HAUSDORFF DIMENSION OF THE WEDDING CAKE TYPE SURFACES

BALÁZS BÁRÁNY AND MANUJ VERMA

Abstract. In this paper, we study the Hausdorff dimension of fractal interpolation surfaces (FISs) over a triangular domain. These FISs are known as 'wedding cake surfaces'. These surfaces are the attractor of some deterministic self-affine iterated function systems (IFS) on $\mathbb{R}^3$ generated by a fractal interpolation algorithm. Due to the recent seminal result of Rapaport (Adv. Math. 449 (2024) 109734), the dimension theory of self-affine IFS on $\mathbb{R}^3$ is known whenever the IFS is strongly irreducible and proximal. However, the self-affine IFSs associated with FIS are not strongly irreducible. We prove that the Hausdorff dimension of the self-affine set (or FIS) is the same as the affinity dimension outside a set of scaling parameters with zero Lebesgue measure. Lastly, by computing the overlapping number for the associated Furstenberg IFS, we determine the Hausdorff dimension for every type of scaling parameter in a certain range of parameters.

1. Introduction and Statements

1.1. **Historical background.** For $n \geq 2$, the system $\mathcal{I} = \{f_1, f_2, \dots, f_n\}$ is called an `iterated function system (IFS)` on $\mathbb{R}^d$, if the map f_i is a contraction map on $\mathbb{R}^d$ for each $i \in \{1, 2, \dots, n\}$. Hutchinson [11] proved that there exists a unique non-empty compact set $K \subset \mathbb{R}^d$ such that $K = \bigcup_{i=1}^n f_i(K)$. Let $\mathbf{p} = (p_1, p_2, \dots, p_n)$ be a probability vector. Hutchinson [11] also proved that there exists a unique Borel probability measure μ supported on K such that $\mu(B) = \sum_{i=1}^n p_i \mu(f_i^{-1}(B))$ for all Borel sets $B \subset \mathbb{R}^d$. The set K is called the `attractor` of IFS $\mathcal{I}$ and the measure μ is known as the `stationary measure` corresponding to IFS $\mathcal{I}$ with the probability vector $\mathbf{p}$. The map $f : \mathbb{R}^d \to \mathbb{R}^d$ such that $f(x) = Ax + a$ is called a self-affine map, where $a \in \mathbb{R}^d$ and $A \in \mathrm{GL}(d, \mathbb{R})$ with $\|A\| < 1$. The IFS $\mathcal{I} = \{f_1, f_2, \dots, f_n\}$ is called a self-affine IFS, if each f_i is a self-affine map on $\mathbb{R}^d$ for each $i \in \{1, 2, \dots, n\}$. The attractor of the self-affine IFS is known as `self-affine set`, and the stationary measure corresponding to the self-affine IFS $\mathcal{I}$ and probability vector $\mathbf{p}$ is known as `self-affine measure`.

In this paper, our focus is on the Hausdorff dimension of the self-affine IFS on $\mathbb{R}^3$, which are generated by the fractal interpolation algorithm on the triangular domain of $\mathbb{R}^2$. In 1986, Barnsley [4] introduced the concept of fractal interpolation functions (FIFs) on $\mathbb{R}$. The FIF is a function which interpolates the given data set, and the graph of this function is the attractor of some iterated function system (IFS). In [4], Barnsley provided

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an algorithm to construct an IFS corresponding to the given data set. Barnsley, Elton and Hardin [5] determined the box dimension of the graph of the FIF for a given data set. Later, Bárány, Simon and Rams [2] determined the Hausdorff dimension of the graph of the FIF by studying the dimension theory of the associated self-affine IFS on $\mathbb{R}^2$ for a given data set on $\mathbb{R}$.

In 1990, Massopust [12] extended Barnsley's FIF theory on the plane and defined the notion of the fractal interpolation surfaces (FISs) and also provided an algorithm to construct a self-affine IFS on $\mathbb{R}^3$ corresponding to a given finite data sets over the triangular domain, where the data points on the boundary of the triangular domain are required to be coplanar. Barnsley called these surfaces "wedding cake" surfaces. Under the consideration of uniform triangulation of the equilateral triangle and the linear part of the self-affine IFS does not contain the rotation matrix, Massopust [12] determined the box dimension of the graph of corresponding FISs. Geronimo and Hardin [8] provided another construction of the FISs over the triangular domain and polygonal domain by considering uniform scaling parameters without assuming the coplanarity condition as in [12], and also determined the box dimension of the FISs under some condition. We note that the construction of the FISs on the rectangular domain was found in [6, 15].

According to our knowledge, the Hausdorff dimension of the FISs has not yet been studied in these cases. The dimension theory of the FISs is equivalent to the dimension theory of the attractor of the corresponding self-affine IFS on $\mathbb{R}^3$. In 1988, Falconer [7] introduced a natural upper bound for the Hausdorff dimension of the self-affine sets in $\mathbb{R}^d$, which is known as the affinity dimension. Falconer [7] proved that if the self-affine IFS $\mathcal{I} = \{f_i(x) = A_i x + a_i\}_{i=1}^n$ with attractor K satisfies $\|A_i\| < \frac{1}{3}\ \forall\ i \in \{1, 2, \dots, n\}$, then for almost all $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{R}^{nd}$

$$\dim_H(K) = \min\{d, t\}, \tag{1.1}$$

where t is the affinity dimension of the self-affine IFS $\mathcal{I}$ (see precise definition later in Section 2). After that, Solomyak [16] showed that the Falconer's formula (1.1) is also valid whenever $\|A_i\| < \frac{1}{2}\ \forall\ i \in \{1, 2, \dots, n\}$ and the bound $\frac{1}{2}$ is strict. In the planar case, Bárány, Hochman and Rapaport [1] proved that if the self-affine IFS $\mathcal{I}$ is strongly irreducible and proximal, and satisfies the strong open set condition (SOSC), i.e. there exists an open and bounded set U such that

$$f_i(U) \subseteq U,\ f_i(U) \cap f_j(U) = \emptyset \text{ for } i \neq j \text{ and } U \cap K \neq \emptyset,$$

then (1.1) holds. Later, Hochman and Rapaport [10] determined the more general result in the planar case when the maps in the self-affine $\mathcal{I}$ do not have a common fixed point, $\mathcal{I}$ is strongly irreducible and proximal, and satisfies the exponential separation condition. In the case $d = 3$, Rapaport [14] proved recently that (1.1) holds if the self-affine IFS $\mathcal{I}$ is strongly irreducible and proximal, and satisfies the SOSC.

Note that the self-affine IFS on $\mathbb{R}^3$ generated by the fractal interpolation algorithm for a given data set is not strongly irreducible; it is actually reducible. Thus, [14] is not applicable for studying the Hausdorff dimension of the FISs on $\mathbb{R}^3$.

1.2. **Massopust's Fractal Interpolation Surfaces.** First, we take the construction of the FISs given by Massopust [12]. For determining the Hausdorff dimension, we consider the same assumption as taken by Massopust [12] for the box dimension. We consider the equilateral triangle Δ with vertices $\{(0,0), (1,0), (\frac{1}{2}, \frac{\sqrt{3}}{2})\}$ and an integer $N \geq 3$. Then we divide each side into N equal parts, we get a uniform triangulation $\{\Delta_i\}_{i=1}^{N^2}$ (see Figure 1

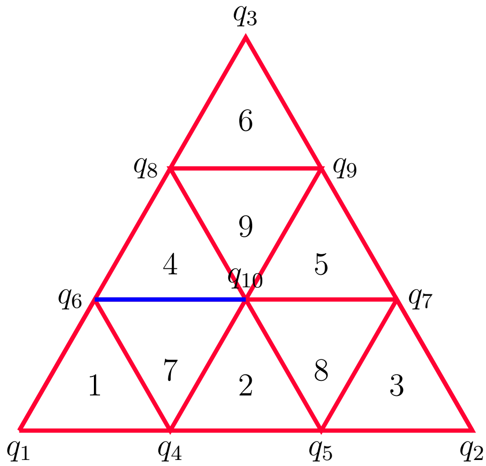


FIGURE 1. Triangularization of the equilateral triangle for $N = 3$. To visualise the sign of the change of the data set on each horizontal edge, we coloured it blue where the data set decreases from left to right, and otherwise, we coloured it red.

for $N = 3$). Without loss of generality, we index the triangles Δ_i, which are pointing up, by indices $i = 1, \ldots, \frac{N(N+1)}{2}$, and the triangles which pointing down by $i = \frac{N(N+1)}{2} + 1, \ldots, N^2$.

Let us denote the vertices of this triangularization on the plane by $\{q_i\}_{i=1}^{L(N)}$, where $L(N) = \frac{(N+1)(N+2)}{2}$ for $N \geq 3$. We will use the convention that q_1, q_2 and q_3 denote the vertices of the original equilateral triangle counted from the bottom left corner in anti-clockwise direction. We consider a data set $\{(q_k, a_k)\}_{k=1}^{L(N)}$ associated with the triangulation $\{\Delta_i\}_{i=1}^{N^2}$. We assume that the data points on the boundary of the equilateral triangle Δ are coplanar. Without loss of generality, we assume that $a_k = 0$ for all k such that the corresponding q_k is on the boundary of the triangle Δ. In particular, the data set is $\{(q_i, 0)\}_{i=1}^{9} \cup \{(q_{10}, a)\}$, for the case $N = 3$, where $a \neq 0$ is a real number. We define the map $f^*\colon \{q_i\}_{i=1}^{L(N)} \to \{a_i\}_{i=1}^{L(N)}$ as follows $f^*(q_k) = a_k$ for $k = 1, \ldots, L(N)$.

For each $i \in \{1, 2, \ldots, N^2\}$, we denote the value of f^* at the left vertex of the horizontal edge of Δ_i by a_1^i, value of f^* at the right vertex of the horizontal edge by a_2^i and value at the other vertex by a_3^i of Δ_i, where $a_1^i, a_2^i, a_3^i \in \{a_k\}_{k=1}^{L(N)}$. For each $i \in \{1, 2, \ldots, N^2\}$, we define a similarity map $U_i : \Delta \to \Delta_i$ such that

$$U_i(x, y) = \begin{cases} \left(\frac{x}{N}, \frac{y}{N}\right) + (e_i, f_i) & \text{if } a_1^i \geq a_2^i \text{ and } 1 \leq i \leq \frac{N(N+1)}{2}, \\ \left(\frac{x}{N}, -\frac{y}{N}\right) + (e_i, f_i) & \text{if } a_1^i \geq a_2^i \text{ and } \frac{N(N+1)}{2} + 1 \leq i \leq N^2, \\ \left(-\frac{x}{N}, \frac{y}{N}\right) + (g_i, f_i) & \text{if } a_1^i < a_2^i \text{ and } 1 \leq i \leq \frac{N(N+1)}{2}, \\ \left(-\frac{x}{N}, -\frac{y}{N}\right) + (g_i, f_i) & \text{if } a_1^i < a_2^i \text{ and } \frac{N(N+1)}{2} + 1 \leq i \leq N^2, \end{cases} \tag{1.2}$$

where (e_i, f_i) and (g_i, f_i) are the left and right vertices of the horizontal line of Δ_i. Furthermore, we define for each $i \in \{1, 2, \ldots, N^2\}$, the hight function as

$$V_i(x, y, z) = \mathbf{a}_i x + b_i y + s_i z + c_i, \tag{1.3}$$

where $s_i \in (0, 1)$ and constants $\{\mathbf{a}_i, b_i, c_i\}$ are uniquely determined by the join-up condition i.e.

$$V_i(q_1, 0) = f^*(U_i(q_1)), V_i(q_2, 0) = f^*(U_i(q_2)) \text{ and } V_i(q_3, 0) = f^*(U_i(q_3)) \tag{1.4}$$

for every $i \in \{1, 2, \ldots, N^2\}$.

We define the affine IFS $\mathcal{I} := \{W_i, i \in \{1, 2, \ldots, N^2\}\}$ on $\mathbb{R}^3$ such that the maps $W_i : \mathbb{R}^3 \to \mathbb{R}^3$ are defined as

$$W_i(x, y, z) = (U_i(x, y), V_i(x, y, z)). \tag{1.5}$$

By [12], the map f^* defined only on the date set can be uniquely extended to a continuous function $f^* : \Delta \to \mathbb{R}$ such that $f^*(q_i) = a_i$ for all $i \in \{1, 2 \ldots, L(N)\}$ and the graph $G(f^*)$ of the function f^* is the attractor of the affine IFS $\mathcal{I}$. In particular, it satisfies the equation:

$$f^*(U_i(x, y)) = V_i(x, y, f^*(x, y)) \text{ for every } i \in \{1, \ldots, L(N)\}.$$

The surfaces $G(f^*)$ are known as fractal interpolation surfaces. The maps of the IFS $\mathcal{I}$ in the case $N = 3$ are precisely as follows:

$$W_1(x,y,z) = \left(\frac{x}{3}, \frac{y}{3}, s_1 z\right), \quad W_4(x,y,z) = \left(\frac{-x}{3} + \frac{1}{2}, \frac{y}{3} + \frac{1}{2\sqrt{3}}, -ax - \frac{a}{\sqrt{3}}y + s_4 z + a\right)$$

$$W_2(x,y,z) = \left(\frac{x+1}{3}, \frac{y}{3}, \frac{2a}{\sqrt{3}}y + s_2 z\right), \; W_9(x,y,z) = \left(\frac{x}{3} + \frac{1}{2}, \frac{-y}{3} + \frac{1}{2\sqrt{3}}, \frac{2a}{\sqrt{3}}y + s_9 z\right)$$

$$W_6(x,y,z) = \left(\frac{x+1}{3}, \frac{y+1}{3}, s_6 z\right) \; W_8(x,y,z) = \left(\frac{x}{3} + \frac{1}{2}, \frac{-y}{3} + \frac{1}{2\sqrt{3}}, -ax - \frac{a}{\sqrt{3}}y + s_8 z + a\right),$$

$$W_3(x,y,z) = \left(\frac{x+2}{3}, \frac{y}{3}, s_3 z\right), \; W_7(x,yz) = \left(\frac{-x}{3} + \frac{1}{2}, \frac{-y}{3} + \frac{1}{2\sqrt{3}}, -ax - \frac{a}{\sqrt{3}}y + s_7 z + a\right)$$

$$W_5(x,y,z) = \left(\frac{x}{3} + \frac{1}{2}, \frac{y}{3} + \frac{1}{2\sqrt{3}}, -ax - \frac{a}{\sqrt{3}}y + s_5 z + a\right).$$

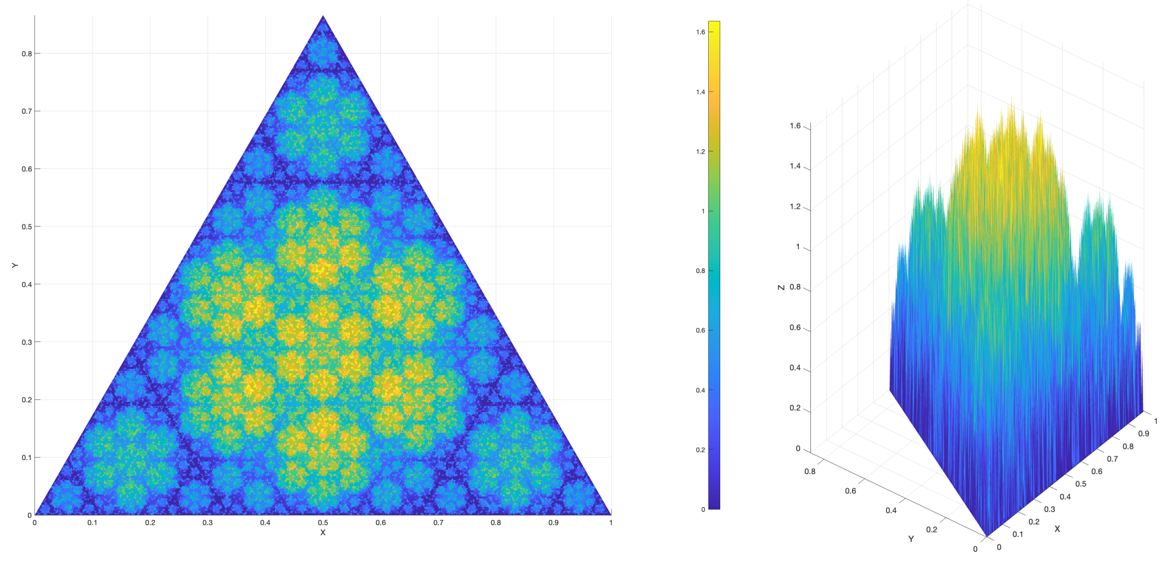


FIGURE 2. Graph of the fractal interpolation surface (top & side view) with parameters $N = 3$, $s_i = s = 0.75$ and $a = 1$.

