OVERLAPPING SELF-AFFINE CARPETS WITH THE WEAK SEPARATION CONDITION

TDK dissertation

AUTHOR
Levente David

SUPERVISOR
Balázs Bárány
Department of Stochastics
BME

November 2024
Budapest University of Technology and Economics

Levente David: *Overlapping self-affine carpets with the weak separation condition,* TDK dissertation, November 2024

Acknowledgments

I would like to address my deepest gratitude to my supervisor, and mentor, Balázs Bárány.

He has always been ready to help with any good or bad questions I had, whether they were about mathematics or the navigation in academic life. Who accurately conjectured many truths right after the start about the path what we will take, and about the achievements we might arrive to. Who guided my sight always whenever I got lost, yet always trusted me with patience, and was always ready to hear my unconventional ideas, even the less fortunate ones.

Without him this would not be.

Contents

Abstract

The study of diagonal self-affine carpets is a main highway in the study of self-affine sets, since they provide one of the simplest examples of sets with differing box-counting and Hausdorff dimension. This in general is a rare occurrence ([6]). With separate cylinders, many previous studies has been done (see [18], [1], [2], [4], [8], [14], [15]), Fraser with Shmerkin ([9]) considered the allowance of some overlapping with the aim of typical type results. To add to this, we will also allow some overlapping, with the constraint that the projection of the IFS to the x- and y-axis satisfies the weak separation condition.

In this thesis we provide formula for the Hausdorff and box-counting dimension for diagonally aligned self-affine carpets whose projections to the x-, and y-axis satisfying the weak separation condition, and who have homogeneous contractions along the x- and y-axes. We also prove various formulas for the upper box-counting dimension in the case when the homogeneity of the contractions along the x- and y-axes is not assumed.

1 Introduction

We begin with the basic concepts of fractal geometry.

Definition 1.1 (IFS) *Let (X, dist) be a complete metric space. We say that a map $S : X \to X$ is a contraction if there exists $\lambda \in (0,1)$ such that for any $x, y \in X : \mathrm{dist}(S(x), S(y)) \leq \lambda \cdot \mathrm{dist}(x, y)$. We call a finite collection of contractions $\mathfrak{F} = \{S_1, S_2, \ldots, S_d\}$ an Iterated Function System. If X is an affine space, and $\mathfrak{F}$ consists of only affinities, then the IFS is called self-affine, in particular if $X = \mathbb{R}^d$, and S_i are similarities, then self-similar.*

From now on, we restrict our view to the later case, when $X = \mathbb{R}^d$ with the Euclidean distance.

The study of self-affine iterated function systems has been in noticeable focus in the past decades, whence the understanding of this greatly harder subfield than the self-similar is, is in constant growth. One of the most basic affinity but not similarity can be believed to be scaling with different values along different axes. Coupled this with translations, one gets the definition of the diagonally aligned or diagonal self-affine IFSs.

Definition 1.2 (Diagonally aligned IFS) *An IFS $\mathfrak{F}$ is said to be diagonally aligned or diagonal, if its functions attain the form*

$$S_i(x_1, x_2) := \begin{bmatrix} r_{i,1} & 0 \\ 0 & r_{i,2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} t_{i,1} \\ t_{i,2} \end{bmatrix}. \tag{Eq. 1.1}$$

Obviously one can extend the definition to arbitrary dimensions, but in this paper we focus on $\mathbb{R}^2$.

By Hutchinson ([11]) we have that for any IFS there exist a unique non-empty set, denoted by Λ throughout the paper (with sometimes subscripts), such that $\Lambda = \bigcup_{S \in \mathfrak{F}} S(\Lambda)$. This is called the attractor. A set is said to be a self-affine/self-similar set if it is an attractor of a self-affine/self-similar IFS. The study of the attractor is one of the main pillars of research in fractal geometry, and we aim for that as well. This can be done trough characterizing its dimensions, most notable of these are the following two, who will be the targets of our paper.

