^\alpha \lambda^{2-k} a_2$$
$$\implies a_0 = \frac{1}{\lambda - 1} \Big(\frac{\lambda}{(\lambda - 1)^2} + \sum_{k=2}^{\infty} k^\alpha \lambda^{2-k} \Big) a_2$$

and

$$\lambda a_2 = \mathbf{L}a_2 = a_0 + a_1, \quad a_1 = a_2 \frac{\lambda}{(\lambda - 1)^2}$$

implies that

$$\lambda = \frac{1}{\lambda - 1} \Big(\frac{\lambda}{(\lambda - 1)^2} + \sum_{k=2}^{\infty} k^\alpha \lambda^{2-k} \Big) + \frac{\lambda}{(\lambda - 1)^2} = \frac{1}{\lambda - 1} \sum_{k=2}^{\infty} k^\alpha \lambda^{2-k} + \frac{\lambda^2}{(\lambda - 1)^3}. \tag{5.2}$$

On $\lambda \in (1,\infty)$ the left-hand side of (5.2) strictly increases from 1 to ∞ continuously, while the right-hand size decreases continuously from ∞ to 0, proving that (5.2) is solved by a unique λ^* on $(1,\infty)$. Hence, choosing a_2 sufficiently large, the claim on the existence of $\mathbf{L}a = \lambda^* a$ also follows. □

Lemma 5.2. $\frac{1}{n}\log\|a^{(n)}\|_1 \to \log\lambda^*$.

Proof: Let $M \in \mathbb{N}^+$, define $\mathbf{L}_M : \mathbb{R}^M \to \mathbb{R}^M$ as $(\mathbf{L}\circ \mathrm{proj}_M)|_M$, where proj_M is the projection of $\mathbb{R}^{\mathbb{N}}$ to the subspace spanned by the first M coordinates. Then $\mathbf{L}_M$ can be represented as the non-negative, irreducible aperiodic, M by M matrix:

$$\mathbf{L}_M = \begin{bmatrix} 1 & 1 & 2^\alpha & 3^\alpha & \cdots & (M-2)^\alpha & (M-1)^\alpha \\ 0 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

By the Perron-Frobenius Theorem, there exists a unique $\lambda_M > 0$ such that $\lim_{n\to\infty}\frac{1}{n}\log\|\mathbf{L}_M^n v\|_1 = \log\lambda_M$ for any $0 \neq v \in \mathbb{R}^M$ with non-negative entries, and there is a $v^* \in \mathbb{R}^M$ with positive entries such that $\mathbf{L}_M v^* = \lambda_M v^*$. Therefore, with computations similar to (5.1)–(5.2) we have:

$$\forall k \in [3, M-1]: \quad \lambda_M v_k^* = v_{k-1}^* \implies v_k^* = \lambda_M^{2-k} v_2^*$$

$$\lambda_M v_1^* = v_1^* + \sum_{k=2}^{M-1} v_k^* = v_1^* + \sum_{k=2}^{M-1} \lambda_M^{2-k} v_2^* \implies v_1^* = v_2^* \frac{1}{\lambda_M - 1}\sum_{k=2}^{M-1}\lambda_M^{2-k}.$$

Finally

$$\lambda_M v_0^* = v_0^* + v_1^* + \sum_{k=2}^{M-1} k^\alpha v_k^* \implies v_0^* = \Big(\frac{1}{(\lambda_M-1)^2}\sum_{k=2}^{M-1}\lambda_M^{2-k} + \frac{1}{\lambda_M-1}\sum_{k=2}^{M-1}k^\alpha\lambda_M^{2-k}\Big)v_2^*$$

and

$$\lambda_M v_2^* = v_0^* + v_1^*, \qquad v_1^* = v_2^*\frac{1}{\lambda_M-1}\sum_{k=2}^{M-1}\lambda_M^{2-k}$$

implies that

$$\lambda_M = \frac{1}{v_2^*}(v_0^* + v_1^*) = \frac{\lambda_M}{(\lambda_M-1)^2}\sum_{k=2}^{M-1}\lambda_M^{2-k} + \frac{1}{\lambda_M-1}\sum_{k=2}^{M-1}k^\alpha\lambda_M^{2-k}. \tag{5.3}$$

From (5.2) and (5.3) we have that $\lim_{M\to\infty}\lambda_M = \lambda^*$, and $\|\mathbf{L}^{n-1}a^{(1)}\|_1 \geq \|\mathbf{L}_M^{n-1}a^{(1)}|_M\|_1$ follows inductively on n, remembering that all entries of $a^{(1)}$ are non-negative. Whence

$$\liminf_{n\to\infty}\frac{1}{n}\log\|\mathbf{L}^{n-1}a^{(1)}\|_1 \geq \log\lambda_M \longrightarrow \log\lambda^*. \tag{5.4}$$

Finally let $a \in \mathbb{R}^{\mathbb{N}}$ be such that $\mathbf{L}a = \lambda^* a$, $a_0, a_1, a_2 \geq 1$ and the rest of the entries of a are positive. Then

$$\|\mathbf{L}^{n-1}a^{(1)}\|_1 \leq \|\mathbf{L}^{n-1}a\|_1 = (\lambda^*)^{n-1}\|a\|_1$$

hence

$$\limsup_{n\to\infty}\frac{1}{n}\log\|\mathbf{L}^{n-1}a^{(1)}\|_1 \leq \log\lambda^*$$

which, along with (5.4), proves the statement. □

Then $\dim_{\mathrm{H}}(\Lambda) = \frac{\log\lambda^*}{\log 3}$ follows by Corollary 1.2, Lemma 5.1 and Lemma 5.2.

