ON THE DIMENSION OF THE GRAPH OF THE CLASSICAL WEIERSTRASS FUNCTION

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Abstract. This paper examines dimension of the graph of the famous Weierstrass non-differentiable function

$$W_{\lambda,b}(x)=\sum_{n=0}^{\infty}\lambda^n\cos(2\pi b^n x)$$

for an integer b larger than 1 and $1/b<\lambda<1$. We prove that for every b there exists (explicitly given) $\lambda_b\in(1/b,1)$ such that the Hausdorff dimension of the graph is equal to $D=2+\frac{\log\lambda}{\log 2}$ for every $\lambda\in(\lambda_b,1)$. We also show that the dimension is equal to D for almost every λ on some larger interval. This partially solves a well-known thirty-year-old conjecture.

1. Introduction and statements

This paper is devoted to the study of dimension of the graph of the famous function

$$W_{\lambda,b}(x)=\sum_{n=0}^{\infty}\lambda^n\cos(2\pi b^n x)$$

for $x\in\mathbb{R}$, where $0<\lambda<1<b$ and $b\lambda>1$, introduced by Weierstrass in 1872 as one of the first examples of a continuous nowhere differentiable function on the real line. In fact, Weierstrass proved the non-differentiability for some values of the parameters, while the complete proof was given by Hardy [11] in 1916. Later, starting from the work of Besicovitch and Ursell [5], the graphs of functions of the form

$$f(x)=\sum_{n=0}^{\infty}b_n^{D-2}\phi(b_n x+\theta_n) \tag{1.1}$$

for non-constant Lipschitz, piecewise C^1, $\mathbb{Z}$-periodic functions $\phi:\mathbb{R}\to\mathbb{R}$ and $1<D<2$, $b_{n+1}/b_n>b>1$, $\theta_n\in\mathbb{R}$ were studied from a geometric point of view as fractal curves in the plane. Much attention was paid to the classical case $b_n=b^n$ for an integer b larger than 1 and $\theta_n=0$. Then the graph of f is an invariant repeller for the expanding dynamical system $\Phi:\mathbb{R}/\mathbb{Z}\times\mathbb{R}\to\mathbb{R}/\mathbb{Z}\times\mathbb{R}$,

$$\Phi(x,y)=\left(bx \pmod 1,\ \frac{y-\phi(x)}{\lambda}\right) \tag{1.2}$$

with Lyapunov exponents $\log 2$, $\log\lambda$ for $\lambda=b^{D-2}$, which allows to use the methods of ergodic theory of smooth dynamical systems.

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The case of the Weierstrass function $W_{\lambda,b}$ for integer b is of particular interest, because then it is the real part of the lacunar (Hadamard gaps) power series

$$w(z) = \sum_{n=0}^{\infty} \lambda^n z^{b^n}, \qquad z \in \mathbb{C},\ |z| \le 1$$

on the unit circle, which relates the problem to harmonic analysis and boundary behaviour of analytic maps. For instance, it was proved by Salem and Zygmund [26] and Kahane, Weiss and Weiss [15] that for λ sufficiently close to 1, the image of the unit circle under w is a Peano curve, i.e. it covers an open subset of the plane. Moreover, Belov [3] and Barański [2] showed that in this case the map w does not preserve (forwardly) Borel sets on the unit circle. The complicated topological boundary behaviour of w was also studied recently by Dong, Lau and Liu in [8].

The graph of a function f of the form (1.1) is approximately self-affine with scales λ and $1/b$, which suggests that its dimension should be equal to

$$D = 2 + \frac{\log \lambda}{\log b}.$$

Indeed, Kaplan, Mallet-Paret and Yorke [14] proved that the box dimension of the graph of f is equal to D. However, the question of determining the Hausdorff dimension turned out to be much more complicated. The conjecture that it is equal to D for the classical Weierstrass case $f = W_{\lambda,b}$ was formulated by Mandelbrot in 1977 [18] and then repeated in many papers, see e.g. [4, 9, 13, 16, 20, 23] and the references therein.

In 1986, Mauldin and Williams [20] proved that if a function f has the form (1.1) with $b_n = b^n$ for an integer b larger than 1, then for given D there exists a constant $C > 0$ such that the Hausdorff dimension of the graph is larger than $D - C/\log b$ for large b. Shortly after, Przytycki and Urbański showed in [23] that if $f = W_{\lambda,b}$ for any integer b larger than 1, then the Hausdorff dimension of the graph is larger than 1. Rezakhanlou [25] proved that the packing dimension of the graph of $W_{\lambda,b}$ is equal to D and in [12], Hu and Lau showed the same for the so-called K-dimension (both are not smaller than the Hausdorff dimension).

In 1992, Ledrappier [16] proved that if f has the form (1.1) with $b_n = 2^n$, $\phi(x) = \mathrm{dist}(x, \mathbb{Z})$ and $\theta = 0$, then the Hausdorff dimension of the graph is equal to D provided the infinite Bernoulli convolutions $\sum_{n=0}^{\infty} \pm 2^{(1-D)n}$, with $\pm$ chosen independently with probability $(1/2, 1/2)$, have absolutely continuous distribution (by the result of Solomyak [29], this holds for almost all $D \in (1, 2)$ with respect to the Lebesgue measure). Analogous result for other functions ϕ was showed by Solomyak in [28].

In 1998, Hunt [13] proved that in the case $b_n = b^n$ for an integer b larger than 1, if one considers the numbers θ_n in (1.1) as independent random variables with uniform distribution on $[0, 1]$, then for many functions ϕ, including $\phi(x) = \cos(2\pi x)$, the Hausdorff dimension of the graph is equal to D almost surely.

It is interesting to notice that in the case $b_{n+1}/b_n \to \infty$ the question of determining the Hausdorff and box dimension of graphs of functions (1.1) can be solved completely, as proved recently by Carvalho [7] and Barański [1]. In this case the Hausdorff and upper box dimension need not coincide.

Recently, Biacino [6] and Fu [10] solved partially the question of determining the Hausdorff dimension of the graph of the classical Weierstrass function $W_{\lambda,b}$, showing that it is equal to D for sufficiently large integers b.

In this paper we make a further step, proving the conjecture for every integer b larger than 1, provided λ is sufficiently close to 1. The proof uses methods of ergodic theory of smooth dynamical systems. In fact, we show that he measure $\mu_{\lambda,b}$ has dimension D, where

$$\mu_{\lambda,b} = ((\mathrm{Id}, W_{\lambda,b})|_{[0,1]})_* \mathcal{L}|_{[0,1]}$$

is the lift of the Lebesgue measure $\mathcal{L}$ on $[0,1]$ to the graph of $W_{\lambda,b}$.

Definition. *We say that a Borel measure μ in a metric space X has local dimension d at a point $x \in X$, if*

$$\lim_{r\to 0} \frac{\log \mu(B_r(x))}{\log r} = d,$$

where $B_r(x)$ denotes the ball of radius r centered at x. If the local dimension of μ exists and is equal to d at μ-almost every x, then we write $\dim \mu = d$.

If $\dim \mu = d$, then every set of positive measure μ has Hausdorff dimension at least d.

Denote by $G_{\lambda,b} \subset \mathbb{R}^2$ the graph of the function $W_{\lambda,b}$ on the interval $[0,1]$, i.e.

$$G_{\lambda,b} = \left\{(x, W_{\lambda,b}(x)) : x \in [0,1]\right\}.$$

Let $\dim_H$ and $\dim_B$ denote, respectively, the Hausdorff and box dimension (for the definition and basic properties of the Hausdorff and box dimension we refer to [9, 19]). As mentioned above, it is well-known that $\dim_B G_{\lambda,b} = D$. Since $\dim_H G_{\lambda,b} \le \dim_B G_{\lambda,b}$, to determine the Hausdorff dimension of $G_{\lambda,b}$ it is sufficient to prove $\dim \mu_{\lambda,b} = D$.

The first of the paper is the following.

Theorem A. *For every positive integer b larger than* 1,

$$\dim \mu_{\lambda,b} = 2 + \frac{\log \lambda}{\log b}$$

for every $\lambda \in (\lambda_b, 1)$, where λ_b is equal to the unique zero of the function

$$h_b(\lambda) = \begin{cases} \frac{1}{4\lambda^2(2\lambda-1)^2} + \frac{1}{16\lambda^2(4\lambda-1)^2} - \frac{5}{64\lambda^2} + \frac{\sqrt{2}}{2\lambda} - 1 & \text{for } b = 2 \\ \frac{1}{(b\lambda-1)^2} + \frac{1}{(b^2\lambda-1)^2} - \sin^2\frac{\pi}{b} & \text{for } b \ge 3 \end{cases}$$

on the interval $(1/b, 1)$. *In particular,*

$$\dim_H G_{\lambda,b} = \dim_B G_{\lambda,b} = 2 + \frac{\log \lambda}{\log b}$$

for every $\lambda \in (\lambda_b, 1)$.