Definition 1.3 (Hausdorff dimension) *Define the Hausdorff dimension of a set E in $\mathbb{R}^d$ as*

$$\dim_{\mathrm{H}}(E) := \sup \left\{ s \geq 0 \;\middle|\; \mathcal{H}^s(E) > 0 \right\} \tag{Eq. 1.2}$$

where the s dimensional Hausdorff measure, $\mathcal{H}^s(.)$ is defined as follows

$$\begin{aligned}\mathcal{H}^s(E) &:= \lim_{\delta\to 0}\mathcal{H}^s_\delta(E)\\ &:= \lim_{\delta\to 0}\inf\left\{\sum_{i\in I}|U_i|^s \;\middle|\; |U_i|\le\delta,\ \bigcup_{i\in I}U_i\supseteq E,\ I \text{ is countable}\right\}.\end{aligned} \tag{Eq. 1.3}$$

Definition 1.4 (Box-counting dimension) *Define the lower and upper box-counting dimension of a set E in $\mathbb{R}^d$ as*

$$\begin{aligned}\overline{\dim}_{\mathrm{B}}(E) &:= \limsup_{\delta\to 0^+}\frac{\log N_\delta(E)}{-\log\delta},\\ \underline{\dim}_{\mathrm{B}}(E) &:= \liminf_{\delta\to 0^+}\frac{\log N_\delta(E)}{-\log\delta},\end{aligned} \tag{Eq. 1.4}$$

if the limit exists, where $N_\delta(E) := \min\left\{m>0 \;\middle|\; \exists x_1,x_2,\dots,x_m : E\subseteq\bigcup_{i=1}^m B(x_i,\delta)\right\}$. In particular, if the upper and lower box-counting dimension agree, then it is said that the box counting dimension exists, and is the agreed upon value.

For the basic properties of these dimensions, one may read the book of Falconer ([5]) or take any introductory fractal geometry course at a university.

1.1 Overview of the various results of the past

To contextualize the statements of our thesis, we begin by a brief introduction of previous advancements regarding special types of diagonal self-affine IFSs, from the perspective of the box-counting and Hausdorff dimension.

1.1.1 Bedford-McMullen carpets

Diagonally aligned IFS's were first studied by Bedford ([4]) and McMullen ([18]) separately, who both studied carpets generated in the following way: let $n>m$ be integers, and R a set of integer pairs (i,j) such that $0\le i<m,\ 0\le j<n$. Then define the IFS

$$\mathfrak{F} := \left\{S_{i,j}(x_1,x_2) := \begin{bmatrix}1/m & 0\\ 0 & 1/n\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}i/m\\ j/n\end{bmatrix}\right\}_{(i,j)\in R}. \tag{Eq. 1.5}$$

Denoting the attractor with Λ, they proved:

$$\begin{aligned}\dim_{\mathrm{H}}(\Lambda) &= \log_m\left(\sum_i \#\left\{j \;\middle|\; (i,j)\in R\right\}^{\log_n m}\right)\\ \dim_{\mathrm{B}}(\Lambda) &= \log_m\left(\#\left\{i \;\middle|\; \exists j:(i,j)\in R\right\}\right) + \log_n\left(\frac{\#R}{\#\left\{i \;\middle|\; \exists j:(i,j)\in R\right\}}\right),\end{aligned} \tag{Eq. 1.6}$$

where $\#A$ denotes the cardinality of the set A.