Define

$$\mathcal{A}_1 := \{i \in \{1, 2, \ldots, N^2\} : a_1^i = a_2^i\}, \quad \mathcal{A}_2 := \{i \in \{1, 2, \ldots, N^2\} : a_1^i > a_2^i\}$$

$$\mathcal{A}_3 := \{i \in \{1, 2, \ldots, N^2\} : a_1^i < a_2^i\}.$$

Moreover, set

$$D = \min_{k,i\in\mathcal{A}_3}\{\frac{|a_1^k - a_2^k|}{|a_1^i - a_2^i|}\} \text{ and } B = \frac{1}{1+\max_{i\in\mathcal{A}_2}\{B_i\}}, \text{ where } B_i = \max_{k\in\mathcal{A}_3}\{\frac{|a_1^i - a_2^i|}{|a_1^k - a_2^k|}\}$$

for all $i \in \mathcal{A}_2$. Our result on the dimension of the graph f^* is as follows:

Theorem 1.1. *Let $\mathcal{I} := \{W_i, i \in \{1, 2, \ldots, N^2\}\}$ be a self-affine IFS on $\mathbb{R}^3$ defined as above. Let $G(f^*)$ be the attractor of the IFS $\mathcal{I}$. For each $i \in \{1, 2, \ldots, N^2\}$, we consider $a_1^i \neq a_2^i$ if a_1^i and a_2^i both are not on the boundary of original triangle Δ. Then,*

$$\dim_H(G(f^*)) = \dim_B(G(f^*)) = 1 + \frac{\log(\sum_{i=1}^{N^2} s_i)}{\log N}$$

for Lebesgue all most every scaling parameters $\underline{s} \in (\frac{1}{N}, 1)^{\#\mathcal{A}_1} \times (\frac{1}{NB}, 1)^{\#\mathcal{A}_2} \times (\frac{1}{ND}, 1)^{\#\mathcal{A}_3}$.

In the case of $N = 3$, one can see that $\#\mathcal{A}_1 = 5, \#\mathcal{A}_2 = 2, \#\mathcal{A}_3 = 2$ and $B = \frac{1}{2}$ and $D = 1$. Thus, we have the following Corollary for the typical type results for the FISs.

Corollary 1.2. For Lebesgue almost every $\underline{s} = (s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9)$ such that $s_1, s_2, s_3, s_4, s_6, s_7, s_9 \in (\frac{1}{3}, 1)$ and $s_5, s_8 \in (\frac{2}{3}, 1)$, we have

$$\dim_H(G(f^*)) = \dim_B(G(f^*)) = 1 + \frac{\log(\sum_{i=1}^{9} s_i)}{\log(3)}.$$

Our second main result gives the Hausdorff dimension of the FIS in the case of $N = 3$ for every parameter in a certain region.

Theorem 1.3. *Let $s_i \in (\frac{2}{3}, 1)$ for every $i \in \{1, 2, \ldots, 9\}$. If $\max\{s_5, s_8\} \leq \min\{s_4, s_7\}$ and $s_2 \leq s_9$, then*

$$\dim_H(G(f^*)) = \dim_B(G(f^*)) = 1 + \frac{\log(\sum_{i=1}^{9} s_i)}{\log(3)}.$$

1.3. **Geronimo-Hardin FISs.** Next, we consider the construction of the FISs given by Geronimo and Hardin [8]. In this construction, the data points on the boundary of the triangle Δ do not need to be coplanar but need to take uniform scaling parameters. Consider the equilateral triangle Δ with vertices $\{q_1 = (0, 0), q_2 = (1, 0), q_3 = (\frac{1}{2}, \frac{\sqrt{3}}{2})\}$. We take the triangulation $\{\Delta_i\}_{i=1}^4$ as shown Figure 3.

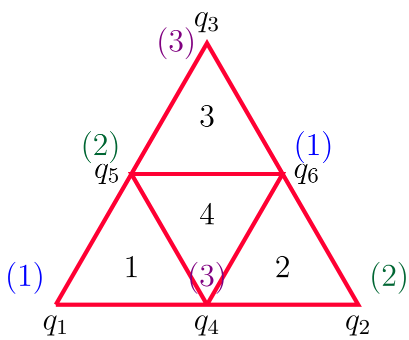


FIGURE 3. Triangularization in the Geronimo-Hardin construction.

We consider a data set as follows

$$\{(q_1, 0), (q_2, 0), (q_3, 0), (q_4, a), (q_5, a), (q_6, a)\},$$

where $a \neq 0$ is a real number, and the basic data function $f^*(q_i) = 0$ if $i = 1, 2, 3$ and $f^*(q_i) = a$ if $i = 4, 5, 6$. The chromatic number for this graph is 3. We can label the vertices with three colours (blue, green, violet). For each $i \in \{1, 2, 3, 4\}$, the similarity map $U_i : \Delta \to \Delta_i$ is defined such that the map U_i maps the vertex of Δ of a color (blue or green or violet) to the vertex of same color (blue or green or violet) of Δ_i. And, for each $i \in \{1, 2, 3, 4\}$, the hight function $V_i : \Delta \times \mathbb{R} \to \mathbb{R}$ is defined by $V_i(x, y, z) = \mathbf{a}_i x + b_i y + sz + c_i$, where $s \in (0, 1)$ and constants $\{\mathbf{a}_i, b_i, c_i\}$ are uniquely determined by the join-up condition i.e. $V_i(q_1, 0) = f^*(U_i(q_1)), V_i(q_2, 0) = f^*(U_i(q_2))$ and $V_i(q_3, 0) = f^*(U_i(q_3))\ \forall\ i \in \{1, 2, \dots, 4\}$. We define the self-affine IFS $\mathcal{I} := \{W_i, i \in \{1, 2, \dots, 4\}\}$, where the map $W_i : \mathbb{R}^3 \to \mathbb{R}^3$ is defined as

$$W_i(x, y, z) = (U_i(x, y), V_i(x, y, z)).$$

By [8], there exists a unique continuous function $f^* : \Delta \to \mathbb{R}$ such that the function f^* interpolates the data sets and the graph $(G(f^*))$ of the function f^* is the attractor of the affine IFS $\mathcal{I}$. Precisely, the maps in the IFS $\mathcal{I}$ are as follows:

$$W_1(x,y,z) = \begin{bmatrix} \frac{1}{4} & \frac{\sqrt{3}}{4} & 0 \\ \frac{\sqrt{3}}{4} & \frac{-1}{4} & 0 \\ a & \frac{a}{\sqrt{3}} & s \end{bmatrix} \begin{pmatrix} x \\ y \\ z, \end{pmatrix}, \quad W_2(x,y,z) = \begin{bmatrix} \frac{1}{4} & \frac{-\sqrt{3}}{4} & 0 \\ \frac{-\sqrt{3}}{4} & \frac{-1}{4} & 0 \\ -a & \frac{a}{\sqrt{3}} & s \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} \frac{3}{4} \\ \frac{\sqrt{3}}{4} \\ a \end{pmatrix},$$

$$W_3(x,y,z) = \begin{bmatrix} \frac{-1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & \frac{-2a}{\sqrt{3}} & s \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} \frac{3}{4} \\ \frac{\sqrt{3}}{4} \\ a \end{pmatrix}, \quad W_4(x,y,z) = \begin{bmatrix} \frac{-1}{2} & 0 & 0 \\ 0 & \frac{-1}{2} & 0 \\ 0 & 0 & s \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} \frac{3}{4} \\ \frac{\sqrt{3}}{4} \\ a \end{pmatrix}.$$

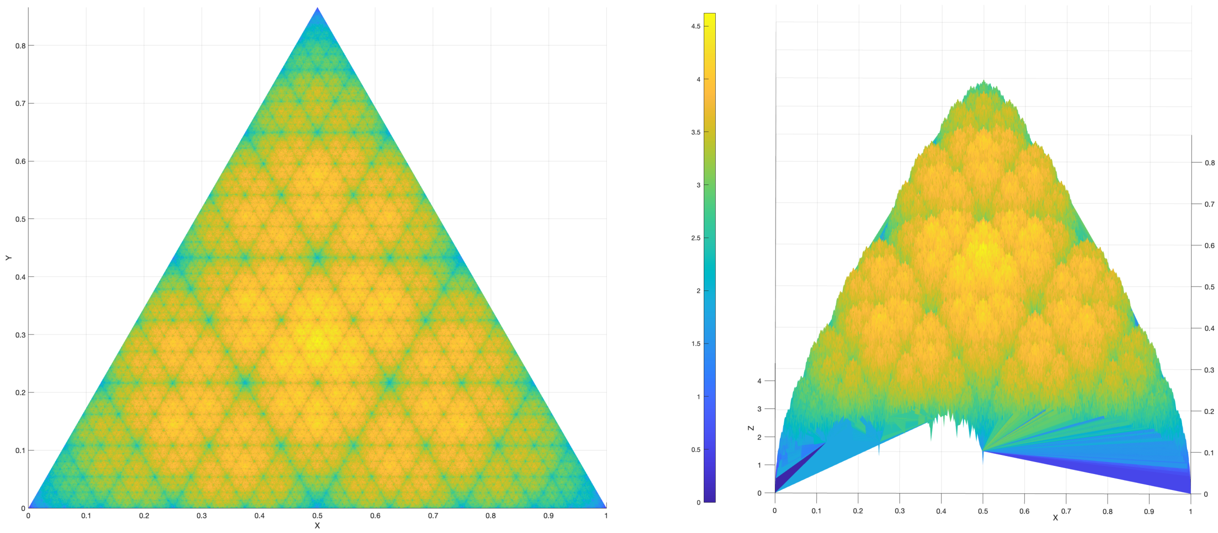


FIGURE 4. Graph of the fractal surfaces (top & aerial view) with parameters $s = 0.82$ and $a = 1$

First, by computing the overlapping number, we determine the dimension for every type of scaling parameter as follows:

Theorem 1.4. *If* $s \in \left[\frac{1+\sqrt{5}}{4}, 1\right)$, *then*

$$\dim_H(G(f^*)) = \dim_B(G(f^*)) = 3 + \frac{\log(s)}{\log(2)}.$$

In this case, we also determine the dimension for typical scaling parameters as follows:

Theorem 1.5. *There exists a set $\mathcal{E} \subset (\frac{1}{2}, 1)$ with $\dim_H(\mathcal{E}) = 0$ such that*

$$\dim_H(G(f^*)) = \dim_B(G(f^*)) = 3 + \frac{\log(s)}{\log(2)} \quad \forall\, s \in \left(\frac{1}{2}, 1\right) \setminus \mathcal{E}.$$

2. Preliminaries

First, we go through some basic definitions and tools we intend to use.

2.1. Dimension concepts.

Definition 2.1. Let $F \subseteq \mathbb{R}^d$. We say that $\{U_i\}$ is a δ-cover of F if $F \subset \bigcup\limits_{i=1}^{\infty} U_i$ and $0 < |U_i| \leq \delta$ for each i, where $|U_i|$ denotes the diameter of the set U_i. For each $\delta > 0$ and $s \geq 0$, we define

$$H^s_\delta(F) = \inf\left\{ \sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } F \right\} \quad \text{and} \quad H^s(F) = \lim_{\delta \to 0+} H^s_\delta(F).$$

We call $H^s(F)$ the s-`dimensional Hausdorff measure` of the set F. Using this, the `Hausdorff dimension` of the set F is defined by

$$\dim_H(F) = \inf\{s \geq 0 : H^s(F) = 0\} = \sup\{s \geq 0 : H^s(F) = \infty\}.$$

Definition 2.2. The `box dimension` of a non-empty bounded subset F of (X, d) is defined as

$$\dim_B F = \lim_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta},$$

where $N_\delta(F)$ denotes the smallest number of sets of diameter at most δ that can cover F, provided the limit exists. If this limit does not exist, then the upper and lower box dimensions, respectively, are defined as

$$\overline{\dim}_B F = \limsup_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta} \text{ and } \underline{\dim}_B F = \liminf_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}.$$

2.2. Symbolic space.

Let $\mathcal{I} = \{f_1, f_2, \ldots, f_n\}$ be an IFS on $\mathbb{R}^d$ such that $\|f_i(x) - f_i(y)\| \leq r_i\|x - y\|$ with $r_i \in (0, 1)$ for all $i \in \{1, 2, \ldots, n\}$. Let $\Sigma := \{1, 2, \ldots, n\}^{\mathbb{N}}$ be the set of all infinite sequences with symbols from $\{1, 2, \ldots, n\}$. The set Σ is the symbolic space corresponding to the IFS $\mathcal{I} = \{f_1, \ldots, f_n\}$. Let $\mathbf{i} = i_1 i_2 \cdots \in \Sigma$. We define $\mathbf{i}_{|m} := i_1 i_2 \ldots i_m$ for all $m \in \mathbb{N}$. We denote the set of all finite sequences of length m with symbols from $\{1, 2, \ldots, n\}$ by Σ_m. Set $\Sigma^* := \bigcup_{m=1}^{\infty} \Sigma_m$. The notation $|\mathbf{i}|$ denotes the length of the finite sequence $\mathbf{i} \in \Sigma^*$. The symbolic space Σ equipped with metric ρ is a compact metric space, where the metric ρ is defined as follows

$$\rho(\mathbf{i}, \mathbf{j}) = 2^{-|\mathbf{i} \wedge \mathbf{j}|}$$

for $\mathbf{i}, \mathbf{j} \in \Sigma$, where $\mathbf{i} \wedge \mathbf{j}$ denotes the initial largest common segment of $\mathbf{i}$ and $\mathbf{j}$.

2.3. **Affinity dimension and Furstenberg measure.** Let A be a $d\times d$ real matrix. For $t\geq 0$, the `singular value function` $\Phi^t(A)$ of A is defined by

$$\Phi^t(A)=\begin{cases}\alpha_1\dots\alpha_{\lfloor t\rfloor}{\alpha_{\lceil t\rceil}}^{t-\lfloor t\rfloor} & \text{if } 0\leq t\leq d\\ |\det(A)|^{t/d} & \text{if } t>d,\end{cases}$$

where $\alpha_1\geq\alpha_2\geq\dots\geq\alpha_d$ are the singular values of A.

The `affinity dimension` of the self-affine IFS $\mathcal{I}=\{f_i(x)=A_ix+a_i\}_{i=1}^n$ is defined by

$$t_0:=\inf\left\{t>0:\sum_{m=1}^{\infty}\sum_{i_1\dots i_m\in\Sigma_m}\Phi^t(A_{i_1}\cdots A_{i_m})<\infty\right\}.$$

If the matrices A_i have the block triangular form

$$A_i=\begin{bmatrix}\lambda_iU_i & \underline{0}\\ \underline{a}_i^T & s_i\end{bmatrix}\text{ for every } i, \tag{2.1}$$

where U_i are 2×2 orthogonal matrices, $0<\lambda_i<|s_i|<1$, $\underline{a}_i\in\mathbb{R}^2$, and $\mathcal{I}$ satisfies the SOSC then by [2, Remark 2.6], the affinity dimension t_0 satisfies the equation $t_0=\min\{r_1,r_2\}$, where

$$\sum_{i=1}^n|s_i|^{r_1}=1\text{ and }\sum_{i=1}^n|s_i|\lambda_i^{r_2-1}=1. \tag{2.2}$$

In particular, when $\sum_{i=1}^n\lambda_i^2=1$, then $t_0=r_2\in[2,3]$.

Following the lines [2, Section 2.4], we define the corresponding Furstenberg IFS induced by the IFS $\mathcal{I}$ with matrices of the form (2.1) as follows:

$$\mathcal{J}=\left\{h_i(\underline{x})=\frac{\lambda_i}{s_i}U_i^T\underline{x}-\frac{1}{s_i}\underline{a}_i\right\}_{i=1}^n. \tag{2.3}$$

The result of Rapaport [13, Section 1.2] gives a sufficient condition to calculate the dimension of the attractor of $\mathcal{I}$, see [2, Section 2.4]. We state it in the special case we require throughout this paper.

Theorem 2.3 (Rapaport)**.** *Let*

$$\mathcal{I}=\left\{f_i(\underline{x})=\begin{bmatrix}\lambda_iU_i & \underline{0}\\ \underline{a}_i^T & s_i\end{bmatrix}\underline{x}+a_i\right\}_{i=1}^n$$

be a self-affine IFS in $\mathbb{R}^3$ with attractor K such that it satisfies the SOSC, U_i are 2×2 orthogonal matrices, $\sum_{i=1}^n\lambda_i^2=1$ and $0<\lambda_i<|s_i|<1$. Let $\mu_F=\sum_{i=1}^n|s_i|\lambda_i^{t_0-1}(h_i)_*\mu_F$ be the Furstenberg measure corresponding to the IFS in (2.3). If $\dim_H\mu_F>3-t_0$ then $\dim_H K=t_0$.