Using Peres–Solomyak transversality methods, we can extend the result for almost every λ on some larger interval. To state the next theorem, we need to recall some definitions related to so-called $(*)$-functions considered in the study of infinite Bernoulli convolutions (see e.g. [21, 22, 28]). For $\beta \ge 1$ let

$$\mathcal{G}_\beta = \left\{ g(t) = 1 + \sum_{n=1}^{\infty} g_n t^n,\ g_n \in [-\beta, \beta] \text{ for } n \ge 1 \right\}.$$

Let $y(\beta)$ be the smallest possible value of positive double roots of functions in $\mathcal{G}_\beta$, i.e.

$$y(\beta) = \inf\left\{ t > 0 : \text{there exists } g \in \mathcal{G}_\beta \text{ such that } g(t) = g'(t) = 0 \right\}.$$

Theorem B. *For every positive integer b larger than* 1,

$$\dim \mu_{\lambda,b} = 2 + \frac{\log \lambda}{\log b}$$

for Lebesgue almost every $\lambda \in (\tilde{\lambda}_b, 1)$, where $\tilde{\lambda}_b$ is equal to the unique root of the equation

$$y\left(\frac{1}{\sqrt{\sin^2(\pi/b) - 1/(b^2\lambda - 1)^2}}\right) = \frac{1}{b\lambda}$$

on the interval $(1/b, 1)$. In particular,

$$\dim_H G_{\lambda,b} = \dim_B G_{\lambda,b} = 2 + \frac{\log \lambda}{\log b}$$

for Lebesgue almost every $\lambda \in (\tilde{\lambda}_b, 1)$.

Estimating the numbers λ_b and $\tilde{\lambda}_b$ in the above theorems, we obtain the following.

Corollary C.

$$\dim_H G_{\lambda,2} = 2 + \frac{\log \lambda}{\log 2} \qquad \textit{for every } \lambda \in (0.9531, 1) \textit{ and almost every } \lambda \in (0.81, 1),$$

$$\dim_H G_{\lambda,3} = 2 + \frac{\log \lambda}{\log 3} \qquad \textit{for every } \lambda \in (0.7269, 1) \textit{ and almost every } \lambda \in (0.55, 1),$$

$$\dim_H G_{\lambda,4} = 2 + \frac{\log \lambda}{\log 4} \qquad \textit{for every } \lambda \in (0.6083, 1) \textit{ and almost every } \lambda \in (0.44, 1).$$

For every $b \geq 5$,

$$\dim_H G_{\lambda,b} = 2 + \frac{\log \lambda}{\log b} \qquad \textit{for every } \lambda \in (0.5448, 1) \textit{ and almost every } \lambda \in (1.04/\sqrt{b}, 1).$$

Obviously, using Theorem A and B, one can get better estimates for $b \geq 5$ (for large b, the numbers λ_b tend to $1/\pi$ and $\tilde{\lambda}_b\sqrt{b}$ tends to $1/\sqrt{\pi}$, see Lemmas 3.4 and 4.1).

2. Background

We consider $G_{\lambda,b}$ as an invariant repeller of the dynamical system (1.2) for $\phi(x) = \cos(2\pi x)$ and use the results of ergodic theory of non-uniformly hyperbolic smooth dynamical systems on manifolds (Pesin theory) developed by Ledrappier and Young in [17] and applied by Ledrappier in [16] to study the graphs of the Weierstrass-type functions. The theory in [17] is valid for smooth diffeomorphisms, so to apply it for Φ one considers the inverse limit (alternatively, it is possible to use analogous theory for smooth endomorphisms developed by Qian, Xie and Zhu in [24]).

For the reader's convenience, let us recall the results of Ledrappier–Young theory from [16, 17] applied for the graph of $W_{\lambda,b}$. (Note that the quoted results are formulated in [16] for $b = 2$. However, the theory is valid for any integer b larger than 1.) Consider the symbolic space

$$\Sigma = \{0, \ldots, b-1\}^{\mathbb{Z}^+}$$

and let

$$\Sigma^* = \bigcup_{n=0}^{\infty} \{0, \ldots, b-1\}^n$$

be the set of finite length words of symbols. For a finite length word $(i_1, \ldots, i_n) \in \Sigma^*$ let $[i_1, \ldots, i_n]$ be the corresponding cylinder set, i.e.

$$[i_1, \ldots, i_n] = \{(j_1, j_2, \ldots) \in \Sigma : j_1 = i_1, \ldots, j_n = i_n\}.$$

Define for $x \in [0, 1]$ and $\gamma \in (1/b, 1)$ a mapping $Y_{x,\lambda}$ from Σ to the real line as follows:

$$Y_{x,\gamma}(\mathbf{i}) = 2\pi \sum_{n=1}^{\infty} \gamma^n \sin\left(2\pi\left(\frac{x}{b^n} + \frac{i_1}{b^n} + \cdots + \frac{i_n}{b}\right)\right), \tag{2.1}$$

where $\mathbf{i} = (i_1, i_2, \ldots)$ and

$$\gamma = \frac{1}{b\lambda}.$$

The latter formula will be used throughout the paper.

Define the inverse of the map Φ from (1.2) as the map $F : [0, 1] \times \mathbb{R} \times \Sigma \to [0, 1] \times \mathbb{R} \times \Sigma$,

$$F(x, y, \mathbf{i}) = \left(\frac{x}{b} + \frac{i_1}{b},\ \lambda y + \phi\left(\frac{x}{b} + \frac{i_1}{b}\right),\ \sigma(\mathbf{i})\right),$$

where $\phi(x) = \cos(2\pi x)$, $\mathbf{i} = (i_1, i_2, \ldots)$ and σ is the left-side shift on Σ. We have

$$F(G_{\lambda,b} \times \Sigma) = G_{\lambda,b} \times \Sigma, \qquad F_*(\mu_{\lambda,b} \times \mathbb{P}) = \mu_{\lambda,b} \times \mathbb{P}.$$

Defining

$$F_i(x, y) = \left(\frac{x}{b} + \frac{i}{b},\ \lambda y + \phi\left(\frac{x}{b} + \frac{i}{b}\right)\right)$$

for $i \in \{0, \ldots, b-1\}$, we have

$$DF_i(x, y) = \begin{bmatrix} 1/b & 0 \\ \phi'(x/b + i/b)/b & \lambda \end{bmatrix}$$

Consider the products of these matrices which arise by composing the maps $F_{i_1}, F_{i_2}, \ldots$ for given $\mathbf{i} = (i_1, i_2, \ldots)$. By the Oseledets multiplicative ergodic theorem, the Lyapunov exponents of the system are equal to $-\log 2$, $\log \lambda$ and there is exactly one strong stable direction in $\mathbb{R}^2$ (corresponding to the exponent $-\log 2$), given by

$$\mathcal{J}_{x,\mathbf{i}} = \begin{bmatrix} 1 \\ -\sum_{n=1}^{\infty} \gamma^n \phi'(x/b^n + i_1/b^n + \cdots + i_n/b) \end{bmatrix} = \begin{bmatrix} 1 \\ Y_{x,\gamma}(\mathbf{i}) \end{bmatrix}$$

for $\gamma = 1/(b\lambda)$. In fact,

$$DF_{i_1}(x, y)(\mathcal{J}_{x,\mathbf{i}}) = \frac{1}{b}\mathcal{J}_{x/b+i_1/b,\, \sigma(\mathbf{i})}.$$

Note that $\mathcal{J}_{x,\mathbf{i}}$ does not depend on y. For given $\mathbf{i}$, the vector field $\mathcal{J}_{x,\mathbf{i}}$ defines a foliation of $(0, 1) \times \mathbb{R}$ into strong stable manifolds, given by parallel smooth curves $\Gamma_{x,y,\mathbf{i}}$ (graphs of functions of the first coordinate).

For the measure $\mu = \mu_{\lambda,b}$ there exists a system of conditional measures $\mu_{x,y,\mathbf{i}}$ on $\Gamma_{x,y,\mathbf{i}}$, associated to this foliation treated as a measurable partition. Take a vertical line ℓ and let $\nu_{x,\mathbf{i}}$ (called transversal measure) be the projection of μ to ℓ along the curves $\Gamma_{x,y,\mathbf{i}}$, $y \in \mathbb{R}$. The following result is a part of the Ledrappier–Young theory from [17] (see also [16, Proposition 2]).