The functions are structured by rows of their images of $[0,1]^2$, these are the level-1 cylinder rectangles or cylinder rectangles. This ordering by the rows phenomena persist in the area, and we will see that it is related to the fact that $1/m > 1/n$. Now imagine the image of $[0,1]^2$ by a high iterate, it will be an "exponentially tall" rectangle. Imaging that we would like to cover this

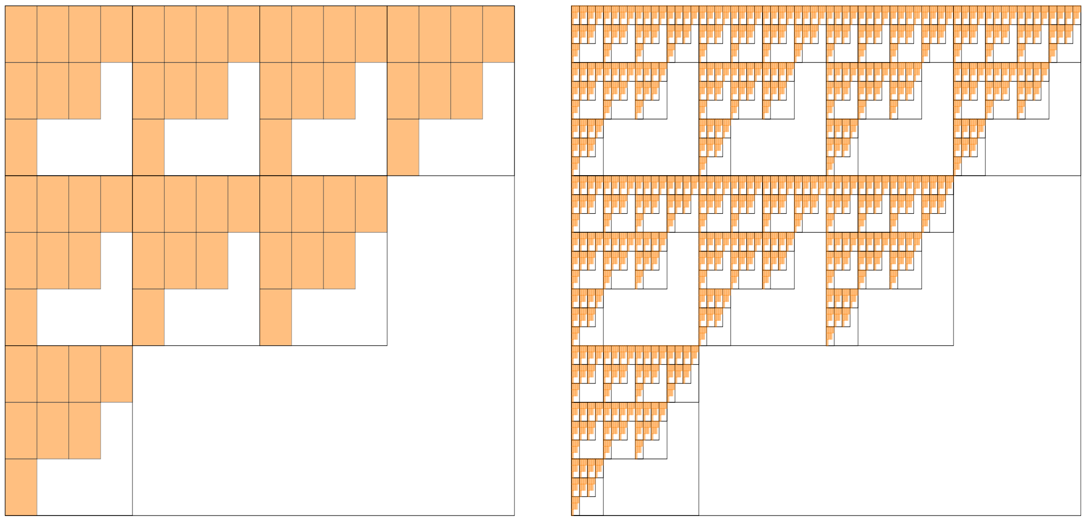


Figure 1: A Bedford-McMullen carpet's second and fourth iteration.

optimally by square-alike objects, we notice that the diversity of the attractor along the y-axis won't be relevant.

Kenyon and Peres proved ([12]) the analogous formula in arbitrary dimension, and in [13] they considered another graph-directed self-affine type construction, not necessarily self-affine but similarly interesting, and the main theorem of this paper has ideas resembling what we will consider.

Then exploration of the area followed with weakening the assumptions of the grid-like structure, with firstly letting the uniform scaling factors to differ, while preserving separation amongst the cylinders rectangles (by separation now think of that any two level-1 cylinder rectangle have intersection with 0 Lebesgue measure), and after that with assuming even weaker versions of separations.

1.1.2 Gatzouras-Lalley carpets

Gatzouras and Lalley ([15]) considered a more general case:

$$\mathfrak{F} := \left\{ S_{i,j}(x_1, x_2) := \begin{bmatrix} a_{i,j} & 0 \\ 0 & b_i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} c_{i,j} \\ d_i \end{bmatrix} \right\}_{(i,j)\in R}, \qquad \text{(Eq. 1.7)}$$

where $0 \leq i \leq m,\ 0 \leq j \leq n_i$, assuming $a_{i,j} < b_i < 1$ for all pairs, $\sum_{i=1}^{m} b_i \leq 1$, $\sum_{j=1}^{n_i} a_{i,j} \leq 1$ for each i. Also $0 \leq d_1 < \cdots < d_m < 1$ with $d_{i+1} - d_i \geq b_i$, $1 - d_m \geq d_m$ and for an i $0 \leq c_{i,1} < \cdots < c_{i,n_i} < 1$ with $c_{i,j+1} - c_{i,j} \geq a_{i,j}$ and $1 - c_{i,n_i} \geq a_{i,n_i}$.
They proved that if a $p \in \mathbb{R}$ is defined by

$$\sum_{i=1}^{m} b_i^p = 1 \quad \text{then} \quad \sum_{i=1}^{m}\sum_{j=1}^{n_i} b_i^p a_{i,j}^{\dim_B(\Lambda)-p} = 1 \qquad \text{(Eq. 1.8)}$$

