Group A

1. We choose uniformly a set of five distinct balls, so each has probability $\frac{1}{\binom{20}{5}}$.

   (a) We choose the 2 yellow balls we pick among 5, and the 3 non-yellow among the 15 total non-yellow:
   $$\mathbb{P}[\text{2 yellow balls}] = \frac{\binom{5}{2} \cdot \binom{15}{3}}{\binom{20}{5}}.$$

   (b)
   $$\begin{aligned}\mathbb{P}[\text{at least two different colors}] &= 1 - \mathbb{P}[\text{the five balls have the same color}] \\ &= 1 - \mathbb{P}[\text{five red}] - \mathbb{P}[\text{five blue}] - \mathbb{P}[\text{five yellow}] - \mathbb{P}[\text{five white}] \\ &= 1 - \frac{4}{\binom{20}{5}}.\end{aligned}$$

   (c) We first choose a color to pick 2 of, there are 4 choices, and we then pick 2 in 5 balls of this color and 1 in 5 of every other color.
   $$\mathbb{P}[\text{one of each color}] = \frac{4 \cdot \binom{5}{1}^3 \cdot \binom{5}{2}}{\binom{20}{5}}.$$

2. We model the number of fishes caught at lake Huron on a day I went to Huron with a Poisson random variable X_H with parameter λ_H, and the number of fishes caught at lake Erie on a day I went to Erie with a Poisson random variable X_E with parameter λ_E.

   (a) The distribution of the number of fishes caught last weekend is the distribution of X_E. We thus just have to determine λ_E.
   $$\frac{1}{3} = \mathbb{P}[X_E = 0] = e^{-\lambda_E} \frac{{\lambda_E}^0}{0!} = e^{-\lambda_E}$$
   so $\lambda_E = \ln(3)$.

   (b) First, we determine the distribution of the number of fishes caught on a trip to lake Huron.
   $$\frac{1}{2} = \mathbb{P}[X_H = 0] = e^{-\lambda_H} \frac{{\lambda_H}^0}{0!} = e^{-\lambda_H}$$
   so $\lambda_H = \ln(2)$.
   Then write Y for the number of fishes caught today, E for the event that I went to lake Erie, and H for the event that I went to lake Huron.
   $$\begin{aligned}\mathbb{P}[Y \geq 2] &= \mathbb{P}[E]\mathbb{P}[Y \geq 2 \mid E] + \mathbb{P}[H]\mathbb{P}[Y \geq 2 \mid H] \\ &= \mathbb{P}[E]\mathbb{P}[X_E \geq 2] + \mathbb{P}[H]\mathbb{P}[X_H \geq 2] \\ &= \mathbb{P}[E](1 - \mathbb{P}[X_E \in \{0,1\}]) + \mathbb{P}[H](1 - \mathbb{P}[X_H \in \{0,1\}]) \\ &= \frac{1}{2}(1 - e^{-\ln(3)}(\ln(3)^0 + \ln(3)^1)) + \frac{1}{2}(1 - e^{-\ln(2)}(\ln(2)^0 + \ln(2)^1)) \\ &= 1 - (\frac{1}{6}(\ln(3) + 1) + \frac{1}{4}(\ln(2) + 1)).\end{aligned}$$

(c) We use Bayes formula:

$$\begin{aligned}\mathbb{P}[E \mid Y=2] &= \frac{\mathbb{P}[Y=2 \mid E]\mathbb{P}[E]}{\mathbb{P}[Y=2]} \\ &= \frac{\mathbb{P}[X_E=2]\mathbb{P}[E]}{\mathbb{P}[E]\mathbb{P}[X_E=2]+\mathbb{P}[H]\mathbb{P}[X_H=2]} \\ &= \frac{\frac{1}{2}e^{-\ln(3)}\ln(3)^2/2}{\frac{1}{2}e^{-\ln(3)}\ln(3)^2/2+\frac{1}{2}e^{-\ln(2)}\ln(2)^2/2} \\ &= \frac{\frac{1}{3}\ln(3)^2}{\frac{1}{3}\ln(3)^2+\frac{1}{2}\ln(2)^2}.\end{aligned}$$

(d) Let's write Z for the amount of fishes caught by my friend.

$$\begin{aligned}\mathbb{P}[Z \geq 1 \mid Y=2] &= \frac{\mathbb{P}[Z \geq 1, Y=2]}{\mathbb{P}[Y=2]} \\ &= \frac{\mathbb{P}[Z \geq 1, Y=2 \mid H]\mathbb{P}[H]+\mathbb{P}[Z \geq 1, Y=2 \mid E]\mathbb{P}[E]}{\mathbb{P}[Y=2]} \\ &= \frac{\mathbb{P}[Z \geq 1 \mid H]\mathbb{P}[Y=2 \mid H]\mathbb{P}[H]+\mathbb{P}[Z \geq 1 \mid E]\mathbb{P}[Y=2 \mid E]\mathbb{P}[E]}{\mathbb{P}[Y=2]} \\ &= \frac{\mathbb{P}[X_H \geq 1]\mathbb{P}[X_H=2]\mathbb{P}[H]+\mathbb{P}[X_E \geq 1]\mathbb{P}[X_E=2]\mathbb{P}[E]}{\mathbb{P}[Y=2]} \\ &= \frac{(1-\mathbb{P}[X_H=0])\mathbb{P}[X_H=2]\mathbb{P}[H]+(1-\mathbb{P}[X_E=0])\mathbb{P}[X_E=2]\mathbb{P}[E]}{\mathbb{P}[H]\mathbb{P}[X_H=2]+\mathbb{P}[E]\mathbb{P}[X_E=2]} \\ &= \frac{\ln(2)^2/4+2\ln(3)^2/9}{\ln(2)^2/2+\ln(3)^2/3}.\end{aligned}$$