Theorem 2.1 (Ledrappier–Young)**.** *The local dimension of the measure μ exists and is constant μ-almost everywhere. The local dimension of the measure $\mu_{x,y,\mathbf{i}}$ exists, is constant $\mu_{x,y,\mathbf{i}}$-almost everywhere, and is constant for $(\mu \times \mathbb{P})$-almost every $(x, y, \mathbf{i})$. The local dimension of the measure $\nu_{x,\mathbf{i}}$ exists, is constant $\nu_{x,\mathbf{i}}$-almost everywhere, and is constant for $(\mathcal{L} \times \mathbb{P})$-almost every $(x, \mathbf{i})$, where $\mathcal{L}$ is the Lebesgue measure. Moreover,*

$$\dim \mu = \dim \mu_{x,y,\mathbf{i}} + \dim \nu_{x,\mathbf{i}}$$

and

$$\log b \dim \mu_{x,y,\mathbf{i}} - \log \lambda \dim \nu_{x,\mathbf{i}} = \log b.$$

The latter is a "conditional" version of the Pesin entropy formula. As a corollary, one gets

$$\dim \mu_{\lambda,b} = 1 + \left(1 + \frac{\log \lambda}{\log b}\right) \dim \nu_{x,\mathbf{i}}. \tag{2.2}$$

In [16], Ledrappier proved a kind of the Marstrand-type projection theorem, showing that if the distribution of angles of directions $\mathcal{J}_{x,\mathbf{i}}$ has dimension 1, then the dimension of the transversal measure is also equal to 1. More precisely, he proved the following.

Let $\mathbb{P} = \left\{\frac{1}{b}, \dots, \frac{1}{b}\right\}^{\mathbb{Z}^+}$ be the uniform Bernoulli measure on Σ and let

$$m_{x,\gamma} = (Y_{x,\gamma})_* \mathbb{P}.$$

Theorem 2.2 (Ledrappier, [16])**.** *Let $\gamma \in (1/b, 1)$. If $\dim m_{x,\gamma} = 1$ for Lebesgue almost every $x \in (0, 1)$, then $\dim \nu_{x,\mathbf{i}} = 1$.*

In view of (2.2), this implies that to have $\dim \mu_{\lambda,b} = 2 + \log \lambda / \log b$, it is enough to prove that $\dim m_{x,\gamma} = 1$ for Lebesgue almost every $x \in (0, 1)$. In fact, we will show that $m_{x,\gamma}$ is absolutely continuous with respect to the Lebesgue measure for Lebesgue almost every $x \in (0, 1)$, which is a stronger property.

3. Proof of Theorem A

In the proof of Theorem A we use a result due to Tsujii [30]. He considered the SBR measure ν for a skew product $T : \mathbb{S}^1 \times \mathbb{R} \to \mathbb{S}^1 \times \mathbb{R}$ of the form

$$T(x, y) = (bx, \gamma y + \varphi(x))$$

for an integer b larger than 1, a real number $\gamma \in (0, 1)$ and a C^2 function φ on $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. We apply here his results for $\varphi(x) = \sin(2\pi x)$.

Definition 3.1 (Tsujii, [30])**.** *Let $\varepsilon, \delta > 0$, $\mathbf{i}, \mathbf{j} \in \Sigma$, $m \in \mathbb{N}$, $k \in \{1, \dots, b^m\}$. The functions $Y_{\cdot,\gamma}(\mathbf{i})$ and $Y_{\cdot,\gamma}(\mathbf{j})$ are called (ε, δ)-transversal on the interval $I_{m,k} = [(k-1)/b^m, k/b^m]$ if for every $x \in I_{m,k}$,*

$$|Y_{x,\gamma}(\mathbf{i}) - Y_{x,\gamma}(\mathbf{j})| > \varepsilon \quad \text{or} \quad \left|\frac{d}{dx} Y_{x,\gamma}(\mathbf{i}) - \frac{d}{dx} Y_{x,\gamma}(\mathbf{j})\right| > \delta.$$

Otherwise they are called (ε, δ)-tangent on $I_{m,k}$.

Let $\mathbf{e}(n, m; \varepsilon, \delta)$ be the maximum over $k \in \{1, \dots, b^m\}$ and $(i_1, \dots, i_n) \in \Sigma^*$ of the maximal number of finite words $(j_1, \dots, j_n) \in \Sigma^*$ for which there exist $\mathbf{i} \in [i_1, \dots, i_n]$ and $\mathbf{j} \in [j_1, \dots, j_n]$ such that the functions $Y_{\cdot,\gamma}(\mathbf{i})$ and $Y_{\cdot,\gamma}(\mathbf{j})$ are (ε, δ)-tangent on $I_{m,k}$.

Remark. *The above definition is suited to the case $\varphi(x) = \sin(2\pi x)$. In general, instead of $Y_{x,\gamma}(\mathbf{i})$ one should take $\sum_{n=1}^{\infty} \gamma^n \varphi(x/b^n + i_1/b^n + \cdots + i_n/b)$.*

In [30], Tsujii proved the following result.

Theorem 3.2 (Tsujii, [30, Proposition 8])**.** *If* $\mathbf{e}(n,m;\varepsilon,\delta) < \gamma^n b^n$ *for some* $\varepsilon,\delta > 0$ *and positive integers* n,m*, then the SBR measure* ν *for* T *is absolutely continuous with respect to the Lebesgue measure on* $\mathbb{S}^1 \times \mathbb{R}$*.*

There is a direct relation between the SBR measure ν for $\varphi(x) = \sin(2\pi x)$ and the measure $m_{x,\gamma}$. More precisely, we have

$$\nu = \Psi_*(\mathcal{L}|_{\mathbb{S}^1} \times \mathbb{P}),$$

where $\Psi : \mathbb{S}^1 \times \Sigma \to \mathbb{S}^1 \times \mathbb{R}$,

$$\Psi(x, \mathbf{i}) = \left(x, \frac{Y_{x,\gamma}(\mathbf{i})}{2\pi\gamma}\right)$$

and $\mathcal{L}$ is the Lebesgue measure (for details, see [30]). Hence, for a measurable $A \subset \mathbb{S}^1 \times \mathbb{R}$, we have

$$\nu(A) = (\mathcal{L}|_{\mathbb{S}^1} \times \mathbb{P})\left(\left\{(x,\mathbf{i}) : \left(x, \frac{Y_{x,\gamma}(\mathbf{i})}{2\pi\gamma}\right) \in A\right\}\right) = \int_{\mathbb{S}^1} m_{x,\gamma}(\{2\pi\gamma y : (x,y) \in A\})dx.$$

This easily implies the following lemma.

Lemma 3.3. *If the SBR measure* ν *for* $T(x,y) = (bx, \gamma y + \sin(2\pi x))$ *is absolutely continuous, then the measure* $m_{x,\gamma}$ *is absolutely continuous for Lebesgue almost every* $x \in (0,1)$*, in particular* $\dim m_{x,\gamma} = 1$ *for Lebesgue almost every* $x \in (0,1)$*.*

Now we will find conditions under which the measure ν is absolutely continuous. To use Theorem 3.2, we check the transversality condition for the functions $Y_{\cdot,\gamma}$. First, we prove the existence of the numbers λ_b defined in Theorem A.

Lemma 3.4. *For every integer* b *larger than* 1*, the function* h_b *is strictly decreasing on the interval* $(1/b, 1)$ *and has a unique zero* $\lambda_b \in (1/b,1)$*. In particular,* $\lambda_2 < 0.9531, \lambda_3 < 0.7269$*,* $\lambda_4 < 0.6083$ *and* $\lambda_b < 0.5448$ *for* $b \geq 5$*. Moreover,* $\lambda_b \to 1/\pi$ *as* $b \to \infty$*.*

Proof. Consider first the case $b = 2$. We easily check

$$\frac{d}{d\lambda}\left(-\frac{5}{64\lambda^2} + \frac{\sqrt{2}}{2\lambda}\right) < 0$$

for $\lambda \in (1/2, 1)$, which immediately implies that the function h_2 is strictly decreasing on the interval $(1/2, 1]$. Moreover, $h_2(\lambda) \to +\infty$ as $\lambda \to (1/2)^+$ and $h_2(1) < 0$. Hence, h_2 has a unique zero $\lambda_2 \in (1/2, 1)$.

Consider now the case $b \geq 3$. It is obvious that h_b is strictly decreasing on the interval $(1/b, 1]$ and tends to $+\infty$ as $\lambda \to (1/b)^+$. Using the inequality $\sin x > x - x^3/6$ for $x > 0$, we get

$$h_b(\lambda) < \frac{1}{(b\lambda - 1)^2} + \frac{1}{(b^2\lambda - 1)^2} + \frac{\pi^4}{3b^4} - \frac{\pi^2}{b^2} = \frac{H_b(\lambda)}{b^2}$$

for

$$H_b(\lambda) = \frac{1}{(\lambda - 1/b)^2} + \frac{1}{(b\lambda - 1/b)^2} + \frac{\pi^4}{3b^2} - \pi^2.$$

For $\lambda \in (1/b, 1]$, the function $b \mapsto H_b(\lambda)$ is strictly decreasing. Moreover, $H_3(1) < 0$, so $h_b(1) < 0$ for $b \geq 3$. This proves the existence of the unique zero $\lambda_b \in (1/b, 1)$ of the function h_b.