and

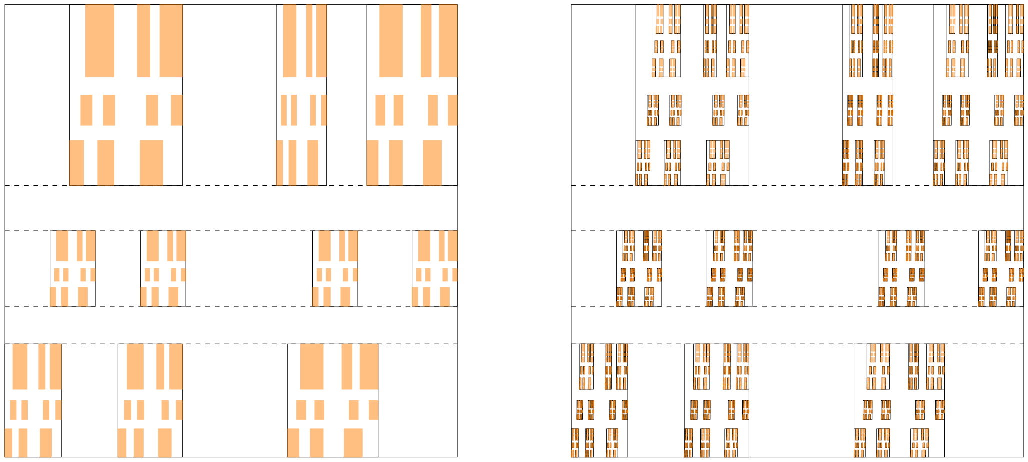


Figure 2: A Gatzouras-Lalley carpet's second and fourth iteration.

$$\dim_{\mathrm{H}}(\Lambda) = \max\left\{ \frac{\sum_i \sum_j p_{i,j} \log p_{i,j}}{\sum_i \sum_j p_{i,j} \log a_{i,j}} + \left(\sum_i (\sum_j p_{i,j}) \log(\sum_j p_{i,j}) \right) \left(\frac{1}{\sum_i \sum_j p_{i,j} \log b_i} - \frac{1}{\sum_i \sum_j p_{i,j} \log a_{i,j}} \right) \right\} \quad \text{(Eq. 1.9)}$$

where we maximize over the probability distributions on R (= discrete set of pairs). They also joined the discussion whether the $\dim_{\mathrm{B}} = \dim_{\mathrm{H}}$ case is typical, but we will avoid this dispute. Their formula for the Hausdorff dimension of Λ is not so surprising, since it connects fairly to the general formula for the Hausdorff dimension of a self-similar measure achieved by Feng-Hu ([7]), which was inspired by the earlier work of Ledrappier and Young ([16], [17]).

1.1.3 Feng-Wang and Barański carpets

Barański ([1]) considered

$$\mathfrak{F} := \left\{ S_{i,j}(x_1, x_2) := \begin{bmatrix} a_i & 0 \\ 0 & b_j \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \sum_{k=1}^{i-1} a_k \\ \sum_{k=1}^{j-1} b_k \end{bmatrix} \right\}_{(i,j)\in R}, \quad \text{(Eq. 1.10)}$$

where $0 \le i \le n,\ 0 \le j \le m$ and $\sum_{i=1}^{n} a_i = \sum_{j=1}^{m} b_j = 1$.
He derived that

$$\dim_{\mathrm{B}}(\Lambda) = \max\{d_1, d_2\} \quad \text{where} \quad \begin{aligned} &\sum_{(i,j)\in R} a_i^{s_x} b_j^{d_1 - s_x} = 1, \quad \sum_i a_i^{s_x} = 1 \\ &\sum_{(i,j)\in R} a_i^{d_2 - s_y} b_j^{s_y} = 1, \quad \sum_j b_j^{s_y} = 1. \end{aligned} \quad \text{(Eq. 1.11)}$$

A more general theorem was proved by Feng-Wang ([8], [2]) about the box-counting dimension:

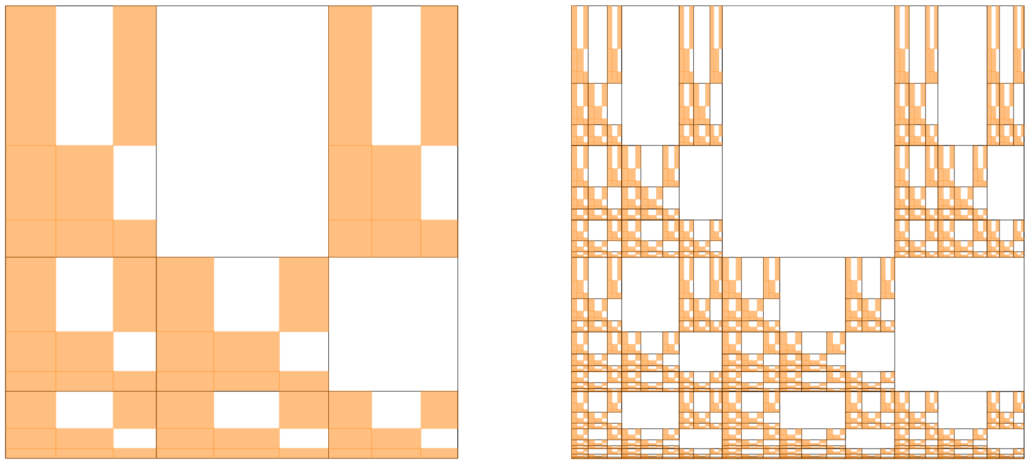


Figure 3: A Barański carpet

Theorem 1.1 (Feng-Wang) *Given that a diagonally aligned IFS*

$$\mathfrak{F} := \left\{ S_i(x_1, x_2) := \begin{bmatrix} r_{i,1} & 0 \\ 0 & r_{i,2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} t_{i,1} \\ t_{i,2} \end{bmatrix} \right\}_{i=1,\dots,\mathfrak{m}} \tag{Eq. 1.12}$$

satisfies the Rectangular Open Set Condition (ROSC), which is that $S_i([0,1]^2) \subset [0,1]^2$ *for any* i *and* $S_i((0,1)^2) \cap S_j((0,1)^2) = \emptyset$ *for any pair* $i \neq j$*, we have that*

$$\dim_{\mathrm{B}}(\Lambda) = \max\{d_1, d_2\} \quad \textit{where} \quad \begin{aligned} \sum_{i=1}^{\mathfrak{m}} |r_{i,1}|^{\dim_{\mathrm{B}}(\underline{\Lambda})} |r_{i,2}|^{d_1 - \dim_{\mathrm{B}}(\underline{\Lambda})} = 1 \\ \sum_{i=1}^{\mathfrak{m}} |r_{i,1}|^{d_2 - \dim_{\mathrm{B}}(|\Lambda)} |r_{i,2}|^{\dim_{\mathrm{B}}(|\Lambda)} = 1 \end{aligned} \tag{Eq. 1.13}$$

where $\underline{\Lambda}$ *and* $|\Lambda$ *are the projections of* Λ *to the* x*- and* y*-axis.*

At this point the nexus between these results is not entirely clear, but later, with the introduction of Hutchinson's Theorem (1.2) for self-similar sets, it will. We stated the nexus between the result of Feng, Wang and the result of Barański, because we will also obtain a similar result to the one Feng and Wang obtained. But by the work of Zerner ([19]) for self-similar sets satisfying the weak separation condition, we will have the luxury that the dimensions of the projection IFSs are computable (Theorem 1.5).

Kolossváry in [14] used the method of types to calculate the box-counting dimension of Barański-like and Gatzouras-Lalley-like carpets (sponges) in arbitrary dimension.