Finally, we state a simple proposition to estimate the dimension of self-similar measures from below.

Proposition 2.4. Let $\mathcal{J}=\{h_i(x)=\lambda_iU_ix+t_i\}_{i=1}^N$ be a self-similar IFS on $\mathbb{R}^d$ with attractor K and let $(p_i)_{i=1}^N$ be a non-degenerate probability vector and let $\mu=\sum_{i=1}^Np_i(h_i)_*\mu$. If $\min_{x\in K}\#\{i\in\{1,\dots,N\}: x\notin h_i(K)\}\geq Q$ then

$$\dim_H\mu\geq\frac{\log(1-Qp_{\min})}{\log\lambda_{\max}},$$

where $\lambda_{\max}=\max_{i=1,\dots,N}|\lambda_i|$ and $p_{\min}=\min_{i=1,\dots,N}p_i$.

Proof. Let $r \ll 1$. Denote $B(x,r)$ the closed ball of radius r with center x. Let $C_x := \{i : x \notin h_i(K)\}$. We have

$$\begin{aligned}
\max_{x\in K} \mu(B(x,r)) &= \max_{x\in K} \sum_{i=1}^{N} p_i \mu(h_i^{-1}(B(x,r)\cap K)) \\
&\le \max_{x\in K} \sum_{i\in C_x^c} p_i \mu(B(h_i^{-1}(x), \lambda_i^{-1} r)) \\
&\le \max_{x\in K} \mu(B(x, \lambda_{\max}^{-1} r)) \max_{x\in K} \left(1 - \sum_{i\in C_x} p_i\right) \\
&\le \max_{x\in K} \mu(B(x, \lambda_{\max}^{-1} r)) \max_{x\in K} \Big(1 - Q p_{\min}\Big).
\end{aligned}$$

This implies by induction

$$\max_{x\in K} \mu(B(x, \lambda_{\max}^{n})) \le (1 - Q p_{\min})^n \text{ for every } n \in \mathbb{N},$$

and so,

$$\liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} = \liminf_{n\to\infty} \frac{\log \mu(B(x, (\lambda_{\max})^n))}{n \log \lambda_{\max}} \ge \frac{\log(1 - Q p_{\min})}{\log \lambda_{\max}}.$$

This completes the proof. □

3. Dimension theory of some self-similar IFS having a common fixed points structure and some negative contraction parameters on line

First, we provide techniques to estimate the dimension of the Furstenberg measure from below. Let us now consider a self-similar IFS as follows

$$\mathcal{G} = \{f_i(x) = \lambda_i x\}_{i=1}^{N_1} \cup \{f_i(x) = \lambda_i x + \gamma_i \lambda_i\}_{i=N_1+1}^{N_2} \cup \{f_i(x) = -\lambda_i x + \gamma_i \lambda_i\}_{i=N_2+1}^{N_3},$$

where $\lambda_i \in (0,1)$ for every $i \in \{1, 2, \ldots, N_1\}$, $\gamma_i > 0$ for every $i \in \{N_1+1, \ldots, N_3\}$. The IFS considered above has a common fixed point structure. The maps f_i with $i \le N_1$ share the same fixed point 0. The authors [3] considered recently such systems. Let us recall some corresponding definitions we need for further analysis.

We denote $I_0 := \{1, 2, \ldots, N_1\}$, $I_1 := \{N_1+1, \ldots, N_2\}$ and $I_2 := \{N_2+1, \ldots, N_3\}$. Set

$$D = \min_{k,i\in I_2} \left\{\frac{\gamma_k}{\gamma_i}\right\} \text{ and } B = \frac{1}{1 + \max_{i\in I_1}\{B_i\}}, \text{ where } B_i = \max_{k\in I_2} \left\{\frac{\gamma_i}{\gamma_k}\right\} \text{ for all } i \in I_1.$$

Let Σ be the symbolic space corresponding to IFS $\mathcal{G}$. For a symbol $i \in I_0 \cup I_1 \cup I_2$ and a finite sequence $\mathbf{i} \in \Sigma^*$, let $\#_i\mathbf{i}$ be the number of the appearances of the symbol i in the sequence $\mathbf{i}$. For $\mathbf{i} \in \Sigma \cup \Sigma^*$, we define the "first block" $b_1^{\mathbf{i}}$ of $\mathbf{i}$ as follows: if $i_1 \ge N_1 + 1$ then $b_1^{\mathbf{i}} = \mathbf{i}|_{|b_1^{\mathbf{i}}|}$ where $|b_1^{\mathbf{i}}| = \min\{k \ge 1 : i_k \ne i_1\} - 1$. Otherwise, $b_1^{\mathbf{i}} := \mathbf{i}|_{|b_1^{\mathbf{i}}|}$ where $|b_1^{\mathbf{i}}| := \min\{k \ge 1 : i_k \ge N_1 + 1\} - 1$. Then we define by induction. Suppose that $b_1^{\mathbf{i}}, \ldots, b_n^{\mathbf{i}}$ are defined and finite. Then let

$$|b_{n+1}^{\mathbf{i}}| := \begin{cases} \max\left\{k \ge 1 : i_{|b_1^{\mathbf{i}}|+\cdots+|b_n^{\mathbf{i}}|+1} = i_{|b_1^{\mathbf{i}}|+\cdots+|b_n^{\mathbf{i}}|+\ell} \ \forall\ 1 \le \ell \le k\right\} & \text{if } i_{|b_1^{\mathbf{i}}|+\cdots+|b_n^{\mathbf{i}}|+1} > N_1 \\ \max\left\{k \ge 1 : i_{|b_1^{\mathbf{i}}|+\cdots+|b_n^{\mathbf{i}}|+\ell} \le N_1 \text{ for all } 1 \le \ell \le k\right\} & \text{if } i_{|b_1^{\mathbf{i}}|+\cdots+|b_n^{\mathbf{i}}|+1} \le N_1 \end{cases}$$

If $|b^{\mathbf{i}}_{n+1}| = \left|\sigma^{|b^{\mathbf{i}}_1|+\cdots+|b^{\mathbf{i}}_n|}\mathbf{i}\right|$ then let $b^{\mathbf{i}}_{n+1} := \sigma^{|b^{\mathbf{i}}_1|+\cdots+|b^{\mathbf{i}}_n|}\mathbf{i}$, and so, $\mathbf{i} = b^{\mathbf{i}}_1 \dots b^{\mathbf{i}}_n b^{\mathbf{i}}_{n+1}$. Otherwise, let $b^{\mathbf{i}}_{n+1} := (\sigma^{|b^{\mathbf{i}}_1|+\cdots+|b^{\mathbf{i}}_n|}\mathbf{i})_{|_{|b^{\mathbf{i}}_{n+1}|}}$. For each $\mathbf{i} \in \Sigma$ have the following unique block representation

$$\mathbf{i} = \underbrace{i_1 i_2 i_3 \dots i_l}_{b^{\mathbf{i}}_1}\underbrace{i_{l+1} \dots i_m}_{b^{\mathbf{i}}_2}\underbrace{i_{m+1} \dots i_n}_{b^{\mathbf{i}}_3} i_{n+1} \dots \tag{3.1}$$

We say that for $\mathbf{i}, \mathbf{j} \in \Sigma$, the first blocks are disjoint if the sets formed by the symbols in the first blocks $b^{\mathbf{i}}_1$ and $b^{\mathbf{j}}_1$ are disjoint. We denote it by $b^{\mathbf{i}}_1 \cap b^{\mathbf{j}}_1 = \emptyset$. In other words, $\min\{\#_i b^{\mathbf{i}}_1, \#_i b^{\mathbf{j}}_1\} = 0$ for every $1 \le i \le N_3$. Let Π be the natural projection corresponding to IFS $\mathcal{G}$.

Definition 3.1. We say that IFS $\mathcal{G}$ satisfies the `Exponential Separation Condition for the Common Fixed Point System (ESC for CFS)`, if there exist $N \in \mathbb{N}$ and $b > 1$ such that for every $n \ge N$ and every $\mathbf{i}, \mathbf{j} \in \Sigma_n$ with $\lambda_{\mathbf{i}} = \lambda_{\mathbf{j}}$, we have the following:

$$\text{either } \mathbf{i}, \mathbf{j} \text{ have the same block structure or } |\Pi(\mathbf{i}) - \Pi(\mathbf{j})| > 2^{-bn}. \tag{3.2}$$

This section aims to show that the IFS $\mathcal{G}$ satisfies the ESC for CFS for typical contraction ratio parameters. The proof follows the lines of [3, Section 4] with only minor changes. We will only give the essential steps and highlight the differences, but we leave the details for the reader.

Proposition 3.2. There exists a set $\mathcal{E} \subset (0,1)^{N_1} \times (0,B)^{N_2-N_1} \times (0,D)^{N_3-N_2}$ such that $\dim_H \mathcal{E} \le N_3 - 1$ such that the IFS $\mathcal{G}$ satisfies ESC for CFS for every parameters $\underline{\lambda} \in (0,1)^{N_1} \times (0,B)^{N_2-N_1} \times (0,D)^{N_3-N_2} \setminus \mathcal{E}$.

We begin the discussion with the following, which makes the structure of the IFS slightly less complicated. Still, studying the dimension theory of wedding cake-type surfaces, particularly to estimate the dimension of the corresponding Furstenberg measure, is sufficient.

Lemma 3.3. *Let $\mathcal{G}$ be the self-similar defined as above. If the parameters $\lambda_i \in (0,1)$ $\forall$ $i \in I_0$, $\lambda_i \in (0,B)$ $\forall$ $i \in I_1$ and $\lambda_i \in (0,D)$ $\forall$ $i \in I_2$, then there exists an $A > 0$ and an $\tilde{\epsilon} > 0$ such that*

$$f_i[0,A] \subset (0,A] \ \forall \ i \in I_1 \cup I_2.$$

Proof. First, we show that there exists a constant $A > 0$ such that $f_i[0,A] \subset (0,A]$ for every $i \in I_1 \cup I_2$.

Let us denote the fixed point of the map f_i by $Fix(f_i)$. One can see that $Fix(f_i) = 0$ for every $i \in I_0$, $Fix(f_i) = \frac{\gamma_i\lambda_i}{1-\lambda_i}$ for every $i \in I_1$ and $Fix(f_i) = \frac{\gamma_i\lambda_i}{1+\lambda_i}$ for every $i \in I_2$. Since $f_i(0) = \gamma_i\lambda_i$ for every $i \in I_1 \cup I_2$, we have

$$A = \max\left\{\max_{i\in I_2} \gamma_i\lambda_i, \max_{i\in I_1} \frac{\gamma_i\lambda_i}{1-\lambda_i}\right\}.$$

First, we consider $A = \gamma_{i_0}\lambda_{i_0}$ for some $i_0 \in I_2$. Then $A = \gamma_{i_0}\lambda_{i_0} \ge \frac{\gamma_k\lambda_k}{1-\lambda_k}$ for every $k \in I_1$. This implies that $f_k[0,A] = [\gamma_k\lambda_k, \lambda_k\gamma_{i_0}\lambda_{i_0} + \gamma_k\lambda_k] \subseteq (0,A]$ for all $k \in I_1$. For $k \ne i_0 \in I_2$, we have

$$f_k(\gamma_{i_0}\lambda_{i_0}) > 0 \Leftarrow -\lambda_k\gamma_{i_0}\lambda_{i_0} + \gamma_k\lambda_k > 0 \Leftarrow \lambda_k(-\gamma_{i_0}\lambda_{i_0} + \gamma_k) > 0 \Leftarrow \lambda_{i_0} < \frac{\gamma_k}{\gamma_{i_0}}.$$

This implies that for the parameters $\lambda_i \in (0,1)$ for all $i \in I_0 \cup I_1$ and $\lambda_i \in (0,D)$ for every $i \in I_2$, then $f_i[0,A] \subset (0,A]$ for every $i \in I_1 \cup I_2$.

On the other hand, if $A = \frac{\gamma_{i_0}\lambda_{i_0}}{1-\lambda_{i_0}}$ for some $i_0 \in I_1$. Then, we have $\frac{\gamma_{i_0}\lambda_{i_0}}{1-\lambda_{i_0}} \geq \frac{\gamma_k\lambda_k}{1-\lambda_k}$ for all $k \in I_1$ and $\gamma_k\lambda_k \leq \frac{\gamma_{i_0}\lambda_{i_0}}{1-\lambda_{i_0}}$ for every $k \in I_2$. This implies that $f_k[0,A] \subseteq (0,A]$ for all $k \in I_1$. For $k \in I_2$, we have

$$f_k(A) > 0 \Leftarrow -\lambda_k \frac{\gamma_{i_0}\lambda_{i_0}}{1-\lambda_{i_0}} + \gamma_k\lambda_k > 0 \Leftarrow \lambda_{i_0} < \frac{1}{1+\frac{\gamma_{i_0}}{\gamma_k}}.$$

Thus, for the parameters $\lambda_i \in (0,1)$ for every $i \in I_0 \cup I_2$ and $\lambda_i \in (0,B)$ for all $i \in I_1$ then $f_i[0,A] \subset (0,A]$ for all $i \in I_1 \cup I_2$. $\square$

For every $0 < \epsilon \leq \min\{\min_{i\in I_0}\{\lambda_i, 1-\lambda_i\}, \min_{i\in I_1}\{\lambda_i, B-\lambda_i\}, \min_{i\in I_2}\{\lambda_i, D-\lambda_i\}, \tilde{\epsilon}\}$, then by Lemma 3.3,

$$f_i[0,A] \subset [\epsilon, A] \ \forall\ i \in I_1 \cup I_2.$$

For $\mathbf{i},\mathbf{j} \in \Sigma$, we define $\Delta_{\mathbf{i},\mathbf{j}}(\underline{\lambda}) = \Pi(\mathbf{i}) - \Pi(\mathbf{j})$ for every vector $\underline{\lambda} \in (0,1)^{N_1} \times (0,B)^{N_2-N_1} \times (0,D)^{N_3-N_2}$. Let us define the following set of pairs:

$$\mathcal{L} := \left\{(\mathbf{i},\mathbf{j}) \in \Sigma \times \Sigma : b_1^{\mathbf{i}} \cap b_1^{\mathbf{j}} = \emptyset \ \&\ b_1^{\mathbf{i}} \neq \mathbf{i} \ \&\ b_1^{\mathbf{j}} \neq \mathbf{j}\right\}. \tag{3.3}$$

We divide the set $\mathcal{L}$ further:

$$\mathcal{L}_1 := \{(\mathbf{i},\mathbf{j}) \in \mathcal{L} : i_1 \neq j_1, i_1 \in I_0 \cup I_1 \cup I_2, j_1 \in I_1 \cup I_2\}, \mathcal{L}_2 := \mathcal{L} \setminus \mathcal{L}_1 = \{(\mathbf{i},\mathbf{j}) \in \mathcal{L} : i_1 \neq j_1 \in I_0\}. \tag{3.4}$$

Now, set $\tilde{N}_0 := \lceil \frac{(1-\epsilon)(2+\epsilon)}{\epsilon^3} \rceil + 1$, and divide $\mathcal{L}_2$ further:

$$\mathcal{L}_3 = \left\{(\mathbf{i},\mathbf{j}) \in \mathcal{L}_2 : \max_{k\in I_0}\left\{\max\left\{\#_k b_1^{\mathbf{i}}, \#_k b_1^{\mathbf{j}}\right\}\right\} \leq \tilde{N}_0\right\} \text{ and } \mathcal{L}_4 = \mathcal{L}_2 \setminus \mathcal{L}_3. \tag{3.5}$$

One can see that $\mathcal{L}_1$ and $\mathcal{L}_3$ are compact subsets of $\Sigma \times \Sigma$.

Lemma 3.4. *Let $\epsilon > 0$ be arbitrary as defined above. Then there exists a constant $C > 0$ such that for every $\underline{\lambda} \in [\epsilon, 1-\epsilon]^{N_1} \times [\epsilon, B-\epsilon]^{N_2-N_1} \times [\epsilon, D-\epsilon]^{N_3-N_2}$ and for every $(\mathbf{i},\mathbf{j}) \in \mathcal{L}_4$*

$$\min\left\{|\Delta_{\mathbf{i},\mathbf{j}}(\underline{\lambda})|, \left|\frac{\partial \Delta_{\mathbf{i},\mathbf{j}}}{\partial \lambda_k}(\underline{\lambda})\right|\right\} \geq C\epsilon^{2\max\{|b_1^{\mathbf{i}}|,|b_1^{\mathbf{j}}|\}},$$

where k is such that $\max\{\#_k b_1^{\mathbf{i}}, \#_k b_1^{\mathbf{j}}\} > \tilde{N}_0$.