One can directly check that $h_2(0.9531)$, $h_3(0.7269)$, $h_4(0.6083) < 0$, which shows $\lambda_2 < 0.9531$, $\lambda_3 < 0.7269$, $\lambda_4 < 0.6083$. Moreover, $H_5(0.5448) < 0$, so $H_b(0.5448) < 0$ for every $b \geq 5$, which implies $\lambda_b < 0.5448$ for $b \geq 5$. The last assertion of the lemma follows easily from the definition of the function h_b and the fact $\lim_{x\to 0} \sin x/x = 1$. □

Now we prove the transversality condition for the functions $Y_{\cdot,\gamma}$.

Proposition 3.5. *If $\gamma \in (1/b, 1/(b\lambda_b))$, then there exists $\delta > 0$ such that for every $\mathbf{i} = (i_1, i_2, \ldots)$, $\mathbf{j} = (j_1, j_2, \ldots) \in \Sigma$ with $i_1 \neq j_1$ and every $x \in [0, 1]$,*

$$|Y_{x,\gamma}(\mathbf{i}) - Y_{x,\gamma}(\mathbf{j})| > \delta \quad or \quad \left|\frac{d}{dx}Y_{x,\gamma}(\mathbf{i}) - \frac{d}{dx}Y_{x,\gamma}(\mathbf{j})\right| > \delta.$$

Proof. Fix $\gamma \in (1/b, 1/(b\lambda_b))$. Suppose the assertion does not hold. Then for every $\delta > 0$ there exist $\mathbf{i} = (i_1, i_2, \ldots)$, $\mathbf{j} = (j_1, j_2, \ldots) \in \Sigma$ with $i_1 \neq j_1$ and $x \in [0, 1]$, such that

$$|Y_{x,\gamma}(\mathbf{i}) - Y_{x,\gamma}(\mathbf{j})| \leq \delta, \quad \left|\frac{d}{dx}Y_{x,\gamma}(\mathbf{i}) - \frac{d}{dx}Y_{x,\gamma}(\mathbf{j})\right| \leq \delta. \tag{3.1}$$

First, consider the case $b \geq 3$. By the definition of $Y_{x,\gamma}$ (see (2.1)),

$$\begin{aligned}
|Y_{x,\gamma}(\mathbf{i}) - Y_{x,\gamma}(\mathbf{j})| &\geq 2\pi\gamma \left|\sin\left(2\pi\frac{x+i_1}{b}\right) - \sin\left(2\pi\frac{x+j_1}{b}\right)\right| - 4\pi\sum_{n=2}^{\infty}\gamma^n \\
&= 4\pi\gamma \sin\left(2\pi\frac{|i_1-j_1|}{2b}\right)\left|\cos\left(2\pi\frac{2x+i_1+j_1}{2b}\right)\right| - \frac{4\pi\gamma^2}{1-\gamma} \\
&\geq 4\pi\gamma \sin\frac{\pi}{b}\left|\cos\left(2\pi\frac{2x+i_1+j_1}{2b}\right)\right| - \frac{4\pi\gamma^2}{1-\gamma},
\end{aligned} \tag{3.2}$$

as $1 \leq |i_1 - j_1| \leq b - 1$. Similarly, since

$$\frac{d}{dx}Y_{x,\gamma}(\mathbf{i}) = 4\pi^2\sum_{n=1}^{\infty}\left(\frac{\gamma}{b}\right)^n \cos\left(2\pi\left(\frac{x}{b^n} + \frac{i_1}{b^n} + \cdots + \frac{i_n}{b}\right)\right),$$

we obtain

$$\begin{aligned}
\left|\frac{d}{dx}Y_{x,\gamma}(\mathbf{i}) - \frac{d}{dx}Y_{x,\gamma}(\mathbf{j})\right| &\geq \frac{4\pi^2\gamma}{b}\left|\cos\left(2\pi\frac{x+i_1}{b}\right) - \cos\left(2\pi\frac{x+j_1}{b}\right)\right| - 8\pi^2\sum_{n=2}^{\infty}\left(\frac{\gamma}{b}\right)^n \\
&= \frac{8\pi^2\gamma}{b}\sin\left(2\pi\frac{|i_1-j_1|}{2b}\right)\left|\sin\left(2\pi\frac{2x+i_1+j_1}{2b}\right)\right| - \frac{8\pi^2\gamma^2}{b(b-\gamma)} \\
&\geq \frac{8\pi^2\gamma}{b}\sin\frac{\pi}{b}\left|\sin\left(2\pi\frac{2x+i_1+j_1}{2b}\right)\right| - \frac{8\pi^2\gamma^2}{b(b-\gamma)}.
\end{aligned} \tag{3.3}$$

By (3.1), (3.2) and (3.3),

$$\begin{aligned}
\sin\frac{\pi}{b}\left|\cos\left(2\pi\frac{2x+i_1+j_1}{2b}\right)\right| &\leq \frac{\gamma}{1-\gamma} + \frac{\delta}{4\pi\gamma}, \\
\sin\frac{\pi}{b}\left|\sin\left(2\pi\frac{2x+i_1+j_1}{2b}\right)\right| &\leq \frac{\gamma}{b-\gamma} + \frac{\delta b}{8\pi^2\gamma}.
\end{aligned}$$

Taking the sum of the squares of the two inequalities, we get

$$\sin^2\frac{\pi}{b} \leq \left(\frac{\gamma}{1-\gamma} + \frac{\delta}{4\pi\gamma}\right)^2 + \left(\frac{\gamma}{b-\gamma} + \frac{\delta b}{8\pi^2\gamma}\right)^2.$$

Since δ is arbitrarily small, in fact this implies

$$0 \leq \frac{\gamma^2}{(1-\gamma)^2} + \frac{\gamma^2}{(b-\gamma)^2} - \sin^2\frac{\pi}{b} = h_b(\lambda)$$

for $\lambda = 1/(b\gamma) > \lambda_b$, which contradicts Lemma 3.4. This ends the proof in the case $b \geq 3$.

Consider now the case $b = 2$. We improve the estimates made by Tsujii in [30, Appendix]. In this case we need to consider also the second term of $Y_{x,\gamma}$. Since $i_1 \neq j_1$, we can assume $i_1 = 1$, $j_1 = 0$. Then

$$\begin{aligned}
&|Y_{x,\gamma}(\mathbf{i}) - Y_{x,\gamma}(\mathbf{j})| \\
&\geq 2\pi\gamma \left| \sin(\pi(x+1)) - \sin(\pi x) + \gamma\left(\sin\left(\pi\frac{x+1+2i_2}{2}\right) - \sin\left(\pi\frac{x+2j_2}{2}\right)\right)\right| - 4\pi\sum_{n=3}^{\infty}\gamma^n \\
&= 4\pi\gamma\left|\sin(\pi x) - \gamma\left(\sin\left(\pi\frac{1+2(i_2-j_2)}{4}\right)\cos\left(\pi\frac{2x+1+2(i_2+j_2)}{4}\right)\right)\right| - \frac{4\pi\gamma^3}{1-\gamma}
\end{aligned}$$

and

$$\begin{aligned}
&\left|\frac{d}{dx}Y_{x,\gamma}(\mathbf{i}) - \frac{d}{dx}Y_{x,\gamma}(\mathbf{j})\right| \\
&\geq 2\pi^2\gamma\left|\cos(\pi(x+1)) - \cos(\pi x) + \frac{\gamma}{2}\left(\cos\left(\pi\frac{x+1+2i_2}{2}\right) - \cos\left(\pi\frac{x+2j_2}{2}\right)\right)\right| - 8\pi^2\sum_{n=3}^{\infty}\left(\frac{\gamma}{2}\right)^n \\
&= 4\pi^2\gamma\left|\cos(\pi x) + \frac{\gamma}{2}\left(\sin\left(\pi\frac{1+2(i_2-j_2)}{4}\right)\sin\left(\pi\frac{2x+1+2(i_2+j_2)}{4}\right)\right)\right| - \frac{2\pi^2\gamma^3}{2-\gamma}
\end{aligned}$$

which together with (3.1) implies

$$\begin{aligned}
\left|\sin(\pi x) - \gamma\left(\sin\left(\pi\frac{1+2(i_2-j_2)}{4}\right)\cos\left(\pi\frac{2x+1+2(i_2+j_2)}{4}\right)\right)\right| &\leq \frac{\gamma^2}{1-\gamma} + \frac{\delta}{4\pi\gamma}, \\
\left|\cos(\pi x) + \frac{\gamma}{2}\left(\sin\left(\pi\frac{1+2(i_2-j_2)}{4}\right)\sin\left(\pi\frac{2x+1+2(i_2+j_2)}{4}\right)\right)\right| &\leq \frac{\gamma^2}{2(2-\gamma)} + \frac{\delta}{4\pi^2\gamma}.
\end{aligned}$$