1.1.4 Fraser-Shmerkin carpets

Generalizing these advancements, one would have to open the possibility to some overlaps between the cylinder rectangles, like what Fraser and Shmerkin did. They considered ([9]) Bedford-McMullen type construction with the change that they translated the rows vertically by a param-

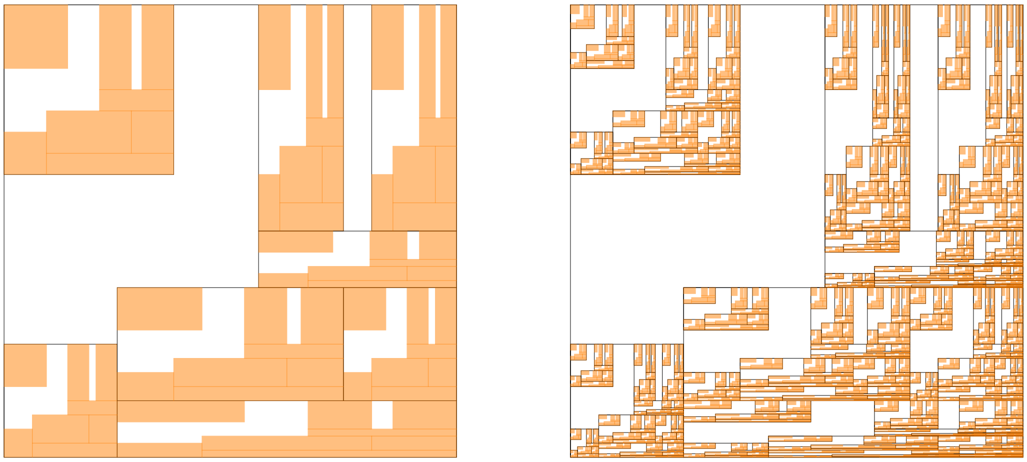


Figure 4: A Feng-Wang carpet's second and fourth iteration.

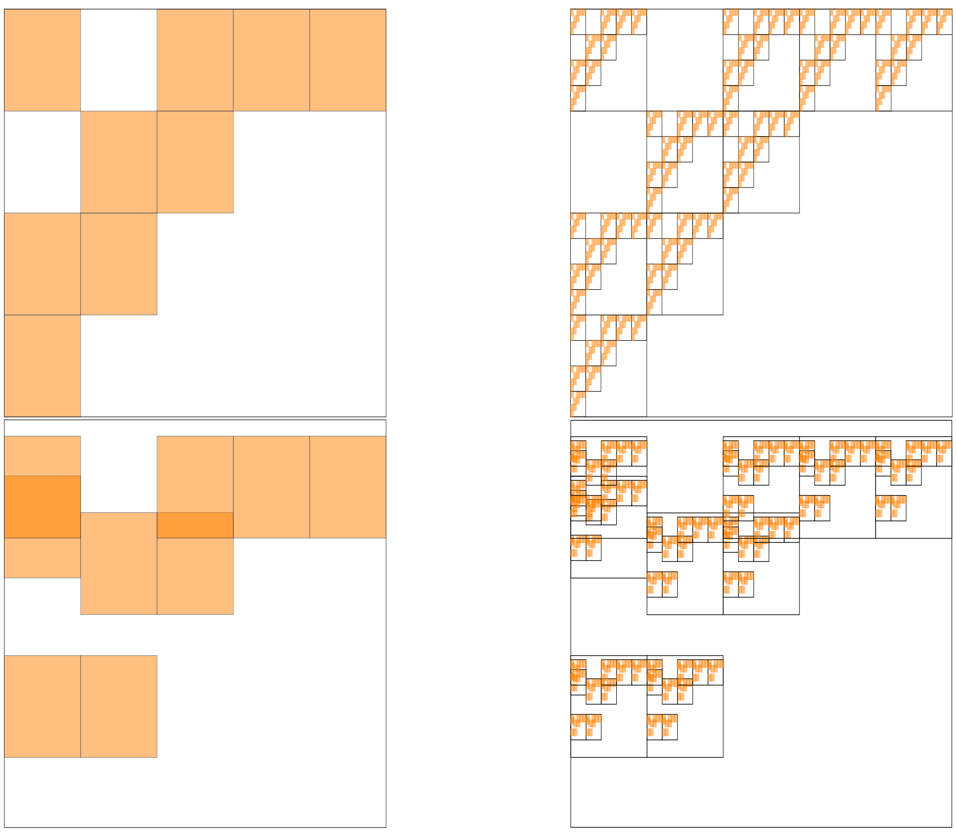


Figure 5: A Bedford-McMullen carpet's first and third iteration in the upper row, while in the lower a perturbed version of it, like what Fraser and Shmerkin considered.