Proof. By Lemma 3.3, the self-similar IFS $\mathcal{G}$ satisfies all the assumptions of the self-similar IFS $\mathcal{F}$ defined in [3] with $c_i = \gamma_i\lambda_i$ for all $i \in \{N_1+1,\ldots,N_3\}$. Thus by [3, Lemma 3.2], we get the claim of our result. Let $(\mathbf{i},\mathbf{j}) \in \mathcal{L}_4$. This implies that $b_1^{\mathbf{i}} \cap b_1^{\mathbf{j}} = \emptyset$ and $i_1 \neq j_1 \in I_0$. First, we assume that $\frac{\lambda_{b_1^{\mathbf{i}}}}{\lambda_{b_1^{\mathbf{j}}}} \notin \left(\frac{\epsilon}{2}, \frac{2}{\epsilon}\right)$. Then, by [3, Lemma 4.1], we get the

$$|\Delta_{\mathbf{i},\mathbf{j}}(\underline{\lambda})| \geq \epsilon^{2\max\{|b_1^{\mathbf{i}}|,|b_1^{\mathbf{j}}|\}}.$$

Lastly, we suppose that $\frac{\lambda_{b_1^{\mathbf{i}}}}{\lambda_{b_1^{\mathbf{j}}}} \in \left(\frac{\epsilon}{2}, \frac{2}{\epsilon}\right)$. Then by [3, Lemma 4.2], we get that

$$\left|\frac{\partial \Delta_{\mathbf{i},\mathbf{j}}}{\partial \lambda_k}(\underline{\lambda})\right| \geq C\epsilon^{\max\{|b_1^{\mathbf{i}}|,|b_1^{\mathbf{j}}|\}}$$

for some uniform constant $C > 0$. $\square$

Lemma 3.5. *Let $\epsilon > 0$ be arbitrary as defined above. Then there exist $p \geq 0$ and $\tilde{C} > 0$ such that for every $(\mathbf{i}, \mathbf{j}) \in \mathcal{L}_3$ and for all $\underline{\lambda} \in [\epsilon, 1-\epsilon]^{N_1} \times [\epsilon, B-\epsilon]^{N_2-N_1} \times [\epsilon, D-\epsilon]^{N_3-N_2}$, there exists $(m_{i,j})_{(i,j)\in I} \in \mathbb{N}^{N_3}$ such that $m = \sum_{(i,j)\in I} m_{i,j} \leq p$ and*

$$\left| \frac{\partial^m \Delta_{\mathbf{i},\mathbf{j}}}{\prod_{(i,j)\in I} \partial^{m_{i,j}} \lambda_{i,j}}(\underline{\lambda}) \right| > \tilde{C}.$$

Proof. The lemma can be proven along the same lines as the proof of [3, Lemma 4.5]. We omit the details. □

Lemma 3.6. *Let $\epsilon > 0$ be arbitrary as defined above. Then there exist $p \geq 0$ and $\tilde{C} > 0$ such that for every $(\mathbf{i}, \mathbf{j}) \in \mathcal{L}_1$ and for all $\underline{\lambda} \in [\epsilon, 1-\epsilon]^{N_1} \times [\epsilon, B-\epsilon]^{N_2-N_1} \times [\epsilon, D-\epsilon]^{N_3-N_2}$, there exists $(m_{i,j})_{(i,j)\in I} \in \mathbb{N}^{N_3}$ such that $m = \sum_{(i,j)\in I} m_{i,j} \leq p$ and*

$$\left| \frac{\partial^m \Delta_{\mathbf{i},\mathbf{j}}}{\prod_{(i,j)\in I} \partial^{m_{i,j}} \lambda_{i,j}}(\underline{\lambda}) \right| > \tilde{C}.$$

Proof. The lemma can be proven along the same lines as the proof of [3, Lemma 4.6]. We leave the details in this case again for the reader. □

Proof of Proposition 3.2. First, we define a set as follows

$$\mathcal{L}^n := \left\{ (\mathbf{i}, \mathbf{j}) \in \Sigma_n \times \Sigma_n : b_1^{\mathbf{i}} \cap b_1^{\mathbf{j}} = \emptyset \ \&\ b_1^{\mathbf{i}} \neq \mathbf{i} \ \&\ b_1^{\mathbf{j}} \neq \mathbf{j} \right\}.$$

Then for every $\epsilon > 0$, let

$$E_\epsilon = \bigcap_{\eta>0} \bigcap_{\tilde{N}\geq 1} \bigcup_{n\geq \tilde{N}} \bigcup_{(\mathbf{i},\mathbf{j})\in\mathcal{L}^{(n)}} \left\{ \underline{\lambda} \in [\epsilon, 1-\epsilon]^{N_1} \times [\epsilon, B-\epsilon]^{N_2-N_1} \times [\epsilon, D-\epsilon]^{N_3-N_2} : |\Delta_{\mathbf{i},\mathbf{j}}(\underline{\lambda})| < \eta^n \right\}. \tag{3.6}$$

Using Lemma 3.4, Lemma 3.5 and Lemma 3.6, and applying the same technique as in [3, Proposition 4.7], we get that $\dim_H(E_\epsilon) \leq N_3 - 1$, and for all $\underline{\lambda} \in [\epsilon, 1-\epsilon]^{N_1} \times [\epsilon, B-\epsilon]^{N_2-N_1} \times [\epsilon, D-\epsilon]^{N_3-N_2} \setminus E_\epsilon$, $\exists\ \eta > 0, \exists\ \tilde{N} \in \mathbb{N}, \forall\ n \geq \tilde{N}, \forall\ (\mathbf{i}, \mathbf{j}) \in (\Sigma_n \times \Sigma_n) \cap \mathcal{L}^n$ such that

$$|\Delta_{\mathbf{i},\mathbf{j}}(\underline{\lambda})| > \eta^n.$$

We define another set as follows:

$$G_\epsilon = \bigcup_{n=1}^{\infty} \bigcup_{\substack{(\mathbf{i},\mathbf{j})\in\Sigma_n\times\Sigma_n \\ \mathbf{i}\cap\mathbf{j}=\emptyset}} \{ \underline{\lambda} \in [\epsilon, 1-\epsilon]^{N_1} \times [\epsilon, B-\epsilon]^{N_2-N_1} \times [\epsilon, D-\epsilon]^{N_3-N_2} : \lambda_{\mathbf{i}} = \lambda_{\mathbf{j}} \}$$

One can show along the lines of [3, Lemma 4.8] that $\dim_H(G_\epsilon) \leq N_3 - 1$. We define the exceptional set $\mathcal{E}$ by $\mathcal{E} := E \cup G$, where $E = \cup_{n\geq 1} E_{1/n}$ and $G = \cup_{n\geq 1} G_{1/n}$. Then one can finish the proof by applying the techniques in [3, Proposition 4.9]. □

4. Hausdorff dimension of Massopust's surfaces

This section is devoted to proving Theorem 1.1 and Theorem 1.3. Let us recall some definitions from Section 1.2. Consider the equilateral triangle Δ with vertices $\{(0,0), (1,0), (\frac{1}{2}, \frac{\sqrt{3}}{2})\}$, and let $\{\Delta_i\}_{i=1}^{N^2}$ be the uniform triangulation for $N \geq 3$. Consider a data set $\{(q_k, a_k)\}_{k=1}^{L(N)}$ associated with the triangulation $\{\Delta_i\}_{i=1}^{N^2}$, where $L(N) = \frac{(N+1)(N+2)}{2}$. We assume that $a_k = 0$ for all k such that the corresponding q_k is on the boundary of the triangle Δ. For each $i \in \{1, 2, \ldots, N^2\}$, we denote the value at the left vertex of the horizontal line of Δ_i

by a_1^i, value at the right vertex of the horizontal line of Δ_i by a_2^i and value at the other vertex by a_3^i. For each $i \in \{1, 2, \dots, N^2\}$, define the similarity map $U_i : \Delta \to \Delta_i$ as in (1.2), the map

$$V_i(x, y, z) = \mathbf{a}_i x + b_i y + s_i z + c_i$$

as in (1.3) such that it satisfies the boundary condition (1.4). Clearly, by using the above conditions, we get

$$\mathbf{a}_i = -|a_1^i - a_2^i| \text{ and } b_i = \frac{2}{\sqrt{3}}\left(\frac{-(a_1^i + a_2^i)}{2} + a_3^i\right)$$

Now, we define an affine IFS $\mathcal{I} := \{W_i, i \in \{1, 2, \dots, N^2\}\}$ on $\mathbb{R}^3$, where the map $W_i : \mathbb{R}^3 \to \mathbb{R}^3$ is defined as

$$W_i(x, y, z) = (U_i(x, y), V_i(x, y, z)).$$

Denote $f^* : \Delta \to \mathbb{R}$ the unique fractal interpolation function of which graph $G(f^*)$ is the attractor of the affine IFS $\mathcal{I}$.

Note 4.1. One can also see that the subspace generated by the vector $(0, 0, 1)$ is invariant under the linear parts of the IFS $\mathcal{I}$. Thus, the IFS $\mathcal{I}$ is not strongly irreducible. So, the dimension theory of self-affine IFS presented in [14] is not applicable here.

We assume throughout the paper that $s_i \in (\frac{1}{N}, 1)$. Thus, by (2.2) the affinity dimension t_0 corresponding to the self-affine IFS $\mathcal{I}$ is the unique solution of the following equation

$$\sum_{i=1}^{N^2} s_i \left(\frac{1}{N}\right)^{t_0 - 1} = 1.$$

By [7], the affinity dimension t is a natural upper bound for the box dimension of the self-affine set, and we have

$$\dim_H(G(f^*)) \leq \overline{\dim}_B(G(f^*)) \leq t_0 = 1 + \frac{\log(\sum_{i=1}^{N^2} s_i)}{\log N} \tag{4.1}$$

for all parameters.

4.1. **The Furstenberg measure and a sufficient condition.** Let $\tilde{\Delta}$ be the interior of the original equilateral triangle Δ. Then, one can see that

$$W_i(\tilde{\Delta} \times \mathbb{R}) \subset \tilde{\Delta} \times \mathbb{R},\ W_i(\tilde{\Delta} \times \mathbb{R}) \cap W_j(\tilde{\Delta} \times \mathbb{R}) = \emptyset$$

for all $i \neq j \in \{1, 2, \dots, N^2\}$. This implies the IFS $\mathcal{I}$ satisfies the SOSC.

Let $\mathbf{p} = (p_1, p_2, \dots, p_{N^2})$ be a probability vector, where $p_i = s_i\left(\frac{1}{N}\right)^{t_0 - 1}$ for every $i \in \{1, 2, \dots, N^2\}$.

Define

$$\mathcal{A}_1 := \{i \in \{1, 2, \dots, N^2\} : a_1^i = a_2^i\}, \quad \mathcal{A}_2 := \{i \in \{1, 2, \dots, N^2\} : a_1^i > a_2^i\}$$

$$\mathcal{A}_3 := \{i \in \{1, 2, \dots, N^2\} : a_1^i < a_2^i\}.$$

Let $\mathbf{p} = (p_1, p_2, \dots, p_{N^2})$ be a probability vector, where $p_i = s_i\left(\frac{1}{N}\right)^{t_0 - 1}$ for every $i \in \{1, 2, \dots, N^2\}$. By applying (2.3), we construct an Furstenberg IFS $\mathcal{J} = \{h_1, h_2, \dots, h_{N^2}\}$

on $\mathbb{R}^2$ by the self-affine IFS $\mathcal{I}$ as follows

$$h_i(x,y) = \begin{cases} \dfrac{1}{Ns_i}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \dfrac{1}{s_i}\begin{bmatrix} \mathbf{a}_i \\ b_i \end{bmatrix} \text{if } a_1^i \geq a_2^i \text{ and } 1 \leq i \leq \dfrac{N(N+1)}{2}, \\ \dfrac{1}{Ns_i}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \dfrac{1}{s_i}\begin{bmatrix} \mathbf{a}_i \\ b_i \end{bmatrix} \text{if } a_1^i \geq a_2^i \text{ and } \dfrac{N(N+1)}{2} + 1 \leq i \leq N^2, \\ \dfrac{1}{Ns_i}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \dfrac{1}{s_i}\begin{bmatrix} \mathbf{a}_i \\ b_i \end{bmatrix} \text{if } a_1^i < a_2^i \text{ and } 1 \leq i \leq \dfrac{N(N+1)}{2}, \\ \dfrac{1}{Ns_i}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \dfrac{1}{s_i}\begin{bmatrix} \mathbf{a}_i \\ b_i \end{bmatrix} \text{if } a_1^i < a_2^i \text{ and } \dfrac{N(N+1)}{2} + 1 \leq i \leq N^2, \end{cases}$$

The Furstenberg measure $\mu_F = \sum_{i=1}^{N^2} p_i (h_i)_* \mu_F$ is the unique invariant Borel probability measure corresponding to the IFS $\mathcal{J}$ with probability vector $\mathbf{p}$. Since the self-affine IFS $\mathcal{I}$ satisfies the SOSC and $s_i \in (\frac{1}{N}, 1)$ $\forall\, i \in \{1, 2, \dots, N^2\}$, to verify the equality in (4.1), we only need to show that

$$\dim_H(\mu_F) > 3 - t_0$$

by Theorem 2.3. In particular, to show Theorem 1.1 and Theorem 1.3, we will prove the following:

Proposition 4.2. For each $i \in \{1, 2, \dots, N^2\}$, let $a_1^i \neq a_2^i$ if a_1^i and a_2^i both are not on the boundary of original triangle Δ. Then,

$$\dim_H(\mu_F) > 3 - t_0$$

for Lebesgue all most every scaling parameters $\underline{s} \in (\frac{1}{N}, 1)^{\#\mathcal{A}_1} \times (\frac{1}{NB}, 1)^{\#\mathcal{A}_2} \times (\frac{1}{ND}, 1)^{\#\mathcal{A}_3}$.

Proposition 4.3. Let $s_i \in \left(\frac{2}{3}, 1\right)$ $\forall\, i \in \{1, 2, \dots, 9\}$. Suppose that $\max\{s_5, s_8\} \leq \min\{s_4, s_7\}$ and $s_2 \leq s_9$. Then,

$$\dim_H(\mu_F) > 3 - t_0,$$

where t_0 is the affinity dimension of the IFS $\mathcal{I}$ in (4.1).

Proof of Theorem 1.1. The claim follows by (4.1), and the combination of Theorem 2.3 and Proposition 4.2. ☐

Proof of Theorem 1.3. The claim follows by (4.1), and the combination of Theorem 2.3 and Proposition 4.3. ☐

Remark 4.4. We note that the method is not applicable in every configuration. In the above construction, if we consider $s_i = s$ for every $i \in \{1, 2, \dots, N^2\}$ and $a_k = a$ for all $k \in \{1, 2, \dots, L(N)\}$, then for the large value of N,

$$\dim_H(\mu_F) \not> 3 - t_0 = \frac{-\log s}{\log N} \;\forall\, s \in \left(\frac{1}{N}, 1\right). \tag{4.2}$$

In this consideration, there are 9 different mapping in Furstenberg IFS $\mathcal{J}$ with multiplicity $\frac{(N-3)(N-2)}{2}+3$, $\frac{(N-4)(N-3)}{2}$, $(N-2), (N-2), (N-2), (N-2), (N-3), (N-2), 1$, respectively.
Examples: For $N = 100, \dim_H(\mu_F) < \frac{-\log s}{\log N}$ for $s \in (0.042, 0.237)$.
For $N = 1000, \dim_H(\mu_F) < \frac{-\log s}{\log N}$ for $s \in (0.001, 0.430)$.
For $N = 10000, \dim_H(\mu_F) < \frac{-\log s}{\log N}$ for $s \in (0.0001, 0.461)$.

Although, if we consider some $s_i \neq s_j$ for some $i, j \in \{1, 2, \ldots, N^2\}$ and $a_k = a$ for all $k \in \{1, 2, \ldots, L(N)\}$, then for the large value of N, the same situation as in (4.2) occurs. For example, we consider $\frac{(N-3)(N-2)}{2} + 3$ many mappings with $s_i = s + 0.001$, $\frac{(N-4)(N-3)}{2}$ many mappings with $s_i = s + 0.002$, $5(N-2)$ many mappings with $s_i = s + 0.003$, $(N-3)$ many mappings with $s_i = s + 0.004$ and 1 map with $s_i = s + 0.002$. In this consideration, we have the followings:

For $N = 100, \dim_H(\mu_F) < 2 - \frac{\log \sum_{i=1}^{N^2} s_i}{\log N}$ for $s \in (0.035, 0.279)$.