Recall that i_2, j_2, x depend on δ. Taking a sequence of δ-s tending to 0 we can choose a subsequence such that i_2, j_2, x converge, so by continuity we can assume

$$\begin{aligned}
\left|\sin(\pi x) - \gamma\left(\sin\left(\pi\frac{1+2(i_2-j_2)}{4}\right)\cos\left(\pi\frac{2x+1+2(i_2+j_2)}{4}\right)\right)\right| &\leq \frac{\gamma^2}{1-\gamma}, \\
\left|\cos(\pi x) + \frac{\gamma}{2}\left(\sin\left(\pi\frac{1+2(i_2-j_2)}{4}\right)\sin\left(\pi\frac{2x+1+2(i_2+j_2)}{4}\right)\right)\right| &\leq \frac{\gamma^2}{2(2-\gamma)}.
\end{aligned}$$

for some $i_2, j_2 \in \{0,1\}$ and $x \in [0,1]$. Taking the sum of the squares of the two inequalities and noting that $\sin^2(\pi(1+2(i_2-j_2))/4) = 1/2$, we obtain

$$g(x) \geq 0, \tag{3.4}$$

where

$$g(t) = \tilde{g}(t) - \frac{3\gamma^2}{8}\cos^2\left(\pi\frac{2t+1+2(i_2+j_2)}{4}\right)$$

for

$$\begin{aligned}\tilde{g}(t) = &\frac{\gamma^4}{(1-\gamma)^2} + \frac{\gamma^4}{4(2-\gamma)^2} - \frac{\gamma^2}{8} - 1 \\ &+ 2\gamma \sin\left(\pi \frac{1 + 2(i_2 - j_2)}{4}\right) \sin(\pi t) \cos\left(\pi \frac{2t + 1 + 2(i_2 + j_2)}{4}\right) \\ &- \gamma \sin\left(\pi \frac{1 + 2(i_2 - j_2)}{4}\right) \cos(\pi t) \sin\left(\pi \frac{2t + 1 + 2(i_2 + j_2)}{4}\right).\end{aligned}$$

We have

$$\begin{aligned}g'(t) = \frac{3\pi\gamma}{8} \cos\left(\pi \frac{2t + 1 + 2(i_2 + j_2)}{4}\right) \\ \left(4 \sin\left(\pi \frac{1 + 2(i_2 - j_2)}{4}\right) \cos(\pi t) + \gamma \sin\left(\pi \frac{2t + 1 + 2(i_2 + j_2)}{4}\right)\right)\end{aligned}$$

and

$$\tilde{g}'(t) = \frac{3\pi\gamma}{2} \sin\left(\pi \frac{1 + 2(i_2 - j_2)}{4}\right) \cos(\pi t) \cos\left(\pi \frac{2t + 1 + 2(i_2 + j_2)}{4}\right).$$

Now we consider four cases, depending on the values of i_2, j_2.

First, let $i_2 = j_2 = 0$. Then

$$\tilde{g}'(t) = \frac{3\sqrt{2}\pi\gamma}{4} \cos(\pi t) \cos\left(\pi \frac{2t + 1}{4}\right) \geq 0$$

for $t \in [0, 1]$. Hence,

$$g(x) \leq \tilde{g}(x) \leq \tilde{g}(1) = \frac{\gamma^4}{(1-\gamma)^2} + \frac{\gamma^4}{4(2-\gamma)^2} - \frac{\gamma^2}{8} + \frac{\gamma}{2} - 1. \tag{3.5}$$

Let now $i_2 = j_2 = 1$. Then

$$\tilde{g}'(t) = -\frac{3\sqrt{2}\pi\gamma}{4} \cos(\pi t) \cos\left(\pi \frac{2t + 1}{4}\right) \leq 0$$

for $t \in [0, 1]$, so

$$g(x) \leq \tilde{g}(x) \leq \tilde{g}(0) = \frac{\gamma^4}{(1-\gamma)^2} + \frac{\gamma^4}{4(2-\gamma)^2} - \frac{\gamma^2}{8} + \frac{\gamma}{2} - 1. \tag{3.6}$$

The third case is $i_2 = 1$, $j_2 = 0$. Then

$$g'(t) = -\frac{3\pi\gamma}{8} \sin\left(\pi \frac{2t + 1}{4}\right) \left(2\sqrt{2} \cos(\pi t) + \gamma \cos\left(\pi \frac{2t + 1}{4}\right)\right) \begin{cases} \leq 0 & \text{for } t \in [0, 1/2] \\ > 0 & \text{for } t \in (1/2, 1], \end{cases}$$

which implies

$$g(x) \leq \max(g(0), g(1)) = \frac{\gamma^4}{(1-\gamma)^2} + \frac{\gamma^4}{4(2-\gamma)^2} - \frac{5\gamma^2}{16} - \frac{\gamma}{2} - 1. \tag{3.7}$$

The last case is $i_2 = 0$, $j_2 = 1$. Then

$$\begin{aligned} g'(t) &= -\frac{3\pi\gamma}{8}\sin\left(\pi\frac{2t+1}{4}\right)\left(-2\sqrt{2}\cos(\pi t) + \gamma\cos\left(\pi\frac{2t+1}{4}\right)\right) \\ &= -\frac{3\sqrt{2}\pi\gamma}{16}\sin\left(\pi\frac{2t+1}{4}\right)\left(\cos\frac{\pi t}{2} - \sin\frac{\pi t}{2}\right)\left(\gamma - 4\left(\cos\frac{\pi t}{2} + \sin\frac{\pi t}{2}\right)\right) \\ &\begin{cases} \geq 0 & \text{for } t \in [0, 1/2] \\ < 0 & \text{for } t \in (1/2, 1], \end{cases} \end{aligned}$$

since $\gamma - 4(\cos(\pi t/2) + \sin(\pi t/2)) \leq \gamma - 4 < 0$ for $t \in [0, 1]$. Hence,

$$g(x) \leq g(1/2) = \frac{\gamma^4}{(1-\gamma)^2} + \frac{\gamma^4}{4(2-\gamma)^2} - \frac{5\gamma^2}{16} + \sqrt{2}\gamma - 1. \tag{3.8}$$

Considering the conditions (3.5)–(3.8) we easily conclude that the largest upper estimate for $g(x)$ appears in (3.8). Therefore, by (3.4), in all cases we have

$$0 \leq \frac{\gamma^4}{(1-\gamma)^2} + \frac{\gamma^4}{4(2-\gamma)^2} - \frac{5\gamma^2}{16} + \sqrt{2}\gamma - 1 = h_2(\lambda)$$

for $\lambda = 1/(2\gamma) > \lambda_2$, which contradicts Lemma 3.4. This ends the proof in the case $b = 2$. □

To conclude the proof of Theorem A, it is enough to notice that by Proposition 3.5, for $\lambda \in (\lambda_b, 1)$ we have $\mathbf{e}(1, 1; \delta, \delta) = 1 < \gamma b$ and use Theorem 3.2, Lemma 3.3, Theorem 2.2 and (2.2). The estimates for λ_2, λ_3 and λ_4 in Corollary C follow from Lemma 3.4.

4. Proof of Theorem B

Using the transversality method developed by Peres and Solomyak in the study of infinite Bernoulli convolutions (see [21, 22]), with a minor modification on the standard argument, we will show that $m_{x,\gamma}$ is absolutely continuous for Lebesgue almost every $(x, \gamma) \in (0, 1) \times (1/b, 1/(b\tilde{\lambda}_b))$. The statement will follow from the Fubini theorem.

First, we prove the existence of the numbers $\tilde{\lambda}_b$ defined in Theorem B.

Lemma 4.1. *For every integer b larger than 1 there exists a unique number $\tilde{\lambda}_b \in (1/b, 1)$ such that*

$$y\left(\frac{1}{\sqrt{\sin^2(\pi/b) - 1/(b^2\tilde{\lambda}_b - 1)^2}}\right) = \frac{1}{b\tilde{\lambda}_b}$$

and for $\lambda \in (1/b, 1)$,

$$y\left(\frac{1}{\sqrt{\sin^2(\pi/b) - 1/(b^2\lambda - 1)^2}}\right) < \frac{1}{b\lambda} \qquad \Longleftrightarrow \lambda \in (1/b, \tilde{\lambda}_b).$$

Moreover, $\tilde{\lambda}_b < \lambda_b$ for every $b \geq 2$, $\tilde{\lambda}_b < 1.04/\sqrt{b}$ for every $b \geq 5$ and $\tilde{\lambda}_b\sqrt{b} \to 1/\sqrt{\pi}$ as $b \to \infty$.

Proof. First, note that

$$\sin\frac{\pi}{b} > \frac{1}{b^2\lambda - 1}$$

for every $\lambda \in (1/b, 1)$. Indeed, for $b = 2$ it is obvious and for $b \geq 3$,

$$\sin\frac{\pi}{b} - \frac{1}{b^2\lambda - 1} > \sin\frac{\pi}{b} - \frac{1}{b-1} > 0$$

since $h_b(1) < 0$ (see the proof of Lemma 3.4). This implies that

$$\beta = \beta(\lambda) = \frac{1}{\sqrt{\sin^2(\pi/b) - 1/(b^2\lambda - 1)^2}}$$

is well-defined for $\lambda \in (1/b, 1)$. Obviously, $\beta > 1$.