Group B

1. There are 4^5 sequences of 4 colors of length 5 (with order). We pick one of them uniformly.

   (a) We have $\binom{5}{2}$ ways to choose which of the balls were yellow, and 3^3 possible combination of colors for the three remaining.

   $$\mathbb{P}[\text{exactly 2 yellows}] = \frac{\binom{5}{2} \cdot 3^3}{4^5}.$$

   (b)

   $$\begin{aligned}\mathbb{P}[\text{at least two colors}] &= 1 - \mathbb{P}[\text{only one color}] \\ &= 1 - \mathbb{P}[5 \text{ red} \cup 5 \text{ blue} \cup 5 \text{ yellow} \cup 5 \text{ white}] \\ &= 1 - \frac{4}{4^5}.\end{aligned}$$

   (c) Seeing the four colors means that every color is picked once except one that is picked twice. We have 4 ways of choosing which is picked twice, 5! ways of ordering the five balls that we divide by 2 because the order of the two balls of the same color does not matter.

   $$\begin{aligned}\mathbb{P}[\text{one of each color}] &= \frac{4 \cdot 5!/2}{4^5} \\ &= \frac{15}{64}.\end{aligned}$$

2. Write X_G for the number of meat pieces on a plate when Gordon is cooking, and X_J for the number of meat pieces on a plate when Jamie is cooking. We assume that the pieces of beef are small and the plate vast, so that X_G is a Poisson random variable with parameter λ_G and X_J a Poisson random variable with parameter λ_J.

   (a) Last week, I went to the canteen on a day when Gordon was the chef, so the number of pieces of meat on my plate was a Poisson variable of parameter λ_G. Recall that the parameter of a Poisson variable is equal to its expectation, and that Gordon puts on average 3 pieces of meat on every plate. This means that $\lambda_G = 3$. Note that the same reasoning also implies $\lambda_J = 1$.

   (b) Let us write Y for the number of pieces of meat on my plate on the day of this week I'm going to the canteen, G for the event that Gordon is cooking on this day and J for the event that Jamie is cooking on this day.

   There are 5 days in the week when the canteen is open, so $\mathbb{P}[G] = 2/5$ and $\mathbb{P}[J] = 3/5$.

   $$\begin{aligned}\mathbb{P}[Y \geq 1] &= \mathbb{P}[G]\mathbb{P}[Y \geq 1 \mid G] + \mathbb{P}[J]\mathbb{P}[Y \geq 1 \mid J] \\ &= \mathbb{P}[G](1 - \mathbb{P}[X_G = 0]) + \mathbb{P}[J](1 - \mathbb{P}[X_J = 0]) \\ &= \frac{2}{5}(1 - e^{-3}\frac{3^0}{0!}) + \frac{3}{5}(1 - e^{-1}\frac{1^0}{0!}) \\ &= 1 - \frac{2e^{-3} + 3e^{-1}}{5}.\end{aligned}$$

(c) We use Bayes formula:

$$\begin{aligned}
\mathbb{P}[G \mid Y=3] &= \frac{\mathbb{P}[Y=3 \mid G]\mathbb{P}[G]}{\mathbb{P}[Y=3]} \\
&= \frac{\mathbb{P}[Y=3 \mid G]\mathbb{P}[G]}{\mathbb{P}[G]\mathbb{P}[Y=3 \mid G] + \mathbb{P}[J]\mathbb{P}[Y=3 \mid J]} \\
&= \frac{e^{-3}\frac{3^3}{3!}\frac{2}{5}}{e^{-3}\frac{3^3}{3!}\frac{2}{5} + e^{-1}\frac{1^3}{3!}\frac{3}{5}}.
\end{aligned}$$

(d) Write Z for the number of pieces of meat on my friend's plate when we go to the canteen this week.

$$\begin{aligned}
\mathbb{P}[Z \geq 2 \mid Y=3] &= \frac{\mathbb{P}[Z \geq 2, Y=3]}{\mathbb{P}[Y=3]} \\
&= \frac{\mathbb{P}[G]\mathbb{P}[Z \geq 2, Y=3 \mid G] + \mathbb{P}[J]\mathbb{P}[Z \geq 2, Y=3 \mid J]}{\mathbb{P}[Y=3]} \\
&= \frac{\mathbb{P}[G]\mathbb{P}[Z \geq 2 \mid G]\mathbb{P}[Y=3 \mid G] + \mathbb{P}[J]\mathbb{P}[Z \geq 2 \mid J]\mathbb{P}[Y=3 \mid J]}{\mathbb{P}[Y=3]} \\
&= \frac{\mathbb{P}[G](1-\mathbb{P}[X_G \in \{0,1\}])\mathbb{P}[X_G=3] + \mathbb{P}[J](1-\mathbb{P}[X_J \in \{0,1\}])\mathbb{P}[X_J=3]}{\mathbb{P}[Y=3]} \\
&= \frac{\frac{2}{5}(1-e^{-3}-3e^{-3})e^{-3}\frac{3^3}{3!} + \frac{3}{5}(1-e^{-1}-e^{-1})e^{-1}\frac{1^3}{3!}}{e^{-3}\frac{3^3}{3!}\frac{2}{5} + e^{-1}\frac{1^3}{3!}\frac{3}{5}}.
\end{aligned}$$