For $N = 100000, \dim_H(\mu_F) < 2 - \frac{\log \sum_{i=1}^{N^2} s_i}{\log N}$ for $s \in (0.00001, 0.463)$.

Remark 4.5. Let us also note that our method might be applied to other data sets when some of the data values over the horizontal edges of the triangles coincide. Still, there are enough maps where there are no coincidences, and in particular, there are enough maps that do not share the same fixed point, which ensures that the lower bound for the dimension of the Furstenberg measure might hold.

4.2. **Dimension for almost every parameter.** Let f_i be the projection of h_i on the X-axis. Let $\mathcal{J}_X = \{f_1, f_2, \ldots, f_{N^2}\}$ be the projection of the IFS $\mathcal{J}$ on the X-axis. Then, for $i \in \{1, 2, \ldots, N^2\}$ the map f_i is as follows

$$f_i(x) = \begin{cases} \frac{x}{Ns_i} & \text{if } a_1^i = a_2^i \\ \frac{x}{Ns_i} - \frac{\mathbf{a}_i}{s_i} & \text{if } a_1^i \geq a_2^i \\ \frac{-x}{Ns_i} - \frac{\mathbf{a}_i}{s_i} & \text{if } a_1^i < a_2^i. \end{cases}$$

Set $\lambda_i = \frac{1}{Ns_i}$ and $\gamma_i = -\mathbf{a}_i N = |a_1^i - a_2^i| N > 0$, one can see that the IFS $\{f_i\}_{i=1}^{N^2}$ is of type considered in Section 3. Since $s_i \in (\frac{1}{N}, 1)$, we have $\lambda_i \in (\frac{1}{N}, 1)$ for all $i \in \{1, 2, \ldots, N^2\}$. Thus, we have

$$\mathcal{J}_X = \{f_i(x) = \lambda_i x\}_{i \in \mathcal{A}_1} \cup \{f_i(x) = \lambda_i x + \gamma_i \lambda_i\}_{i \in \mathcal{A}_2} \cup \{f_i(x) = -\lambda_i x + \gamma_i \lambda_i\}_{i \in \mathcal{A}_3}$$

Proof of Proposition 4.2. Let $P_{X*}\mu_F$ be the projection of the measure μ_F on the X-axis. The measure $P_{X*}\mu_F$ is the invariant measure corresponding to the IFS $\mathcal{J}_X$ with probability vector $\mathbf{p}$. By Proposition 3.2, there exists a set $\mathcal{E} \subset (0,1)^{\#\mathcal{A}_1} \times (0, B)^{\#\mathcal{A}_2} \times (0, D)^{\#\mathcal{A}_3}$ such that $\dim_H \mathcal{E} \leq \#\mathcal{A}_1 + \#\mathcal{A}_2 + \#\mathcal{A}_3 - 1$ such that the IFS $\mathcal{J}_X$ satisfies ESC for CFS for every parameters $\underline{\lambda} \in (0,1)^{\#\mathcal{A}_1} \times (0, B)^{\#\mathcal{A}_2} \times (0, D)^{\#\mathcal{A}_3} \setminus \mathcal{E}$. Thus, by [3, Theorem 5.1,Theorem 3.5], we get

$$\dim_H(P_{X*}\mu_F) = \min\left\{1, \frac{-\sum_{i=1}^{N^2} p_i \log p_i + \Phi(\mathbf{p})}{-\sum_{i=1}^{N^2} p_i \log \lambda_i}\right\}$$

for every parameters $\underline{s} \in (\frac{1}{N}, 1)^{\#\mathcal{A}_1} \times (\frac{1}{NB}, 1)^{\#\mathcal{A}_2} \times (\frac{1}{ND}, 1)^{\#\mathcal{A}_3} \setminus \mathcal{E}$. Now, by [3, Proposition 2.2], we have

$$\Phi(\mathbf{p}) \geq \sum_{i \in \mathcal{A}_1} p_i \log\left(p_i + \sum_{j \in \mathcal{A}_2 \cup \mathcal{A}_3} p_j\right).$$

Given that for each $i \in \{1, 2, \ldots, N^2\}$, $a_1^i \neq a_2^i$ if a_1^i and a_2^i both are not on the boundary of original triangle Δ. Under this consideration, one can see that

$$\#\mathcal{A}_1 = N + 2 \text{ and } \#\mathcal{A}_2 + \#\mathcal{A}_3 = N^2 - N - 2.$$

Now, our aim is to show that

$$\frac{-\sum_{i=1}^{N^2} p_i \log p_i + \Phi(\mathbf{p})}{-\sum_{i=1}^{N^2} p_i \log \lambda_i} > 3 - t_0 = 2 - \frac{\log \sum_{i=1}^{N^2} s_i}{\log N}.$$

Since $\lambda_i = \frac{1}{Ns_i}$ and $p_i = s_i(\frac{1}{N})^{t_0-1}$, we have

$$\frac{-\sum_{i=1}^{N^2} p_i \log p_i + \Phi(\mathbf{p})}{-\sum_{i=1}^{N^2} p_i \log \lambda_i} > 2 - \frac{\log \sum_{i=1}^{N^2} s_i}{\log N}$$

$$\Leftarrow \frac{-\sum_{i=1}^{N^2} p_i \log p_i + \sum_{i\in\mathcal{A}_1} p_i \log\left(p_i + \sum_{j\in\mathcal{A}_2\cup\mathcal{A}_3} p_j\right)}{\sum_{i=1}^{N^2} p_i \log N s_i} > 2 - \frac{\log \sum_{i=1}^{N^2} s_i}{\log N}$$

$$\Leftarrow \frac{-\sum_{i=1}^{N^2} s_i \log p_i + \sum_{i\in\mathcal{A}_1} s_i \log\left(p_i + \sum_{j\in\mathcal{A}_2\cup\mathcal{A}_3} p_j\right)}{\sum_{i=1}^{N^2} s_i \log N s_i} > 2 - \frac{\log \sum_{i=1}^{N^2} s_i}{\log N}$$

$$\Leftarrow \frac{\sum_{i\in\mathcal{A}_1} s_i \log\left(\frac{s_i+\sum_{j\in\mathcal{A}_2\cup\mathcal{A}_3} s_j}{s_i}\right) + \sum_{i\in\mathcal{A}_2\cup\mathcal{A}_3} s_i \log\left(\frac{\sum_{i=1}^{N^2} s_i}{s_i}\right)}{\sum_{i=1}^{N^2} s_i \log N s_i} > 2 - \frac{\log \sum_{i=1}^{N^2} s_i}{\log N}$$

$$\Leftarrow \sum_{i\in\mathcal{A}_1} s_i \log\left(\frac{s_i+\sum_{j\in\mathcal{A}_2\cup\mathcal{A}_3} s_j}{s_i}\right) + \sum_{i\in\mathcal{A}_2\cup\mathcal{A}_3} s_i \log\left(\frac{\sum_{i=1}^{N^2} s_i}{s_i}\right)$$
$$> \sum_{i=1}^{N^2} s_i \log(N s_i)^2 - \sum_{i=1}^{N^2} s_i \log(N s_i) \frac{\log \sum_{i=1}^{N^2} s_i}{\log N}$$

$$\Leftarrow \sum_{i\in\mathcal{A}_1} s_i \log\left(\frac{s_i+\sum_{j\in\mathcal{A}_2\cup\mathcal{A}_3} s_j}{N^2 s_i^3}\right) + \sum_{i\in\mathcal{A}_2\cup\mathcal{A}_3} s_i \log\left(\frac{\sum_{i=1}^{N^2} s_i}{N^2 s_i^3}\right)$$
$$> -\sum_{i=1}^{N^2} s_i \log N s_i \frac{\log \sum_{i=1}^{N^2} s_i}{\log N}$$

$$\Leftarrow \frac{\log(N s_{\min})}{\log N}\left(\sum_{i\in\mathcal{A}_1} s_i \log\left(\frac{s_i+\sum_{j\in\mathcal{A}_2\cup\mathcal{A}_3} s_j}{N^2 s_i^3}\sum_{i=1}^{N^2} s_i\right) + \sum_{i\in\mathcal{A}_2\cup\mathcal{A}_3} s_i \log\left(\frac{(\sum_{i=1}^{N^2} s_i)^2}{N^2 s_i^3}\right)\right) > 0$$

$$\Leftarrow \sum_{i\in\mathcal{A}_1} s_i \log\left(\frac{s_i+\sum_{j\in\mathcal{A}_2\cup\mathcal{A}_3} s_j}{N^2 s_i^3}\sum_{i=1}^{N^2} s_i\right) + \sum_{i\in\mathcal{A}_2\cup\mathcal{A}_3} s_i \log\left(\frac{(\sum_{i=1}^{N^2} s_i)^2}{N^2 s_i^3}\right) > 0.$$

Clearly $\frac{(\sum_{i=1}^{N^2} s_i)^2}{N^2 s_i^3} \geq \frac{1}{s_i^3} > 1$. This proves our claim. Since $3 - t_0 \in (0,1)$, we have

$$\dim_H(\mu_F) \geq \dim_H(P_{X*}\mu_F) > 3 - t_0$$

for Lebesgue all most every scaling parameters $\underline{s} \in (\frac{1}{N}, 1)^{\#\mathcal{A}_1} \times (\frac{1}{NB}, 1)^{\#\mathcal{A}_2} \times (\frac{1}{ND}, 1)^{\#\mathcal{A}_3}$. The proof is completed by Theorem 2.3. □

4.3. **Hausdorff dimension of the FIS in the case of $N = 3$ for every parameter.** Here, we prove Theorem 1.3 by computing the overlapping number.

Since the upper bound

$$\dim_H(G(f^*)) \leq \overline{\dim}_B(G(f^*)) \leq t = 1 + \frac{\log(\sum_{i=1}^{9} s_i)}{\log(3)}$$

holds for every parameter value, it is enough to show the lower bound.

For the lower bound, consider the corresponding Furstenberg IFS $\mathcal{J} = \{h_1, h_2, \dots, h_9\}$, where $h_i : \mathbb{R}^2 \to \mathbb{R}^2$ are defined as follows:

$$h_1(x,y) = \frac{1}{3s_1}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} - \frac{1}{s_1}\begin{bmatrix}0\\ 0\end{bmatrix}, \quad h_2(x,y) = \frac{1}{3s_2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} - \frac{1}{s_2}\begin{bmatrix}0\\ \frac{2a}{\sqrt{3}}\end{bmatrix},$$

$$h_3(x,y) = \frac{1}{3s_3}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} - \frac{1}{s_3}\begin{bmatrix}0\\ 0\end{bmatrix}, \quad h_4(x,y) = \frac{1}{3s_4}\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} - \frac{1}{s_4}\begin{bmatrix}-a\\ -\frac{a}{\sqrt{3}}\end{bmatrix},$$

$$h_5(x,y) = \frac{1}{3s_5}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} - \frac{1}{s_5}\begin{bmatrix}-a\\ -\frac{a}{\sqrt{3}}\end{bmatrix}, \quad h_6(x,y) = \frac{1}{3s_6}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} - \frac{1}{s_6}\begin{bmatrix}0\\ 0\end{bmatrix},$$

$$h_7(x,y) = \frac{1}{3s_7}\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} - \frac{1}{s_7}\begin{bmatrix}-a\\ -\frac{a}{\sqrt{3}}\end{bmatrix}, \quad h_8(x,y) = \frac{1}{3s_8}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} - \frac{1}{s_8}\begin{bmatrix}-a\\ -\frac{a}{\sqrt{3}}\end{bmatrix},$$

$$h_9(x,y) = \frac{1}{3s_9}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} - \frac{1}{s_9}\begin{bmatrix}0\\ \frac{2a}{\sqrt{3}}\end{bmatrix}.$$

Let $\mu_F = \sum_{i=1}^{9} s_i \left(\frac{1}{3}\right)^{t-1} (h_i)_*\mu_F$ be the invariant Borel probability measure for the IFS $\mathcal{J}$ with probabilities $p_i = s_i \left(\frac{1}{3}\right)^{t-1}$. By Theorem 2.3, to show that

$$\dim_H(G(f^*)) \geq t = 1 + \frac{\log(\sum_{i=1}^{9} s_i)}{\log(3)},$$

it is enough to prove that

$$\dim_H(\mu_F) > 3 - t.$$

Now, we will estimate the Hausdorff dimension of the Furstenberg measure.

Let f_i be the projection of h_i on the X-axis. Let $\mathcal{J}_X = \{f_1, f_2, \dots, f_9\}$ be the projection of the IFS $\mathcal{J}$ on the X-axis. Precisely, the maps $f_i's$ are as follows:

$$f_1(x) = \frac{x}{3s_1}, \quad f_2(x) = \frac{x}{3s_2}, \quad f_3(x) = \frac{x}{3s_3}, \quad f_6(x) = \frac{x}{3s_6}, \quad f_9(x) = \frac{x}{3s_9},$$

$$f_4(x) = -\frac{x}{3s_4} + \frac{a}{s_4}, \quad f_5(x) = \frac{x}{3s_5} + \frac{a}{s_5}, \quad f_7(x) = -\frac{x}{3s_7} + \frac{a}{s_7}, \quad f_8(x) = \frac{x}{3s_8} + \frac{a}{s_8}.$$

The fixed point of the map f_i is denoted by $\mathrm{Fix}(f_i)$ for all $i \in \{1, 2, \dots, 9\}$. Thus, we have

$$\mathrm{Fix}(f_1) = \mathrm{Fix}(f_2) = \mathrm{Fix}(f_3) = \mathrm{Fix}(f_6) = \mathrm{Fix}(f_9) = 0,$$

$$\mathrm{Fix}(f_4) = \frac{3a}{3s_4+1}, \quad \mathrm{Fix}(f_5) = \frac{3a}{3s_5-1}, \quad \mathrm{Fix}(f_7) = \frac{3a}{3s_7+1}, \quad \mathrm{Fix}(f_8) = \frac{3a}{3s_8-1}.$$

Without loss of generality, we assume that $s_5 \leq s_8$.

Lemma 4.6. *Let $s_i \in \left(\frac{2}{3}, 1\right)$ $\forall\, i \in \{1, 2, \dots, 9\}$. Let $[\tilde{a}, \tilde{b}]$ be the invariant interval for the IFS $\mathcal{J}_X$. Then, $\tilde{a} = 0$ and $\tilde{b} = Fix(f_5)$. Moreover, if $\max\{s_5, s_8\} \leq \min\{s_4, s_7\}$, then*

$$(f_4[0,\tilde{b}] \cup f_7[0,\tilde{b}]) \cap (f_5[0,\tilde{b}] \cup f_8[0,\tilde{b}]) = \emptyset. \tag{4.3}$$

For a visualisation, see Figure 5.

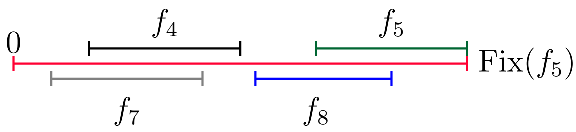


FIGURE 5. Visualisation of the configuration in (4.3).