It is known (see [22]) that for $\beta \geq 1$ the function $\beta \mapsto y(\beta)$ is strictly decreasing, continuous and satisfies

$$1 > y(\beta) \geq \frac{1}{1+\sqrt{\beta}}. \tag{4.1}$$

Moreover,

$$y(\beta) = \frac{1}{1+\sqrt{\beta}} \qquad \text{for } \beta \geq 3 + \sqrt{8}. \tag{4.2}$$

This implies that $y(\beta) - 1/(b\lambda)$ strictly increases with respect to $\lambda \in (1/b, 1)$, moreover $y(\beta) - 1/(b\lambda) < 0$ for λ sufficiently close to $1/b$ and

$$y(\beta) - \frac{1}{(b\lambda)} > \frac{1}{1+\sqrt{\beta}} - \frac{1}{b\lambda} \tag{4.3}$$

for $\lambda \in (1/b, 1)$. By the definition of β, the inequality

$$\frac{1}{1+\sqrt{\beta}} - \frac{1}{(b\lambda)} > 0 \tag{4.4}$$

is equivalent to $\tilde{h}_b(\lambda) < 0$ for

$$\tilde{h}_b(\lambda) = \frac{1}{(b\lambda - 1)^4} + \frac{1}{(b^2\lambda - 1)^2} - \sin^2\frac{\pi}{b}.$$

We have $\tilde{h}_b(\lambda) < h_b(\lambda)$, so by Lemma 3.4, the inequality (4.4) holds for λ sufficiently close to 1. By (4.3), $y(\beta) - 1/(b\lambda) > 0$ for λ sufficiently close to 1. This implies that there exists a unique number $\tilde{\lambda}_b \in (1/b, 1)$ such that $\tilde{\lambda}_b < \lambda_b$ and $y(\beta) = 1/(b\tilde{\lambda})$.

Like in the proof of Lemma 3.4, using the inequality $\sin x - x^3/6$ for $x > 0$, we obtain

$$\tilde{h}_b(\lambda) < \frac{1}{(b\lambda - 1)^4} + \frac{1}{(b^2\lambda - 1)^2} + \frac{\pi^4}{3b^4} - \frac{\pi^2}{b^2} = \frac{\tilde{H}_b(\lambda)}{b^2}$$

for

$$\tilde{H}_b(\lambda) = \frac{1}{(\sqrt{b}\lambda - 1/\sqrt{b})^4} + \frac{1}{(b\lambda - 1/b)^2} + \frac{\pi^4}{3b^2} - \pi^2.$$

Substituting $\lambda = c/\sqrt{b}$ for $c > 0$, we get

$$\tilde{H}_b(c/\sqrt{b}) = \frac{1}{(c - 1/\sqrt{b})^4} + \frac{1}{(c\sqrt{b} - 1/b)^2} + \frac{\pi^4}{3b^2} - \pi^2.$$

The function $\tilde{H}_b(c/\sqrt{b})$ is strictly decreasing with respect to c and b and one can directly check $\tilde{H}_5(1.04/\sqrt{5}) < 0$. This implies that $\tilde{\lambda}_b < 1.04/\sqrt{b}$ for every $b \geq 5$.

For $\beta \geq 19$,

$$\beta > \frac{1}{\sin(\pi/19)} > \frac{19}{\pi} > 3 + \sqrt{8},$$

so by (4.2), the number $\tilde{\lambda}_b$ is equal to the unique zero of the function $\tilde{h}_b$ on the interval $(1/b, 1)$. This easily implies that $\tilde{\lambda}_b\sqrt{b} \to 1/\sqrt{\pi}$ as $b \to \infty$ (the details are left to the reader). □

Let

$$\tilde{\gamma}_b = \frac{1}{b\tilde{\lambda}_b}.$$

Now we prove a modified transversality condition for the functions $Y_{\cdot,\cdot}(\mathbf{i})$. The trick we use is to consider transversality with respect to two variables x, γ.

Proposition 4.2. *For every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $\mathbf{i} = (i_1, i_2, \dots)$, $\mathbf{j} = (j_1, j_2, \dots) \in \Sigma$ with $i_1 \neq j_1$,*

$$|Y_{x,\gamma}(\mathbf{i}) - Y_{x,\gamma}(\mathbf{j})| > \delta \quad \text{or} \quad \left|\frac{d}{dx}Y_{x,\gamma}(\mathbf{i}) - \frac{d}{dx}Y_{x,\gamma}(\mathbf{j})\right| + \left|\frac{d}{d\gamma}Y_{x,\gamma}(\mathbf{i}) - \frac{d}{d\gamma}Y_{x,\gamma}(\mathbf{j})\right| > \delta$$

for every $x \in (0,1)$ and $\gamma \in (1/b, \tilde{\gamma}_b - \varepsilon)$.

Proof. The proof is similar to the proof of Proposition 3.5. Suppose that the statement does not hold. Then for every $\delta > 0$ there exist $\mathbf{i} = (i_1, i_2, \dots)$, $\mathbf{j} = (j_1, j_2, \dots) \in \Sigma$ with $i_1 \neq j_1$, $x \in (0,1)$ and $\gamma \in (1/b + \varepsilon, \tilde{\gamma}_b - \varepsilon)$, such that

$$|Y_{x,\gamma}(\mathbf{i}) - Y_{x,\gamma}(\mathbf{j})| \leq \delta, \quad \left|\frac{d}{dx}Y_{x,\gamma}(\mathbf{i}) - \frac{d}{dx}Y_{x,\gamma}(\mathbf{j})\right| \leq \delta, \quad \left|\frac{d}{d\gamma}Y_{x,\gamma}(\mathbf{i}) - \frac{d}{d\gamma}Y_{x,\gamma}(\mathbf{j})\right| \leq \delta. \tag{4.5}$$

Repeating the estimates in (3.3), we obtain

$$\left|\frac{d}{dx}Y_{x,\gamma}(\mathbf{i}) - \frac{d}{dx}Y_{x,\gamma}(\mathbf{j})\right| \geq \frac{8\pi^2\gamma}{b}\sin\frac{\pi}{b}\left|\sin\left(2\pi\frac{2x+i_1+j_1}{2b}\right)\right| - \frac{8\pi^2\gamma^2}{b(b-\gamma)}. \tag{4.6}$$

By (4.5) and (4.6),

$$\sin\frac{\pi}{b}\left|\sin\left(\frac{\pi(2x+i_1+j_1)}{b}\right)\right| \leq \frac{\gamma}{b-\gamma} + \frac{\delta b}{8\pi^2\gamma} < \frac{\gamma}{b-\gamma} + \frac{\delta b^2}{8\pi^2} < \frac{1}{b-1} + \frac{\delta b^2}{8\pi^2}. \tag{4.7}$$

By the definition of $Y_{x,\gamma}$ (see (2.1)), we have

$$Y_{x,\gamma}(\mathbf{i}) - Y_{x,\gamma}(\mathbf{j}) = 2\pi\sum_{n=1}^{\infty} y_n\gamma^n,$$

where

$$y_1 = \sin\left(2\pi\frac{x+i_1}{b}\right) - \sin\left(2\pi\frac{x+j_1}{b}\right) = 2\sin\left(2\pi\frac{i_1-j_1}{2b}\right)\cos\left(2\pi\frac{2x+i_1+j_1}{2b}\right)$$

and $|y_n| \leq 2$ for $n \geq 2$. Using the fact $i_1 \neq j_1$ and (4.7), we obtain

$$\begin{aligned} |y_1| &\geq 2\sin\frac{\pi}{b}\left|\cos\left(2\pi\frac{2x+i_1+j_1}{2b}\right)\right| \\ &> 2\sqrt{\sin^2\frac{\pi}{b} - \left(\frac{\gamma}{b-\gamma} + \frac{\delta b}{8\pi^2\gamma}\right)^2} \\ &> 2\sqrt{\sin^2\frac{\pi}{b} - \left(\frac{1}{b-1} + \frac{\delta b^2}{8\pi^2}\right)^2}, \end{aligned} \tag{4.8}$$

in particular $y_1 \neq 0$ for sufficiently small δ (because $h_b(1) < 0$, see the proof of Lemma 3.4). Hence, for the function

$$g(t) = \frac{Y_{x,t}(\mathbf{i}) - Y_{x,t}(\mathbf{j})}{2\pi y_1 t}$$

we have

$$g(t) = 1 + \sum_{n=1}^{\infty} g_n t^n,$$

where

$$|g_n| = \frac{|y_{n+1}|}{|y_1|} < \frac{1}{\sqrt{\sin^2(\pi/b) - (\gamma/(b-\gamma) + \delta b/(8\pi^2\gamma))^2}}.$$

This implies that $g \in \mathcal{G}_\beta$ for

$$\beta = \frac{1}{\sqrt{\sin^2(\pi/b) - (\gamma/(b-\gamma) + \delta b/(8\pi^2\gamma))^2}}.$$