5.2 Derivation of Example 1.5

Remark 5.3. The dimension defining structure of Λ is bipartite:

- It has exact overlaps on all levels after 1, but these are generated only by 2 equalities: $S_{14} = S_{21}$ and $S_{24} = S_{31}$.
- $\mathrm{p}_2(\Lambda)$ has additional exact overlaps, generated by the equality: $\mathrm{p}_2(S_{34}) = \mathrm{p}_2(S_{41})$.

Now for Theorem 3.1 to be applied, it is enough to compute the quantities

$$\frac{1}{n}\log\big(\#\mathrm{p}_2(\mathbb{G}_n)\big), \quad \frac{1}{n}\log\big(\#\mathbb{G}_n\big).$$

For the growth rate of $\mathbb{G}_n$ and $\mathrm{p}_2(\mathbb{G}_n)$, we notice that the cylinders form two types of objects: a disjoint cylinder square, and the 3 overlapping ones, building some kind of stair. Call these two constellations type 1 and type 2. Now we can see that a type produces after one iteration exactly one type 1 and a type 2, while a type 2 gives rise in the next level to a type 1 and 3 type 2. Hence,

$$\#\mathbb{G}_n = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & 3\end{bmatrix}^n\begin{bmatrix}1\\ 3\end{bmatrix}.$$

where the left vector is for the initial square, and the right vector finally decomposes the constellations into not entirely overlapping nth level cylinders. Therefore, by the Perron-Frobenius Theorem $\frac{1}{n}\log\big(\#\mathbb{G}_n\big)$ is $\log(\lambda)$, where λ is the largest eigenvalue, $2+\sqrt{2}$ of the matrix in the middle. For $\mathrm{p}_2(\mathbb{G}_n)$ recognise that

$$\#\mathrm{p}_2(\mathbb{G}_n) = 3\cdot\#\mathrm{p}_2(\mathbb{G}_{n-1}) + 1,\ \mathrm{p}_2(\mathbb{G}_0) = 1 \quad\Longrightarrow\quad \mathrm{p}_2(\mathbb{G}_n) = \sum_{i=0}^{n} 3^i$$

and hence $\frac{1}{n}\log\big(\#\mathrm{p}_2(\mathbb{G}_n)\big) = \log 3$ (which agrees with the observation that $\mathrm{p}_2(\Lambda) = [0,1]$, and hence $\dim_{\mathrm{B}}(\mathrm{p}_2(\Lambda)) = 1$, while $r_2 = 1/3$). We conclude that

$$\begin{aligned}\dim_{\mathrm{B}}(\Lambda) &= -\lim_{n\to\infty}\frac{1}{n}\left\{\frac{\log\big(\#\mathrm{p}_2(\mathbb{G}_n)\big)}{\log|r_2|}\Big(1-\frac{\log|r_2|}{\log|r_1|}\Big) + \frac{\log\big(\#\mathbb{G}_n\big)}{\log|r_1|}\right\}\\ &= \frac{\log 3}{\log 3}\Big(1-\frac{\log 3}{\log 4}\Big) + \frac{\log(2+\sqrt{2})}{\log 4} = 1.093295401221\ldots\end{aligned}$$

For the Hausdorff dimension we proceed similarly to the computation of Example 1.4. We partition Σ^* into types: words of type 1 end with 1, words of type 2 end in 2, words of type 4 end with a 4, and words of type 3-ℓ end with ℓ consecutive 3-s.

Now, the transition can be described as an operator $\mathbf{L}:\mathbb{R}^{\mathbb{N}}\to\mathbb{R}^{\mathbb{N}}$, where, respecting the overlaps, we avoid the combinations 14 and 31, and similar to Example 1.4 we prohibit 41 in favour of 34.

$$\mathbf{L} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots\\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots\\ 0 & 1 & 1 & 2^\alpha & 3^\alpha & 4^\alpha & 5^\alpha & 6^\alpha & \cdots\\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots\\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots\\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Then the equation, $\lambda a = \mathbf{L}a$ defines a system of equations for λ and the elements of a. Expressing λ we obtain to the equation

$$\lambda(\lambda-1) = 2 + (\lambda+1)\sum_{k=3}^{\infty}(k-1)^\alpha(\lambda)^{2-k}$$

if $\lambda > 1$. This allow us to compute λ^* numerically.

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