Proof. The maps f_4 and f_7 are flipping the orientation and other maps are orientation preserving maps. Since $s_5 \leq s_8$, the fixed point $\text{Fix}(f_5)$ is the largest fixed points. Since $s_i \in \left(\frac{2}{3}, 1\right)$ $\forall$ $i \in \{1, 2, \dots, 9\}$, we have

$$f_4[0, \text{Fix}(f_5)] = \left[\frac{a(3s_5 - 2)}{s_4(3s_5 - 1)}, \frac{a}{s_4}\right] \subset [0, \text{Fix}(f_5)],$$
$$f_7[0, \text{Fix}(f_5)] = \left[\frac{a(3s_5 - 2)}{s_7(3s_5 - 1)}, \frac{a}{s_7}\right] \subset [0, \text{Fix}(f_5)].$$

This implies that $[0, \tilde{b}]$ is the invariant interval for the IFS $\mathcal{J}_X$, where $\tilde{b} = \text{Fix}(f_5)$. One can see that

$$f_5[0, \tilde{b}] = \left[\frac{a}{s_5}, \tilde{b}\right] \text{ and } f_8[0, \tilde{b}] = \left[\frac{a}{s_8}, \frac{3s_5 a}{s_8(3s_5 - 1)}\right].$$

Now, we assume that $\max\{s_5, s_8\} \leq \min\{s_4, s_7\}$. Then, we get

$$(f_4[0, \tilde{b}] \cup f_7[0, \tilde{b}]) \cap (f_5[0, \tilde{b}] \cup f_8[0, \tilde{b}]) = \emptyset.$$

This completes the proof. □

Now, we will see the projection of the Furstenberg IFS $\mathcal{J}$ on the Y-axis. Let g_i be the projection of h_i on the Y-axis. Let $\mathcal{J}_Y = \{g_1, g_2, \dots, g_9\}$ be the projection of the IFS $\mathcal{J}$ on the y-axis. Precisely, the maps $g_i's$ are as follows:

$$g_1(y) = \frac{y}{3s_1}, \quad g_3(y) = \frac{y}{3s_3}, \quad g_6(y) = \frac{y}{3s_6},$$
$$g_2(y) = \frac{y}{3s_2} - \frac{2a}{\sqrt{3}s_2}, \quad g_4(y) = \frac{y}{3s_4} + \frac{a}{\sqrt{3}s_4}, \quad g_5(y) = \frac{y}{3s_5} + \frac{a}{\sqrt{3}s_5},$$
$$g_7(y) = \frac{-y}{3s_7} + \frac{a}{\sqrt{3}s_7}, \quad g_8(y) = \frac{-y}{3s_8} + \frac{a}{\sqrt{3}s_8}, \quad g_9(y) = \frac{-y}{3s_9} - \frac{2a}{\sqrt{3}s_9}.$$

Next, we will examine the invariant interval for the IFS $\mathcal{J}_Y = \{g_1, g_2, \dots, g_9\}$. The fixed points of the maps $g_i's$ are as follows:

$$\text{Fix}(g_1) = \text{Fix}(g_3) = \text{Fix}(g_6) = 0,$$
$$\text{Fix}(g_2) = \frac{-6a}{\sqrt{3}(3s_2 - 1)}, \quad \text{Fix}(g_4) = \frac{3a}{\sqrt{3}(3s_4 - 1)}, \quad \text{Fix}(g_5) = \frac{3a}{\sqrt{3}(3s_5 - 1)},$$
$$\text{Fix}(g_7) = \frac{3a}{\sqrt{3}(3s_7 + 1)}, \quad \text{Fix}(g_8) = \frac{3a}{\sqrt{3}(3s_8 + 1)}, \quad \text{Fix}(g_9) = \frac{-6a}{\sqrt{3}(3s_9 + 1)}.$$

Without loss of generality, we assume that $s_7 \leq s_8$ and $s_4 \leq s_5$.

Lemma 4.7. *Let $s_i \in (\frac{2}{3}, 1)$ $\forall$ $i \in \{1, 2, \dots, 9\}$. We assume that if $\max\{s_5, s_8\} \leq \min\{s_4, s_7\}$ and $s_2 \leq s_9$. Let $\tilde{I}$ be the invariant interval for the IFS $\mathcal{J}_Y$. Then, the invariant interval $\tilde{I}$ is either $[Fix(g_2), Fix(g_5)]$ or $[Fix(g_2), g_8(Fix(g_2))]$.*

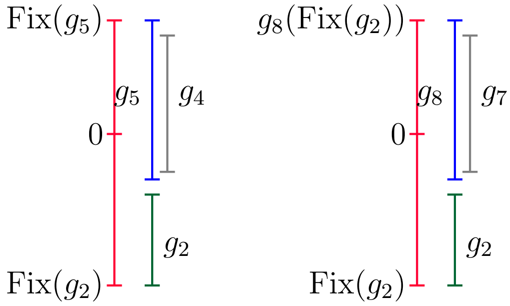


FIGURE 6. Visualisation of the configuration in Lemma 4.7.

For a visualisation, see Figure 6.

Proof. The maps g_2, g_4 and g_5 are preserve the orientation, however the maps g_7, g_8 and g_9 are flipping the orientation about Y- axis. The $\mathrm{Fix}(g_2)$ is the lowest fixed point. Since $s_8 \leq s_7$ and $s_5 \leq s_4$, the fixed point $\mathrm{Fix}(g_5)$ is the largest fixed point. All the fixed points are always in the invariant interval $\tilde{I}$. Thus, $g_8(\mathrm{Fix}(g_2)) \in \tilde{I}$. And, we have

$$g_8(\mathrm{Fix}(g_2)) = \frac{-1}{3s_8}\left(\frac{-6a}{\sqrt{3}(3s_2-1)}\right) + \frac{a}{\sqrt{3}s_8} = \frac{a}{\sqrt{3}s_8}\left(\frac{3s_2+1}{3s_2-1}\right),$$

$$g_7(\mathrm{Fix}(g_2)) = g_8(\mathrm{Fix}(g_2))\left(\frac{s_8}{s_7}\right) < g_8(\mathrm{Fix}(g_2)).$$

Furthermore, $g_8(g_8(\mathrm{Fix}(g_2))) \in \tilde{I}$ and $g_8(\mathrm{Fix}(g_5)) \in \tilde{I}$. Thus, we have

$$g_8(g_8(\mathrm{Fix}(g_2))) = \frac{a}{\sqrt{3}s_8}\left(\frac{3s_8(3s_2-1)-(3s_2+1)}{3s_8(3s_2-1)}\right), g_8(\mathrm{Fix}(g_5)) = \frac{a}{\sqrt{3}s_8}\left(\frac{3s_5-2}{3s_5-1}\right).$$

One can see that

$$g_8(g_8(\mathrm{Fix}(g_2))) > \mathrm{Fix}(g_2)\ , 0 < g_8(\mathrm{Fix}(g_5)) < g_8(\mathrm{Fix}(g_2)) \text{ and } 0 < g_8(\mathrm{Fix}(g_5)) < \mathrm{Fix}(g_5).$$

The map g_9 is also flipping the orientation. So,

$$g_9(\mathrm{Fix}(g_5)) = \frac{-1}{3s_9}\left(\frac{3a}{\sqrt{3}(3s_5-1)}\right) - \frac{2a}{\sqrt{3}s_9} = \frac{-a}{\sqrt{3}s_9}\left(\frac{6s_5-1}{3s_5-1}\right),$$

$$g_9(g_8(\mathrm{Fix}(g_2))) = \frac{-a}{\sqrt{3}s_9}\left(\frac{(3s_2+1)+6s_8(3s_2-1)}{3s_8(3s_2-1)}\right),$$

$$g_9(\mathrm{Fix}(g_2)) = \frac{-1}{3s_9}\left(\frac{-6a}{\sqrt{3}(3s_2-1)}\right) - \frac{2a}{\sqrt{3}s_9} = \frac{-2a}{\sqrt{3}s_9}\left(\frac{3s_2-2}{3s_2-1}\right).$$

One can see that

$$0 > g_9(\mathrm{Fix}(g_2)) > \mathrm{Fix}(g_2).$$

For $s_2 \leq s_9$, we have the following relation for the map g_9:

$$0 > g_9(\mathrm{Fix}(g_5)) > \mathrm{Fix}(g_2) \quad \text{and} \quad 0 > g_9(g_8(\mathrm{Fix}(g_2))) > \mathrm{Fix}(g_2).$$

Thus, for $s_2 \leq s_9$, the invariant interval $\tilde{I}$ is either $[\mathrm{Fix}(g_2), \mathrm{Fix}(g_5)]$ or $[\mathrm{Fix}(g_2), g_8(\mathrm{Fix}(g_2))]$ depending on the relation between $\mathrm{Fix}(g_5)$ and $g_8(\mathrm{Fix}(g_2))$. This completes the proof. □

Proposition 4.8. Let $s_i \in \left(\frac{2}{3}, 1\right)$ $\forall\ i \in \{1, 2, \dots, 9\}$. Suppose that $\max\{s_5, s_8\} \leq \min\{s_4, s_7\}$ and $s_2 \leq s_9$. For $x \in A_F$, let $\underline{d}_{\mu_F}(x)$ denotes the lower local dimension of μ_F at x. Then,

$$\dim_H \mu_F \geq \frac{\log(1 - \frac{s_{\min}}{3^{t_0-2}})}{-\log(3s_{\max})}.$$

Proof. For each $x \in A_F$, we define $C_x := \{i : x \notin h_i(A_F) \text{ and } i \in \{1, 2, \dots, 9\}\}$. First, we show that that $\#C_x \geq 3$ for every $x \in A_F$, where $\#C_x$ denotes the cardinality of C_x.

By Lemma 4.6 and Lemma 4.7, either the rectangle $[0, \tilde{b}] \times [\text{Fix}(g_2), \text{Fix}(g_5)]$ or $[0, \tilde{b}] \times [\text{Fix}(g_2), g_8(\text{Fix}(g_2))]$ mapped into itself by all the maps of the Furstenberg IFS $\mathcal{J}$.

In the first situation, $\tilde{I} = [\text{Fix}(g_2), \text{Fix}(g_5)]$. In this case, the cylinder corresponding to h_2 is placed at the bottom of the invariant set (on the Y-axis) and the cylinder corresponding to h_5 is placed at the top of the invariant set (on the left side of the Y-axis). Since $\frac{1}{3s_2} + \frac{1}{3s_5} < 1$ and $s_5 \leq s_4$, we have

$$h_2([0, \tilde{b}] \times \tilde{I}) \cap h_5([0, \tilde{b}] \times \tilde{I}) = \emptyset \text{ and } h_2([0, \tilde{b}] \times \tilde{I}) \cap h_4([0, \tilde{b}] \times \tilde{I}) = \emptyset.$$

By Equation 4.3, we have

$$\left(h_4([0, \tilde{b}] \times \tilde{I}) \cup h_7([0, \tilde{b}] \times \tilde{I})\right) \cap \left(h_5([0, \tilde{b}] \times \tilde{I}) \cup h_8([0, \tilde{b}] \times \tilde{I})\right) = \emptyset.$$

Thus, from the above, it is clear that a point can be contained in at most six cylinders, and so $\#C_x \geq 3$.

In the another situation $\tilde{I} = [\text{Fix}(g_2), g_8(\text{Fix}(g_2))]$, then by using same idea as above, and using the conditions $\frac{1}{3s_2} + \frac{1}{3s_8} < 1$ and $s_8 \leq s_7$, one can get $\#C_x \geq 3$. The claim then follows from Proposition 2.4. □

Proof of Proposition 4.3. Since $\dim_H(\mu_F) = \text{ess}\inf(\underline{d}_{\mu_F}(x))$ and by Proposition 4.8, we obtain

$$\dim_H(\mu_F) \geq \frac{\log(1 - \frac{s_{\min}}{3^{t_0-2}})}{-\log(3s_{\max})}.$$

For proving $\dim_H(\mu_F) > 3 - t_0$, it is enough to show that $\frac{\log(1 - \frac{s_{\min}}{3^{t_0-2}})}{-\log(3s_{\max})} > 3 - t_0$. We have

$$\begin{aligned}
&\frac{\log(1 - \frac{s_{\min}}{3^{t_0-2}})}{-\log(3s_{\max})} > 3 - t_0 \\
\Leftarrow &\frac{\log(1 - \frac{2}{3^{t_0-1}})}{-\log(3)} > 3 - t_0 \\
\Leftarrow &\left(1 - \frac{6}{3^{t_0}}\right) < \frac{3^{t_0}}{27} \\
\Leftarrow &0 < 3^{2t_0} - 27 \times 3^{t_0} + 162.
\end{aligned}$$

This implies that if $3^{t_0} > 18$, then the inequality $\frac{\log(1 - \frac{s_{\min}}{3^{t_0-2}})}{-\log(3s_{\max})} > 3 - t_0$ holds. Since $s_i \in \left(\frac{2}{3}, 1\right)$ and the affinity dimension $t_0 = 1 + \frac{\log(\sum_{i=1}^{9} s_i)}{\log(3)} > 1 + \frac{\log 6}{\log 3}$, and so, we have $3^{t_0} > 18$. This completes the proof. □

Remark 4.9. The method of overlapping numbers might again be applicable for other cases when $N \geq 4$ and for general data sets. However, since there are many maps in very general positions, verifying that the overlapping number is sufficiently small would clearly require tedious calculations.

4.4. **Uniform scaling factors.** Next, we will consider a uniform scaling factor $s = s_i \; \forall \, i \in \{1, 2, \ldots, 9\}$ in the construction of fractal surfaces. For that, we have the following result:

Corollary 4.10. If $s \in (\frac{2}{3}, 1)$, then

$$\dim_H(G(f^*)) = \dim_B(G(f^*)) = 3 + \frac{\log(s)}{\log(3)}.$$

The proof of this corollary follows from Theorem 1.3.
Next, we show some dimension results for the typical choice of the uniform scaling factor.

Theorem 4.11. *If $s = s_i \; \forall \; i \in \{1, 2, \ldots, 9\}$, then*

$$\dim_H(G(f^*)) = \dim_B(G(f^*)) = 3 + \frac{\log(s)}{\log(3)} \text{ for a.e. } s \in (1/3, 1).$$

Proof. If $s = s_i$ for every $i \in \{1, 2, \ldots, 9\}$, then the projected IFS $\mathcal{J}_X = \{f_1, f_2, \ldots, f_9\}$ of the Furstenberg IFS is as follows.

$$f_1(x) = f_2(x) = f_3(x) = f_6(x) = f_9(x) = \frac{x}{3s},$$
$$f_4(x) = f_7(x) = -\frac{x}{3s} + \frac{a}{s}, \quad f_5(x) = f_8(x) = \frac{x}{3s} + \frac{a}{s}.$$

And in this case $p_i = \frac{1}{9}$ for every $i \in \{1, 2, \ldots, 9\}$. Thus, the IFS $\mathcal{J}_X$ is equivalent to the IFS $\tilde{\mathcal{J}}_X = \{\tilde{f}_1(x) = \frac{x}{3s}, \tilde{f}_2(x) = -\frac{x}{3s} + \frac{a}{s}, \tilde{f}_3(x) = \frac{x}{3s} + \frac{a}{s}\}$ with probability vector $(\frac{5}{9}, \frac{2}{9}, \frac{2}{9})$. Let $P_{X*}\mu_F$ be the projection of the measure μ_F on the X-axis. Let $\epsilon > 0$ and M be a sufficiently large natural number. Now, we define real analytic maps $r_i : [\frac{1}{3} + \epsilon, M] \to (-1, 1) \setminus \{0\}\}$ and $d_i : [\frac{1}{3} + \epsilon, M] \to \mathbb{R}\}$ are as follows:

$$r_1(s) = r_3(s) = \frac{1}{3s}, \; r_2(s) = -\frac{1}{3s}, \; d_1(s) = 0, \; d_2(s) = d_3(s) = \frac{a}{s}.$$

The IFS $\tilde{\mathcal{J}}_X$ is same as a parametric family of self-similar IFS $\mathcal{I}_s = \{r_i(s)x + d_i(s)\}_{i=1}^3$. For $s \in (1, M]$, the parametric IFS $\mathcal{I}_s$ satisfies the strong separation condition. Let A_s be the attractor of the parametric IFS $\mathcal{I}_s$. Let $\Pi_s : \Sigma = \{1, 2, 3\}^{\mathbb{N}} \to A_s$ be the associated natural projection. Then,

$$\forall \, \mathbf{i}, \mathbf{j} \in \Sigma, \quad \Pi_s(\mathbf{i}) = \Pi_s(\mathbf{j}) \quad \text{on} \quad \left[\frac{1}{3} + \epsilon, M\right] \iff \mathbf{i} = \mathbf{j}.$$

Thus by the result of Hochman [9, Theorem 1.10], we get

$$\dim_H(P_{X*}\mu_F) = \min\left\{1, \frac{-\sum_{i=1}^3 p_i \log(p_i)}{-\sum_{i=1}^3 p_i \log(|r_i|)} = \frac{\frac{-5}{9}\log(\frac{5}{9}) - \frac{4}{9}\log(\frac{2}{9})}{\log(3s)}\right\} \text{ for } a.e. \; s \in \left[\frac{1}{3} + \epsilon, M\right].$$

The ϵ is arbitrarily small. Thus, we have

$$\dim_H(\mu_F) \geq \dim_H(P_{X*}\mu_F) = \min\left\{1, \frac{\frac{-5}{9}\log(\frac{5}{9}) - \frac{4}{9}\log(\frac{2}{9})}{\log(3s)}\right\} \text{ for } a.e. \; s \in \left(\frac{1}{3}, 1\right).$$

One can easily see that $\frac{\frac{-5}{9}\log(\frac{5}{9}) - \frac{4}{9}\log(\frac{2}{9})}{\log(3s)} > 3 - t = \frac{\log(\frac{1}{s})}{\log(3)}$ and $3 - t \in (0, 1)$ for all $s \in (\frac{1}{3}, 1)$. Thus,

$$\dim_H(\mu_F) > 3 - t \text{ for } a.e. \; s \in \left(\frac{1}{3}, 1\right).$$