On the other hand, by (4.5) and (4.8),

$$|g(\gamma)| \leq \frac{\delta}{2\pi|y_1|\gamma} < \frac{\delta b}{4\pi\sqrt{\sin^2(\pi/b) - (1/(b-1) + \delta b^2/(8\pi^2))^2}} \tag{4.9}$$

and

$$|g'(\gamma)| \leq \frac{(\gamma+1)\delta}{2\pi|y_1|\gamma^2} < \frac{\delta b^2}{2\pi\sqrt{\sin^2(\pi/b) - (1/(b-1) + \delta b^2/(8\pi^2))^2}} \tag{4.10}$$

Note that g, γ and β depend on δ. Take a sequence of δ-s tending to 0. Then we can choose a subsequence such that $\gamma \to \gamma_* \in [1/b, \tilde{\gamma}_b - \varepsilon]$, $\beta \to \beta_*$ for

$$\beta_* = \frac{1}{\sqrt{\sin^2(\pi/b) - (\gamma_*/(b-\gamma_*))^2}} < \frac{1}{\sqrt{\sin^2(\pi/b) - (\tilde{\gamma}_b/(b-\tilde{\gamma}_b))^2}}$$

and g converges uniformly in $[1/b, \tilde{\gamma}_b]$ to a function $g_* \in \mathcal{G}_{\beta_*}$. Since the right-hand sides of (4.9) and (4.10) tend to 0 as $\delta \to 0$, we obtain

$$g_*(\gamma_*) = g_*'(\gamma_*) = 0,$$

so $y(\beta_*) \leq \gamma^*$. This is a contradiction, because by Lemma 4.1,

$$y(\beta_*) = y\left(\frac{1}{\sqrt{\sin^2(\pi/b) - 1/(b^2\lambda_* - 1)^2}}\right) > \frac{1}{b\lambda_*} = \gamma_*$$

for $\lambda_* = 1/(b\gamma_*) > 1/(b\tilde{\gamma}_b) = \tilde{\lambda}_b$. This ends the proof. □

As a simple consequence of the previous proposition one can prove the following statement (for the proof we refer to [27, Lemma 7.3]).

Lemma 4.3. *For every $\varepsilon > 0$ there exists a constant $C > 0$ such that for every $\mathbf{i} = (i_1, i_2, \ldots,)$, $\mathbf{j} = (j_1, j_2, \ldots,) \in \Sigma$ with $i_1 \neq j_1$,*

$$\mathcal{L}_2\left(\{(x,\gamma) \in (0,1) \times (1/b, \tilde{\gamma}_b - \varepsilon) : |Y_{x,\gamma}(\mathbf{i}) - Y_{x,\gamma}(\mathbf{j})| < r\}\right) \leq Cr$$

for every $r > 0$, where $\mathcal{L}_2$ is the Lebesgue measure on the plane.

To state next results, we need to introduce some notation. For $\mathbf{i} = (i_1, i_2, \dots) \in \Sigma$ let $\mathbf{i}|_n = (i_1, \dots, i_n)$. For $\mathbf{i} = (i_1, i_2, \dots,)$, $\mathbf{j} = (j_1, j_2, \dots,) \in \Sigma$ let

$$\mathbf{i} \wedge \mathbf{j} = \min\{n \geq 0 : i_{n+1} \neq j_{n+1}\}.$$

For a finite length word $(l_1, \dots, l_n) \in \Sigma^*$ let

$$A_{(l_1,\dots,l_n)} = \left\{(\mathbf{i}, \mathbf{j}) \in \Sigma^2 : \mathbf{i} \wedge \mathbf{j} = n\right\}.$$

We note that for the empty word we have $A_\emptyset = \left\{(\mathbf{i}, \mathbf{j}) \in \Sigma^2 : i_1 \neq j_1\right\}$. We will write

$$A_{(l_1,\dots,l_n)}\big|_N = \{(\mathbf{i}|_N, \mathbf{j}|_N) : (\mathbf{i}, \mathbf{j}) \in A_{(l_1,\dots,l_n)}\}$$

for $N \geq 1$. For a finite length word $\bar{i} = (i_1, \dots, i_n) \in \Sigma^*$ let

$$v_{\bar{i}}(x) = \frac{x}{b^n} + \frac{i_1}{b^n} + \dots + \frac{i_n}{b}.$$

Let us observe that for any $\mathbf{i}, \mathbf{j} \in A_{\bar{i}}$,

$$|Y_{x,\gamma}(\mathbf{i}) - Y_{x,\gamma}(\mathbf{j})| = \gamma^n \left| Y_{v_{\bar{i}}(x),\gamma}(\sigma^n \mathbf{i}) - Y_{v_{\bar{i}}(x),\gamma}(\sigma^n \mathbf{j}) \right|, \tag{4.11}$$

where σ denotes the left-side shift on Σ and n is the length of $\bar{i}$.

Unfortunately, because of the structure of the measure $m_{x,\gamma}$, it is not possible to apply directly the transversality method and Lemma 4.3. To avoid this difficulty, we introduce the following lemma.

Lemma 4.4. *Let $\mathbf{i} = (i_1, i_2, \dots,)$, $\mathbf{j} = (j_1, j_2, \dots,) \in \Sigma$ with $i_1 \neq j_1$. Then for every $r > 0$ there exists $N = N(r)$ such that*

$$|Y_{x,\gamma}(\mathbf{i}) - Y_{x,\gamma}(\mathbf{j})| < r \quad \Rightarrow \quad |Y_{x,\gamma}(\mathbf{i}|_N \mathbf{0}) - Y_{x,\gamma}(\mathbf{j}|_N \mathbf{0})| < 2r \tag{4.12}$$

for every $x \in (0, 1)$ and $\gamma \in (1/b, \tilde{\gamma}_b)$, where $\mathbf{0} = (0, 0, \dots)$.

Proof. We have

$$\begin{aligned}
&||Y_{x,\gamma}(\mathbf{i}) - Y_{x,\gamma}(\mathbf{j})| - |Y_{x,\gamma}(\mathbf{i}|_N \mathbf{0}) - Y_{x,\gamma}(\mathbf{j}|_N \mathbf{0})|| \\
&\leq |(Y_{x,\gamma}(\mathbf{i}) - Y_{x,\gamma}(\mathbf{i}|_N \mathbf{0})) - (Y_{x,\gamma}(\mathbf{j}) - Y_{x,\gamma}(\mathbf{j}|_N \mathbf{0}))| \\
&\leq \gamma^N \left| Y_{v_{\mathbf{i}|_N}(x),\gamma}(\sigma^N \mathbf{i}) - Y_{v_{\mathbf{i}|_N}(x),\gamma}(\mathbf{0}) \right| + \gamma^N \left| Y_{v_{\mathbf{j}|_N}(x),\gamma}(\sigma^N \mathbf{j}) - Y_{v_{\mathbf{j}|_N}(x),\gamma}(\mathbf{0}) \right| \\
&\leq \gamma^N \frac{8\pi\gamma}{1-\gamma} < \tilde{\gamma}_b^N \frac{8\pi\tilde{\gamma}_b}{1-\tilde{\gamma}_b} \leq r,
\end{aligned}$$

which implies the inequality (4.12) for sufficiently large $N = N(r)$. □

Proposition 4.5. *For Lebesgue almost every $\gamma \in (1/b, \tilde{\gamma}_b)$ the measure $m_{x,\gamma}$ is absolutely continuous (in particular, $\dim m_{x,\gamma} = 1$) for Lebesgue almost every $x \in (0, 1)$.*

Proof. Take $\varepsilon > 0$. We will prove that $m_{x,\gamma}$ is absolutely continuous with respect to the Lebesgue measure, with density in L^2, for Lebesgue almost every $(x, \gamma) \in R_\varepsilon$, where

$$R_\varepsilon = (0, 1) \times (1/b + \varepsilon, \tilde{\gamma}_b - \varepsilon).$$

Since $\varepsilon > 0$ is arbitrarily small, this will imply the statement. Denote by

$$\underline{D}(m_{x,\gamma}, y) = \liminf_{r \to 0} \frac{m_{x,\gamma}(B_r(y))}{2r}$$

the lower density of the measure $m_{x,\gamma}$ at the point y, where $B_r(y)$ denotes the ball with radius r centered at y. By [19, Theorem 2.12], if $\underline{D}(m_{x,\gamma},y)<\infty$ for $m_{x,\gamma}$-almost every y, then the measure $m_{x,\gamma}$ is absolutely continuous. It is enough to show that

$$\mathcal{I}:=\iint_{R_\varepsilon}\int_{\mathbb{R}}\underline{D}(m_{x,\gamma},y)\,dm_{x,\gamma}(y)d\mathcal{L}_2(x,\gamma)<\infty.$$

