1)

$$\begin{pmatrix} 1 & 1 & 3 & -2 \\ 2 & 0 & 2 & 1 \\ -2 & 1 & 0 & 0 \\ -1 & 1 & 1 & 2 \end{pmatrix} \xrightarrow[\underline{IV}+I]{\underline{II}-2I,\ \underline{III}+2I} \begin{pmatrix} 1 & 1 & 3 & -2 \\ 0 & -2 & -4 & 5 \\ 0 & 3 & 6 & -4 \\ 0 & 2 & 4 & 0 \end{pmatrix} \xrightarrow{II/(-2)}$$

$$\to \begin{pmatrix} 1 & 1 & 3 & 2 \\ 0 & 1 & 2 & 5/2 \\ 0 & 3 & 6 & -4 \\ 0 & 2 & 4 & 0 \end{pmatrix} \xrightarrow[\underline{IV}-2II]{\underline{III}-3II} \begin{pmatrix} 1 & 1 & 3 & 2 \\ 0 & 1 & 2 & 5/2 \\ 0 & 0 & 0 & -23/2 \\ 0 & 0 & 0 & -5 \end{pmatrix}, \xrightarrow{\underline{III}/(-\frac{23}{2})}$$

$$\begin{pmatrix} 1 & 1 & 3 & 2 \\ 0 & 1 & 2 & 5/2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -5 \end{pmatrix} \xrightarrow[\underline{IV}+5\,\underline{III}]{\underline{I}-2\underline{III},\ \underline{II}-\frac{5}{2}\underline{III}} \begin{pmatrix} 1 & 1 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{\underline{I}-\underline{II}}$$

$$\underline{\underline{\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}}} \Rightarrow$$

a) $B = \left\{ \begin{pmatrix} 1 \\ 2 \\ -2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \\ 2 \end{pmatrix} \right\}$

Forms a basis, and

$$\left[\begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right]_B = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

b) $\operatorname{rank}(A) = \underline{\underline{3}}$

$\text{nullity}(A^T) = 4 - \operatorname{rank}(A^T) = 4 - \operatorname{rank}(A) = \underline{\underline{1}}$

$\text{nullity}(A) = 4 - \operatorname{rank}(A) = \underline{\underline{1}}$

2)
$$V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x - 2y - z = 0 \right\} = \left\{ \begin{pmatrix} 2y+z \\ y \\ z \end{pmatrix} : y, z \in \mathbb{R} \right\}$$

$$\Rightarrow \text{Basis of } V \text{ is } \left\{ \underbrace{\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}}_{\underline{v}_1}, \underbrace{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}_{\underline{v}_2} \right\}$$

Using Gram-Schmidt:

$$\underline{u}_1' := \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \ \& \ \underline{u}_2' = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - t \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \text{ where}$$

$$t = \frac{\left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right\rangle}{\left\langle \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right\rangle} = \frac{2}{5} \Rightarrow \underline{u}_2' = \begin{pmatrix} \frac{1}{5} \\ -\frac{2}{5} \\ \frac{1}{5} \end{pmatrix}$$

$$\Rightarrow \underline{u}_1 = \frac{\underline{u}_1'}{\|\underline{u}_1'\|} = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \end{pmatrix} \ \& \ \underline{u}_2 = \frac{\underline{u}_2'}{\|\underline{u}_2'\|} = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$$

3)
$$A = \begin{pmatrix} 4 & 3 \\ 0 & 5 \end{pmatrix} \Rightarrow A^T A = \begin{pmatrix} 4 & 0 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 16 & 12 \\ 12 & 34 \end{pmatrix}$$

$$\det \begin{pmatrix} 16-\lambda & 12 \\ 12 & 34-\lambda \end{pmatrix} = (16-\lambda)(34-\lambda) - 12^2 =$$
$$= \lambda^2 - 50\lambda + 400 = 0$$

$$\Rightarrow \lambda_{1,2} = \frac{50 \pm \sqrt{2500 - 1600}}{2} = \frac{50 \pm 30}{2} = \begin{cases} 40 \\ 10 \end{cases}$$

$$\Rightarrow \alpha_1(A) = \sqrt{40} \ \& \ \alpha_2(A) = \sqrt{10}$$

$\lambda_1 = 40$

$$\left(\begin{array}{cc|c} -24 & 12 & 0 \\ 12 & -6 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \underline{v}_1 = \begin{pmatrix} \frac{y}{2} \\ y \end{pmatrix}$$

$$\left(\frac{y}{2}\right)^2 + y^2 = 1 \Rightarrow y = \pm \frac{2}{\sqrt{5}}$$

$$\Rightarrow \underline{v}_1 = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix} \Rightarrow \underline{v}_2 = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{pmatrix}$$

$$\underline{u}_1 = \frac{1}{\alpha_1} A \underline{v}_1 = \frac{1}{\sqrt{40}} \begin{pmatrix} 4 & 3 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}$$

$$\underline{u}_2 = \frac{1}{\alpha_2} A \underline{v}_2 = \frac{1}{\sqrt{10}} \begin{pmatrix} 4 & 3 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{pmatrix}$$

$$\Rightarrow A = UDV^T = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} \sqrt{40} & 0 \\ 0 & \sqrt{10} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix}$$