Thus, by Theorem 2.3, $\dim_H(\mu) = t = 3 + \frac{\log(s)}{\log(3)}$ for a.e. $s \in (1/3, 1)$. This completes the proof. $\square$

5. Geronimo-Hardin surfaces

In this section, we will prove Theorem 1.4 and Theorem 1.5. In these results, the self-affine IFS is $\mathcal{I} = \{W_1, W_2, W_3, W_4\}$, where the maps W_i are as follows:

$$W_1(x,y,z) = \begin{bmatrix} \frac{1}{4} & \frac{\sqrt{3}}{4} & 0 \\ \frac{\sqrt{3}}{4} & \frac{-1}{4} & 0 \\ a & \frac{a}{\sqrt{3}} & s \end{bmatrix} \begin{pmatrix} x \\ y \\ z, \end{pmatrix}, \quad W_2(x,y,z) = \begin{bmatrix} \frac{1}{4} & \frac{-\sqrt{3}}{4} & 0 \\ \frac{-\sqrt{3}}{4} & \frac{-1}{4} & 0 \\ -a & \frac{a}{\sqrt{3}} & s \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} \frac{3}{4} \\ \frac{\sqrt{3}}{4} \\ a \end{pmatrix},$$

$$W_3(x,y,z) = \begin{bmatrix} \frac{-1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & \frac{-2a}{\sqrt{3}} & s \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} \frac{3}{4} \\ \frac{\sqrt{3}}{4} \\ a \end{pmatrix}, \quad W_4(x,y,z) = \begin{bmatrix} \frac{-1}{2} & 0 & 0 \\ 0 & \frac{-1}{2} & 0 \\ 0 & 0 & s \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} \frac{3}{4} \\ \frac{\sqrt{3}}{4} \\ a \end{pmatrix}.$$

Thus, the IFS $\mathcal{I}$ is a block triangular self-affine IFS. Let $s \in (\frac{1}{2}, 1)$. Then, the affinity dimension (t) for the IFS $\mathcal{I}$ is uniquely given by the following equation

$$\sum_{i=1}^{4} s\left(\frac{1}{2}\right)^{t-1} = 1,$$

and by [7], we get

$$\dim_H(G(f^*)) \leq \overline{\dim}_B(G(f^*)) \leq t = 3 + \frac{\log(s)}{\log(2)}. \tag{5.1}$$

Since bi-Lipschitz conjugation preserves the fractal dimension, we may assume without loss of generality that $a = 1$. For the lower bound, again, we construct the corresponding Furstenberg IFS $\mathcal{J} = \{h_1, h_2, h_3, h_4\}$ by (2.3), where $h_i : \mathbb{R}^2 \to \mathbb{R}^2$ are defined as follows:

$$\begin{aligned} h_1(x,y) &= \frac{1}{2s}\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \quad h_2(x,y) = \frac{1}{2s}\begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & \frac{-1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \\ h_3(x,y) &= \frac{1}{2s}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{-2}{\sqrt{3}} \end{bmatrix}, \quad h_4(x,y) = \frac{1}{2s}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}. \end{aligned} \tag{5.2}$$

For $s \in (\frac{1}{2}, 1)$, the Furstenberg IFS $\mathcal{J}$ is contractive. Let A_F be the attractor of the Furstenberg IFS $\mathcal{J}$. The Furstenberg measure μ_F is the invariant Borel probability measure for the IFS $\mathcal{J}$ with the uniform probability vector $p = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. By taking bi-Lipschitz conjugate of the IFS $\mathcal{J}$ with the map $g(x,y) = \frac{-a}{s}(x,y)$, the resultant IFS is denoted by $\mathcal{J} = \{h_1, h_2, h_3, h_4\}$, which is as follows.

5.1. **Every-type result.** We can not proceed with the argument as in the previous section, because the projection of the Furstenberg IFS $\mathcal{J}$ on the X-axis and Y-axis is not an IFS. In this case, we need to directly determine the invariant set for the IFS $\mathcal{J}$ and with the help of that, we estimate the overlapping number for the IFS $\mathcal{J}$. The fixed points of the maps h_i's are as follows.

$$\mathrm{Fix}(h_1) = \left(\frac{4s}{4s-2}, \frac{4s}{\sqrt{3}(4s-2)}\right), \quad \mathrm{Fix}(h_2) = \left(-\frac{4s}{4s-2}, \frac{4s}{\sqrt{3}(4s-2)}\right),$$

$$\mathrm{Fix}(h_3) = \left(0, -\frac{8s}{\sqrt{3}(4s-2)}\right), \quad \mathrm{Fix}(h_4) = (0,0).$$

Proposition 5.1. Let $s \in \left[\frac{1+\sqrt{5}}{4}, 1\right)$. Then, $\min_{x\in K} \#\{i \in \{1,\dots,N\} : x \notin h_i(A_F)\} \geq 1$.

Proof. Let $\mathrm{Fix}(h_1) = A, \mathrm{Fix}(h_2) = B$ and $\mathrm{Fix}(h_3) = C$. Let $\mathcal{S}$ be the invariant set for the IFS $\mathcal{J}$. The map h_4 flips the orientation about the X-axis. Since $A, B, C \in \mathcal{S}$, we have $A' = h_4(A) = \left(-\frac{2}{4s-2}, -\frac{2}{\sqrt{3}(4s-2)}\right) \in \mathcal{S}, B' = h_4(B) = \left(\frac{2}{4s-2}, -\frac{2}{\sqrt{3}(4s-2)}\right) \in \mathcal{S}$ and $C' = h_4(C) = \left(0, \frac{4}{\sqrt{3}(4s-2)}\right) \in \mathcal{S}$. For $s \in (\frac{1}{2}, 1)$, our claim is that the convex set with vertices A, B, C, A', B' and C' is the invariant set for the IFS $\mathcal{J}$. We denote that convex set by $\mathrm{Con}(ABCA'B'C')$. One can see that the set $\mathrm{Con}(ABCA'B'C')$ is symmetric about Y-axis, see Figure 7.

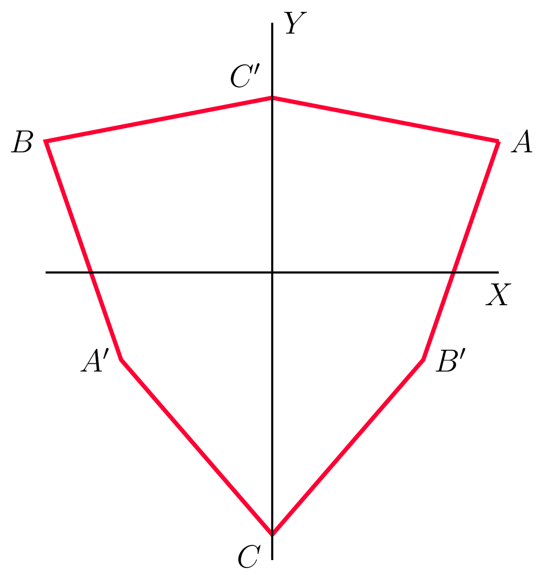


FIGURE 7. The invariant convex hull of the Furstenberg IFS (5.2).

Now, we prove our claim. We have

$$h_1(A) = A,\ h_1(B) = \left(1, \frac{4s-6}{\sqrt{3}(4s-2)}\right) = B_1,\ h_1(C) = \left(\frac{4s-4}{4s-2}, \frac{4s}{\sqrt{3}(4s-2)}\right) = C_1,$$

$$h_1(A') = \left(\frac{-1+s(4s-2)}{s(4s-2)}, \frac{-1+s(4s-2)}{s\sqrt{3}(4s-2)}\right) = A_1', h_1(B') = \left(1, \frac{2+s(4s-2)}{s\sqrt{3}(4s-2)}\right) = B_1'$$

$$h_1(C') = \left(\frac{1+s(4s-2)}{s(4s-2)}, \frac{-1+s(4s-2)}{s\sqrt{3}(4s-2)}\right) = C_1'.$$

One can see that the points B_1' and C_1' are on the line AC' and AB', respectively. For $s \in (\frac{1}{2}, 1)$, we get $1 < \frac{2}{(4s-2)}$. And the point $\left(1, \frac{(4s-1)(4s-2)-8s}{\sqrt{3}(4s-2)}\right)$ is on the line CB'. Since $-\frac{2}{\sqrt{3}(4s-2)} > \frac{4s-6}{\sqrt{3}(4s-2)} > \frac{(4s-1)(4s-2)-8s}{\sqrt{3}(4s-2)}$, the point B_1 is on the above side of line CB'. For $s \in (\frac{1}{2}, 1)$, we have $-\frac{2}{4s-2} < \frac{4s-4}{4s-2} < 0$. Thus the point C_1 is in the left side of the set $\text{Con}(ABCA'B'C')$ and on the line joining A and B. This implies that $h_1(\text{Con}(ABCA'B'C')) \subset \text{Con}(ABCA'B'C')$. For, the mapping h_2, we have

$$h_2(B) = B,\ h_2(A) = \Big(-1, \frac{4s-6}{\sqrt{3}(4s-2)}\Big) = A_2,\ h_2(C) = \Big(-\frac{4s-4}{4s-2}, \frac{4s}{\sqrt{3}(4s-2)}\Big) = C_2,$$

$$h_2(B') = \Big(-\frac{-1+s(4s-2)}{s(4s-2)}, \frac{-1+s(4s-2)}{s\sqrt{3}(4s-2)}\Big) = B_2', h_2(A') = \Big(-1, \frac{2+s(4s-2)}{s\sqrt{3}(4s-2)}\Big) = A_2'$$

$$h_2(C') = \Big(-\frac{1+s(4s-2)}{s(4s-2)}, \frac{-1+s(4s-2)}{s\sqrt{3}(4s-2)}\Big) = C_2'.$$

Thus, one can see that $h_2(\text{Con}(ABCA'B'C'))$ is the mirror image of $h_1(\text{Con}(ABCA'B'C'))$ with respect to Y-axis. Thus, due to symmetry of the set $\text{Con}(ABCA'B'C')$ with respect to the Y-axis, we get $h_2(\text{Con}(ABCA'B'C')) \subset \text{Con}(ABCA'B'C')$. For, the mapping h_3, we have

$$h_3(C) = C,\ h_3(C') = \Big(0, \frac{2(1-s(4s-2))}{\sqrt{3}s(4s-2)}\Big) = C_3',$$

$$h_3(A) = \Big(\frac{-2}{4s-2}, \frac{6-8s}{\sqrt{3}(4s-2)}\Big) = A_3, h_3(A') = \Big(\frac{1}{s(4s-2)}, \frac{-1-2s(4s-2)}{\sqrt{3}s(4s-2)}\Big) = A_3',$$

$$h_3(B) = \Big(\frac{2}{4s-2}, \frac{6-8s}{\sqrt{3}(4s-2)}\Big) = B_3, h_3(B') = \Big(\frac{-1}{s(4s-2)}, \frac{-1-2s(4s-2)}{\sqrt{3}s(4s-2)}\Big) = B_3'.$$

One can see that the points A_3' and B_3' are on the line CB' and CA', respectively. For $s \in (\frac{1}{2}, 1)$, we have $\frac{-2}{\sqrt{3}(4s-2)} < \frac{6-8s}{\sqrt{3}(4s-2)} < \frac{-1-2s(4s-2)}{\sqrt{3}s(4s-2)}$ and $\frac{6-8s}{\sqrt{3}(4s-2)} < \frac{-1-2s(4s-2)}{\sqrt{3}s(4s-2)} < \frac{4}{\sqrt{3}(4s-2)}$. This implies that $h_3(\text{Con}(ABCA'B'C')) \subset \text{Con}(ABCA'B'C')$. For, the mapping h_4, we have

$$h_4(A) = A', h_4(B) = B', h_4(C) = C',$$

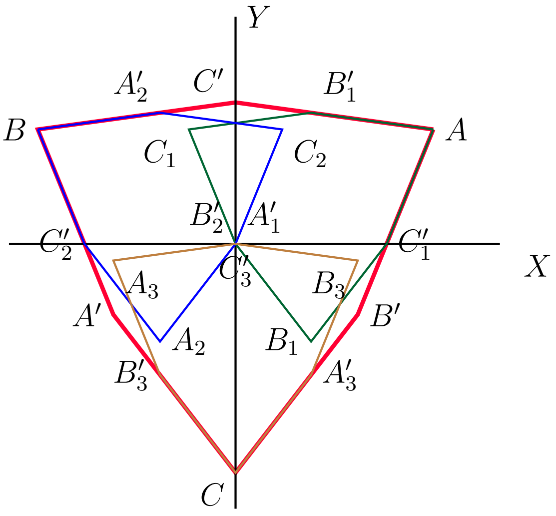
$$h_4(A') = \Big(\frac{1}{s(4s-2)}, \frac{1}{\sqrt{3}s(4s-2)}\Big) = A_4', h_4(B') = \Big(\frac{-1}{s(4s-2)}, \frac{1}{\sqrt{3}s(4s-2)}\Big) = B_4',$$

$$h_4(C') = \Big(0, \frac{-2}{\sqrt{3}s(4s-2)}\Big).$$

For $s \in (\frac{1}{2}, 1)$, we have $\frac{-8s}{\sqrt{3}(4s-2)} < \frac{-2}{\sqrt{3}s(4s-2)} < \frac{-2}{\sqrt{3}(4s-2)}, 0 < \frac{1}{s(4s-2)} < \frac{2}{4s-2}$ and $0 < \frac{1}{\sqrt{3}s(4s-2)} < \frac{2}{\sqrt{3}(4s-2)}$. This implies that $h_4(\text{Con}(ABCA'B'C')) \subset \text{Con}(ABCA'B'C')$. Thus the set $\text{Con}(ABCA'B'C')$ is an invariant set for the IFS $\mathcal{J}$.

Now, we will estimate the overlapping number. For $s = \frac{1+\sqrt{5}}{4}$, one can see that $A_1' = B_2' = C_3' = (0,0)$, $h_1(\text{Con}(ABCA'B'C'))$ is on the right side, $h_2(\text{Con}(ABCA'B'C'))$ is one the left side (mirror image of $h_1(\text{Con}(ABCA'B'C'))$ with respect to Y-axis) and $h_3(\text{Con}(ABCA'B'C'))$ is below the X-axis. Thus, only at $(0,0)$, the overlapping number is 4. This implies that $K \leq 3$ a.e. $x \in A_F$. For a visulalisation, see Figure 8.

Now, we assume that $s \in (\frac{1+\sqrt{5}}{4}, 1)$. In this case, we have $s(4s-2) > 1$. Thus, A_1' and B_1' are in the 1st and 2nd quadrant of the set $\text{Con}(ABCA'B'C')$, respectively. And C_3' is on negative Y-axis. The points C_1' and C_2' are on the above of the X-axis and on line AB' and BA', respectively. Thus, any point of $\text{Con}(ABCA'B'C')$ can lie atmost two of the sets $h_1(\text{Con}(ABCA'B'C'))$, $h_2(\text{Con}(ABCA'B'C'))$ and $h_3(\text{Con}(ABCA'B'C'))$. Thus, $K \leq 3$. This completes the proof. $\square$



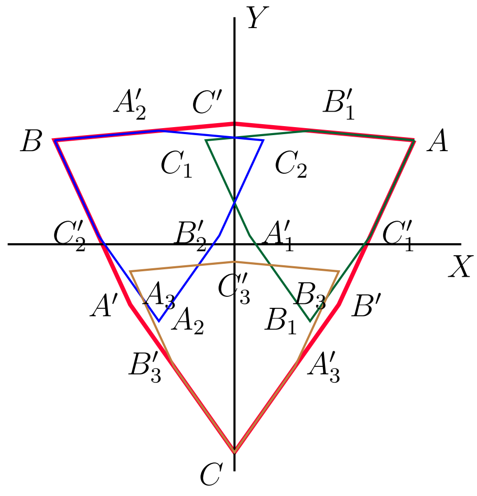


FIGURE 8. The first level cylinder sets for the Furstenberg IFS (5.2).

Proof of Theorem 1.4. Combining Proposition 2.4 and Proposition 5.1, we get $\dim_H(\mu_F) > 3 - t$, where t is the affinity dimension of the IFS $\mathcal{I}$ for every $s \in \left[\frac{1+\sqrt{5}}{4}, 1\right)$.. Then the claim follows by Theorem 2.3. $\square$

5.2. **Almost every-type result.** Next, we discuss almost surely results for the above-constructed fractal surfaces.