The statement follows from the Fubini theorem. By standard manipulations we have

$$\mathcal{I}\leq\liminf_{r\to0}\frac{1}{2r}\iint_{\Sigma\times\Sigma}\mathcal{L}_2\left(\{(x,\gamma)\in R_\varepsilon:|Y_{x,\gamma}(\mathbf{i})-Y_{x,\gamma}(\mathbf{j})|<r\}\right)d\mathbb{P}(\mathbf{i})d\mathbb{P}(\mathbf{j}).$$

Then

$$\begin{aligned}&\iint_{\Sigma\times\Sigma}\mathcal{L}_2\left(\{(x,\gamma)\in R_\varepsilon:|Y_{x,\gamma}(\mathbf{i})-Y_{x,\gamma}(\mathbf{j})|<r\}\right)d\mathbb{P}(\mathbf{i})d\mathbb{P}(\mathbf{j})\\&\qquad=\sum_{n=0}^{\infty}\sum_{\bar{i}\in\{0,\dots,b-1\}^n}\iint_{A_{\bar{i}}}\mathcal{L}_2\left(\{(x,\gamma)\in R_\varepsilon:|Y_{x,\gamma}(\mathbf{i})-Y_{x,\gamma}(\mathbf{j})|<r\}\right)d\mathbb{P}(\mathbf{i})d\mathbb{P}(\mathbf{j}).\end{aligned}$$

By (4.11), for any $\mathbf{i},\mathbf{j}\in A_{\bar{i}}$,

$$\begin{aligned}&\mathcal{L}_2\left(\{(x,\gamma)\in R_\varepsilon:|Y_{x,\gamma}(\mathbf{i})-Y_{x,\gamma}(\mathbf{j})|<r\}\right)\\&=\mathcal{L}_2\left(\left\{(x,\gamma)\in R_\varepsilon:\left|Y_{v_{\bar{i}}(x),\gamma}(\sigma^n\mathbf{i})-Y_{v_{\bar{i}}(x),\gamma}(\sigma^n\mathbf{j})\right|<\gamma^{-n}r\right\}\right)\\&=b^n\mathcal{L}_2\left(\left\{(x,\gamma)\in R_{\bar{i},\varepsilon}:|Y_{x,\gamma}(\sigma^n\mathbf{i})-Y_{x,\gamma}(\sigma^n\mathbf{j})|<\gamma^{-n}r\right\}\right)\\&\leq b^n\mathcal{L}_2\left(\left\{(x,\gamma)\in R_{\bar{i},\varepsilon}:|Y_{x,\gamma}(\sigma^n\mathbf{i})-Y_{x,\gamma}(\sigma^n\mathbf{j})|<\left(\frac{1}{b}+\varepsilon\right)^{-n}r\right\}\right),\end{aligned}$$

where $R_{\bar{i},\varepsilon}=(v_{\bar{i}}(0),v_{\bar{i}}(1))\times(1/b+\varepsilon,\tilde{\gamma}_b)$. Applying Lemma 4.4, we get

$$\begin{aligned}&b^n\mathcal{L}_2\left(\left\{(x,\gamma)\in R_{\bar{i},\varepsilon}:|Y_{x,\gamma}(\sigma^n\mathbf{i})-Y_{x,\gamma}(\sigma^n\mathbf{j})|<\left(\frac{1}{b}+\varepsilon\right)^{-n}r\right\}\right)\\&\qquad\leq b^n\mathcal{L}_2\left(\left\{(x,\gamma)\in R_{\bar{i},\varepsilon}:|Y_{x,\gamma}(\sigma^n\mathbf{i}|_N\,\mathbf{0})-Y_{x,\gamma}(\sigma^n\mathbf{j}|_N\,\mathbf{0})|<2\left(\frac{1}{b}+\varepsilon\right)^{-n}r\right\}\right),\end{aligned}$$

where N depends on n,r. Hence,

$$\begin{aligned}&\sum_{\bar{i}\in\{0,\dots,b-1\}^n}\iint_{A_{\bar{i}}}\mathcal{L}_2\left(\{(x,\gamma)\in R_\varepsilon:|Y_{x,\gamma}(\mathbf{i})-Y_{x,\gamma}(\mathbf{j})|<r\}\right)d\mathbb{P}(\mathbf{i})d\mathbb{P}(\mathbf{j})\\&\leq\sum_{\bar{i}\in\{0,\dots,b-1\}^n}\sum_{(\bar{k},\bar{l})\in A_\emptyset|_N}\frac{b^n}{b^{2n+2N}}\mathcal{L}_2\left(\left\{(x,\gamma)\in R_{\bar{i},\varepsilon}:\left|Y_{x,\gamma}(\bar{k}\mathbf{0})-Y_{x,\gamma}(\bar{l}\mathbf{0})\right|<2\left(\frac{1}{b}+\varepsilon\right)^{-n}r\right\}\right)\\&=\sum_{(\bar{k},\bar{l})\in A_\emptyset|_N}\frac{b^n}{b^{2n+2N}}\mathcal{L}_2\left(\left\{(x,\gamma)\in R_\varepsilon:\left|Y_{x,\gamma}(\bar{k}\mathbf{0})-Y_{x,\gamma}(\bar{l}\mathbf{0})\right|<2\left(\frac{1}{b}+\varepsilon\right)^{-n}r\right\}\right),\end{aligned}$$

where in the last inequality we used that $R_\varepsilon = \bigcup_{\bar{i}\in\{0,\dots,b-1\}^n} R_{\bar{i},\varepsilon}$. Using Lemma 4.3 we get

$$\mathcal{I} \le \liminf_{r\to 0} \frac{1}{2r}\sum_{n=0}^{\infty}\sum_{(\bar{k},\bar{l})\in A_\emptyset|_N} \frac{b^n}{b^{2n+2N}}\mathcal{L}_2\left(\left\{(x,\gamma)\in R_\varepsilon : \left|Y_{x,\gamma}(\bar{k}\mathbf{0}) - Y_{x,\gamma}(\bar{l}\mathbf{0})\right| < 2\left(\frac{1}{b}+\varepsilon\right)^{-n} r\right\}\right)$$

$$\le \liminf_{r\to 0} \frac{1}{2r}\sum_{n=0}^{\infty}\sum_{(\bar{k},\bar{l})\in A_\emptyset|_N} \frac{b^n}{b^{2n+2N}} 2Cr\left(\frac{1}{b}+\varepsilon\right)^{-n} \le C\sum_{n=0}^{\infty}(1+b\varepsilon)^{-n},$$

which is finite since $\varepsilon > 0$. □

Proof of Theorem B. The result is a consequence of Proposition 4.5, Proposition 2.2 and (2.2). □

To obtain more precise estimates of $\tilde{\lambda}_2$, $\tilde{\lambda}_3$, $\tilde{\lambda}_4$ presented in Corollary C, one needs to find suitable $(*)$-functions. To do it, we use the following result.

Lemma 4.6 (Peres, Solomyak [22, Lemma 5.1])**.** *Let $\beta \ge 1$. Suppose that for some positive integer $k = k(\beta)$ and a real number $\eta = \eta(\beta)$ there exists a function $g_\beta : \mathbb{R} \to \mathbb{R}$,*

$$g_\beta(t) = 1 - \beta\sum_{n=1}^{k-1} t^n + \eta t^k + \beta\sum_{n=k+1}^{\infty} t^n$$

such that for some $t_\beta \in (0,1)$,

$$g_\beta(t_\beta) > 0 \quad \text{and} \quad g'_\beta(t_\beta) < 0.$$

Then $y(\beta) > t_\beta$. More precisely, there exists $\varepsilon > 0$ such that for every $g \in \mathcal{G}_\beta$ and every $t \in (0, t_\beta)$,

$$g(t) < \varepsilon \quad \Rightarrow \quad g'(t) < -\varepsilon.$$

Let

$$\beta = \frac{1}{\sqrt{\sin^2(\pi/b) - 1/(b^2\lambda - 1)^2}}.$$

and consider functions g_β defined in Lemma 4.6.

For $b = 2$ take $k = 4$, $\eta = 0.81$, $\lambda = 0.81$. Then $g_\beta(0.62) > 0$ and $g'_\beta(0.62) < 0$, so $y(\beta) > 0.62$. On the other hand, $1/(2\lambda) = 1/1.62 < 0.62$. By Lemma 4.1, $\tilde{\lambda}_2 < 0.81$.

For $b = 3$ take $k = 4$, $\eta = 1.43398$, $\lambda = 0.55$. Then $g_\beta(0.6061) > 0$ and $g'_\beta(0.6061) < 0$, so $y(\beta) > 0.6061$. On the other hand, $1/(3\lambda) = 1/1.65 < 0.6061$. By Lemma 4.1, $\tilde{\lambda}_3 < 0.55$.

For $b = 4$ take $k = 3$, $\eta = -0.298$, $\lambda = 0.44$. Then $g_\beta(0.569) > 0$ and $g'_\beta(0.569) < 0$, so $y(\beta) > 0.569$. On the other hand, $1/(4\lambda) = 1/1.76 < 0.569$. By Lemma 4.1, $\tilde{\lambda}_4 < 0.44$.

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