Construction of Graph directed IFS associated with the projection of the Furstenberg IFS. One can see that the group $(\mathcal{G})$ generated by the linear part of the Furstenberg IFS $\mathcal{J} = \{h_1, h_2, h_3, h_4\}$ is a finite group of order 12. Precisely, the group elements are as follows:

$$Q_1 = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} \end{bmatrix}, Q_2 = \begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & \frac{-1}{2} \end{bmatrix}, Q_3 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, Q_4 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$Q_5 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, Q_6 = \begin{bmatrix} \frac{-1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}, Q_7 = \begin{bmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}, Q_8 = \begin{bmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & \frac{-1}{2} \end{bmatrix},$$

$$Q_9 = \begin{bmatrix} \frac{-1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} \end{bmatrix}, Q_{10} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, Q_{11} = \begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}, Q_{12} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}.$$

First, we consider a direction $v = (1,1) \in \mathbb{R}^2$. Now, we construct an graph directed IFS associated with the Furstenberg IFS $\mathcal{J}$. First, we define a set of vertices $\mathcal{V} := \{v_1, v_2, \dots, v_{12}\}$ such that $v_i = Q_i^T v$. The set $\mathcal{E}_{l,m}$ denotes the set of all directed edges from vertex v_l to v_m. For $l, m \in \mathcal{V}$, if these exists $k \in \{1,2,3,4\}$ such that $v_l = Q_k v_m$, then we define a directed edge $e \in \mathcal{E}_{l,m}$. Since $\#\{Q_l Q_k : k \in 1,2,3,4\} = 4$ for each $l \in \{1,2,\dots,12\}$, there are only 4 directed edges from the vertex v_l and only 4 directed edges toward the vertex v_l. The set of directed edges is defined by $\mathcal{E} := \{\mathcal{E}_{l,m} : 1 \le l, m \le 12\}$. The directed graph is denoted by $G(\mathcal{V}, \mathcal{E})$. For $e \in \mathcal{E}_{l,m}$, the associated map f_e is defined by

$$f_e(x) = \frac{1}{2s}x + v_l \cdot t_k,$$

where $k \in \{1, 2, 3, 4\}$ such that $v_l = Q_k v_m$ and t_k is the translation vector corresponding to the map h_k in the the Furstenberg IFS $\mathcal{J}$. Let A_F be the attractor of the IFS $\mathcal{J}$. One can also see that for each $l \in \{1, 2, \dots, 12\}$, we have $v_l \cdot t_{k_1} \neq v_l \cdot t_{k_2}$ for all $k_1 \neq k_2 \in \{1, 2, 3, 4\}$. The notation $\mathrm{Proj}_v(A)$ denotes the projection of the set A in the direction v. Thus, we have

$$\begin{aligned}\mathrm{Proj}_{v_l}(A_F) &= \mathrm{Proj}_{v_l}\left(\bigcup_{k=1}^{4} h_k(A_F)\right) = \bigcup_{k=1}^{4} \mathrm{Proj}_{v_l}\left(\frac{1}{2s}Q_k(A_F) + t_k\right) \\ &= \bigcup_{k=1}^{4}\left(\frac{1}{2s}\mathrm{Proj}_{Q_k^T v_l}(A_F) + \mathrm{Proj}_{v_l}(t_k)\right) = \bigcup_{m=1}^{12} \bigcup_{e \in \mathcal{E}_{l,m}} f_e(\mathrm{Proj}_{v_m}(A_F)).\end{aligned}$$

Thus, $\bigcup_{l=1}^{12} \mathrm{Proj}_{v_l}(A_F)$ is the attractor of the graph directed IFS $\{f_e : e \in \mathcal{E}\}$ with directed graph $G(\mathcal{V}, \mathcal{E})$.

Induced Markov chain. Let X_n be a Markov chain on the group $\mathcal{G}$. For each $l \in \{1, 2, \dots, 12\}$, one can see that $\#\{Q_l Q_k : k \in 1, 2, 3, 4\} = 4$. Thus, we define the transition probability for the Markov chain by

$$\mathbb{P}(X_{n+1} = Q_m | X_n = Q_l) = \begin{cases} \frac{1}{4} & \text{if} \quad \exists\, k \in \{1, 2, 3, 4\} \text{ with } Q_l Q_k = Q_m \\ 0 & \text{otherwise.} \end{cases}$$

Let P be the transition matrix associated with the Markov chain X_n. Thus, the matrix P is as follows

$$[P]_{i,j} = \begin{cases} \frac{1}{4} & \text{if} \quad \exists\, k \in \{1, 2, 3, 4\} \text{ with } Q_i Q_k = Q_j \\ 0 & \text{otherwise,} \end{cases}$$

where $[P]_{i,j}$ is the ijth entry of the matrix P for $1 \leq i, j \leq 12$.

We have

$$Q_2 \cdot Q_1 = Q_8,\ Q_1 \cdot Q_2 = Q_9,\ Q_2 \cdot Q_4 = Q_7,\ Q_1 \cdot Q_4 = Q_6 = Q_4 \cdot Q_1,$$

$$Q_1 \cdot Q_3 = Q_8,\ Q_2 \cdot Q_3 = Q_9,\ Q_{11} \cdot Q_1 = Q_7, Q_{12} \cdot Q_1 = Q_{10},$$

$$Q_1 \cdot Q_1 = Q_2 \cdot Q_2 = Q_3 \cdot Q_3 = Q_4 \cdot Q_4 = Q_5 = Id,\ Q_4 \cdot Q_3 = Q_{10},\ Q_1 \cdot Q_{10} = Q_{11},\ Q_2 \cdot Q_{10} = Q_{12}.$$

Now, we define two sets $A = \{Q_5, Q_6, Q_7, Q_8, Q_9, Q_{10}\}$ and $B = \{Q_1, Q_2, Q_3, Q_4, Q_{11}, Q_{12}\}$. By the above one can see that for each $Q_i \in A$, there exist a $k \in \{1, 2, 3, 4\}$ such that $Q_i Q_k \in B$ and vice versa. Thus, the directed graph associated with the transition matrix P is bi-bipartite graph. So, by considering the arrangement $\{Q_5, Q_6, Q_7, Q_8, Q_9, Q_{10}, Q_1, Q_2, Q_3, Q_4,$

$Q_{11}, Q_{12}\}$, the transition matrix P can be rewritten in the following form

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/4 & 0 & 0 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/4 & 0 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/4 & 1/4 & 1/4 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/4 & 1/4 & 1/4 & 0 & 0 & 1/4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 0 & 1/4 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/4 & 0 & 1/4 & 1/4 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/4 & 0 & 0 & 1/4 & 1/4 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 0 & 0 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/4 & 1/4 & 1/4 & 0 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/4 & 1/4 & 0 & 1/4 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since Q_5 is identity matrix (Id), we have

$$\mathbb{P}(X_n = Id | X_0 = Id) = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} P^n \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \frac{\#\{\tau \in \{1,2,3,4\}^n : Q_\tau = Id\}}{4^n}.$$

Clearly, $\mathbb{P}(X_{2n+1} = Id | X_0 = Id) = 0$ for $n \in \mathbb{N} \cup \{0\}$ and $\mathbb{P}(X_2 = Id | X_0 = Id) = \frac{1}{4}$. The period of the directed graph associated with the transition matrix P is defined as follows

$$\text{period} = \text{l.c.d.}\{n \in \mathbb{N} : \mathbb{P}(X_n = Id | X_0 = Id) > 0\}.$$

This implies that period is 2. But, $P^2 = \begin{bmatrix} R & 0 \\ 0 & S \end{bmatrix}$, where R and S are 6×6 matrices. The matrix R is as follows

$$R = \begin{bmatrix} 1/4 & 1/8 & 1/8 & 3/16 & 3/16 & 1/8 \\ 1/8 & 1/4 & 3/16 & 1/8 & 1/8 & 3/16 \\ 1/8 & 3/16 & 1/4 & 1/8 & 1/8 & 3/16 \\ 3/16 & 1/8 & 1/8 & 1/4 & 3/16 & 1/8 \\ 3/16 & 1/8 & 1/8 & 3/16 & 1/4 & 1/8 \\ 1/8 & 3/16 & 3/16 & 1/8 & 1/8 & 1/4 \end{bmatrix}$$

The above matrix R is irreducible and aperiodic (period is 1). Thus by Perron-Frobenius theorem, we get

$$R^n \to \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \end{bmatrix} \quad \text{as} \quad n \to \infty.$$

This implies that

$$\mathbb{P}(X_{2n} = Id | X_0 = Id) = \frac{\#\{\tau \in \{1,2,3,4\}^{2n} : Q_\tau = Id\}}{4^{2n}} \to \frac{1}{6} \quad \text{as} \quad n \to \infty.$$

Thus, these exists a $N_0 \in \mathbb{N}$ such that

$$\#\{\tau \in \{1,2,3,4\}^{2n} : Q_\tau = Id\} \geq \frac{4^{2n}}{12} \quad \forall\, n \geq N_0. \tag{5.3}$$

Non-degeneracy of constructed GD-IFS. Now, we will show that the constructed graph directed IFS is non-degenerate, which will be useful for determining almost surely type results. The set of all infinite-length edges is denoted by $\mathcal{E}^*$. Let $\mathbf{e} = (e_1, e_2, \dots), \mathbf{e}' = (e_1', e_2', \dots) \in \mathcal{E}^*$ such that $\mathbf{e} \neq \mathbf{e}'$ and $e_1 \in \mathcal{E}_{l,m_1}, e_1' \in \mathcal{E}_{l,m_2}$. Let $m \in \mathbb{N}$ be smallest number such that $e_m \neq e_m'$. Then, there exist $k_1 \neq k_2 \in \{1, 2, \dots, 12\}$ such that $e_m \in \mathcal{E}_{l_1,m_{k_1}}$ and $e_m' \in \mathcal{E}_{l_1,m_{k_2}}$. Let Π be the natural projection corresponding to the constructed graph-directed IFS. Thus, we have

$$\Pi(\sigma^{m-1}\mathbf{e}) = f_{e_m}(\Pi(\sigma^m \mathbf{e})) = \frac{1}{2s}(\Pi(\sigma^m \mathbf{e})) + v_{l_1} \cdot t_{r_1}$$

$$\Pi(\sigma^{m-1}\mathbf{e}') = f_{e_m'}(\Pi(\sigma^m \mathbf{e}')) = \frac{1}{2s}(\Pi(\sigma^m \mathbf{e}')) + v_{l_1} \cdot t_{r_2},$$

where $r_1 \neq r_2$. Since $v_{l_1} \cdot t_{r_1} \neq v_{l_1} \cdot t_{r_2}$, we have $\Pi(\sigma^{m-1}\mathbf{e}) \not\equiv \Pi(\sigma^{m-1}\mathbf{e}')$ as analytic functions of s on $(1/2, \infty)$, and in particular, on $(1/2, 1/)$. Thus, we get $\Pi(\mathbf{e}) \not\equiv \Pi(\mathbf{e}')$. This implies that the graph-directed IFS is non-degenerate.

Now, by using the above constructed GD-IFS, we will prove typical type results for the Hausdorff dimension.

Proof of Theorem 1.5. The upper bound follows by (5.1). Now, we show the lower bound.

Set $\mathcal{N}_n := \#\{\tau \in \{1,2,3,4\}^{2n} : Q_\tau = Id\}$. For $n \geq N_0$, we define an self-affine IFS

$$\Phi_n := \{W_\tau : Q_\tau = Id \text{ and } \tau \in \{1,2,3,4\}^{2n}\}.$$

Let A_{Φ_n} be the attractor of the IFS Φ_n. The affinity dimension (t_n) of the self-affine IFS Φ_n is uniquely determined by following equation

$$\sum_{\substack{\tau \in \{1,2,3,4\}^{2n} \\ Q_\tau = Id}} s^{2n} \left(\frac{1}{2^{2n}}\right)^{t_n - 1} = 1$$

Let $(p_i)_{i=1}^{\mathcal{N}_n}$ be a probability vector such that $p_i = \frac{1}{\mathcal{N}_n}$ for all $1 \leq i \leq \mathcal{N}_n$. Let $\mathcal{F}_n := \{h_\tau : Q_\tau = Id \text{ and } \tau \in \{1,2,3,4\}^{2n}\}\}$ be the Furstenberg IFS corresponding to the IFS Φ_n. Let $\mu_{\mathcal{F}}^n$ be the invariant measure corresponding to the IFS $\mathcal{F}_n$ with probability vector $(p_i)_{i=1}^{\mathcal{N}_n}$. Since the graph directed IFS $\{f_e : e \in \mathcal{E}\}$ with directed graph $G(\mathcal{V}, \mathcal{E})$ corresponding to the Furstenberg IFS $\mathcal{J}$ is non-degenerate, the projection of the $\mathcal{F}_n$ on the direction $v = (1,1) \in \mathbb{R}^2$ is also non-degenerate. Then, by the result of Hochman [9], there exist a set $\mathcal{E}_n \subset (\frac{1}{2}, 1)$ with $\dim_H(\mathcal{E}_n) = 0$ such that

$$\dim_H(\mu_{\mathcal{F}}^n) \geq \min\left\{1, \frac{-\sum_{i=1}^{\mathcal{N}_n} p_i \log p_i}{\sum_{i=1}^{\mathcal{N}_n} p_i \log(2s)^{2n}}\right\} = \min\left\{1, \frac{\log \mathcal{N}_n}{2n\log(2s)}\right\} \quad \forall\, s \in \left(\frac{1}{2}, 1\right) \setminus \mathcal{E}_n.$$

Now, by using the estimate (5.3) of $\mathcal{N}_n$, we get

$$\dim_H(\mu_{\mathcal{F}}^n) \geq \min\left\{1, \frac{\log 4}{\log(2s)} - \frac{\log 12}{2n\log 2s}\right\} \quad \forall\, s \in \left(\frac{1}{2}, 1\right) \setminus \mathcal{E}_n.$$

Let $\hat{t}_n \in \mathbb{R}$ be such that

$$\sum_{\substack{\tau \in \{1,2,3,4\}^{2n} \\ Q_\tau = Id}} s^{2n} \left(\frac{1}{2^{2n}}\right)^{\hat{t}_n - 1} \geq \frac{4^{2n}}{12} s^{2n} \left(\frac{1}{2^{2n}}\right)^{\hat{t}_n - 1} = 1.$$

This implies that $t_n \geq \hat{t}_n$ and

$$\hat{t}_n = 1 + \frac{\log 4s}{\log 2} - \frac{\log 12}{2n \log 2} = 3 + \frac{\log s}{\log 2} - \frac{\log 12}{2n \log 2}.$$

Clearly, $\hat{t}_n \to t$ as $n \to \infty$. Since $\frac{\log 4}{\log(2s)} > -\frac{\log s}{\log 2}$ $\forall\ s \in \left(\frac{1}{2}, 1\right)$, we have

$$\min\left\{1, \frac{\log 4}{\log(2s)} - \frac{\log 12}{2n \log 2s}\right\} > 3 - \hat{t}_n \geq 3 - t_n\ \forall\ s \in \left(\frac{1}{2}, 1\right)$$

for all $n \geq N_1$, where $N_1 \in \mathbb{N}$ is some large number. This implies that for $n \geq \max\{N_0, N_1\}$ we get

$$\dim_H(\mu^n_{\mathcal{F}}) > 3 - t_n \quad \forall\ s \in \left(\frac{1}{2}, 1\right) \setminus \mathcal{E}_n.$$

Then, by Theorem 2.3, for $n \geq \max\{N_0, N_1\}$ we obtain

$$\dim_H(A_{\Phi_n}) = t_n \geq \hat{t}_n \quad \forall\ s \in \left(\frac{1}{2}, 1\right) \setminus \mathcal{E}_n.$$

Set $\mathcal{E} := \bigcup_{n=\max\{N_0,N_1\}} \mathcal{E}_n$. Thus, $\dim_H(\mathcal{E}) = 0$. This implies that

$$\dim_H(G(f^*)) \geq \dim_H(A_{\Phi_n}) \geq \hat{t}_n \quad \forall\ s \in \left(\frac{1}{2}, 1\right) \setminus \mathcal{E}.$$

Thus, for all $s \in \left(\frac{1}{2}, 1\right) \setminus \mathcal{E}$, we get

$$\dim_H(G(f^*)) \geq t \quad \forall\ s \in \left(\frac{1}{2}, 1\right) \setminus \mathcal{E}.$$

Since $\dim_H(G(f^*)) \leq \overline{\dim}_B(G(f^*)) \leq t = 3 + \frac{\log(s)}{\log(2)}$, we have

$$\dim_H(G(f^*)) = \dim_B(G(f^*)) = t = 3 + \frac{\log(s)}{\log(2)}$$

for all $s \in \left(\frac{1}{2}, 1\right) \setminus \mathcal{E}$. This completes the proof. $\square$

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