eter $t \in [0, 1-1/m]^{\#\{\text{rows}\}}$. By this we mean that they considered the IFS:

$$\mathfrak{F} := \left\{ S_{i,j}(x_1, x_2) := \begin{bmatrix} 1/m & 0 \\ 0 & 1/n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} t_i \\ j/n \end{bmatrix} \right\}_{(i,j)\in R}. \tag{Eq. 1.14}$$

With this they allowed the overlapping of cylinder rectangles, what makes the study generally more challenging. They proved that unless t is in the set of exceptional parameters E of Hausdorff dimension $\#\{\text{rows}\}-1$ in the parameter space, we have that the formulas of Bedford and McMullen work.

They built upon the work of Hochman ([10]), who investigated, in a broader context and specifically in the self-similar case, the phenomenon of super-exponential concentration of higher level cylinders. Hochman also re-examined the situation of exact overlaps between cylindrical rectangles (i.e., $S_i([0,1]^2) = S_j([0,1]^2)$). These two phenomena are closely linked to the dimensional reduction observed in Bedford's and McMullen's formulas.

1.2 Self-similar sets as tools

The theory of self-similar sets is well-developed compared to the self-affine case, we only state few constructions from here.

Theorem 1.2 (Hutchinson, [11]) *Given a self-similar IFS of contraction rations $\{\lambda_i\}_{i\in I}$ satisfies the Open Set Condition, we have that*

$$\dim_{\mathrm{H}}(\Lambda) = \dim_{\mathrm{B}}(\Lambda) = s_o, \tag{Eq. 1.15}$$

where s_o, the so-called similarity dimension is defined implicitly by the equation

$$\sum_{i\in I} \lambda^{s_o} = 1. \tag{Eq. 1.16}$$

From this theorem follows that the Barański case is a consequence of the Feng-Wang Theorem, since in Barański's construction the projection IFSs are self-similar and satisfy the Open Set Condition, that is there exists an open set U such that $S_i(U) \subset U$ and $S_i(U) \cap S_j(U) = \emptyset$ for any $i \neq j$. We will utilize the following theorem:

Theorem 1.3 (Falconer, [5]) *Let Λ be a self-similar set, then*

$$\dim_{\mathrm{B}}(\Lambda) = \dim_{\mathrm{H}}(\Lambda). \tag{Eq. 1.17}$$

1.2.1 Weak Separation Condition

The case, when the super-exponential concentration of cylinders is forbidden, is called exponential separation condition. A slightly different condition was introduced earlier by Lau Ngai, what is called the Weak Separation Condition (or at some occurrences Weak Separation Property), where not only the at most exponentiality is assumed for the concentration of cylinders, but it needs to be comparable to the overlapping cylinder rectangle's length. Generally the WSC may be defined in many ways, we adopt a topological one.

Definition 1.5 (Weak Separation Condition) *We say that a self-similar IFS $\mathfrak{F}$ satisfies the WSC, iff the identity map is not an accumulation point of the following group of transformations*

$$\mathfrak{H} := \left\{ S_{i_1}^{-1} \circ \cdots \circ S_{i_k}^{-1} \circ S_{i_{k+1}} \circ \cdots \circ S_{i_\ell} \;\middle|\; S_i \in \mathfrak{F},\ k \leq \ell \in \mathbb{N} \right\} \tag{Eq. 1.18}$$

with the metric defined between two similarities

$$d(f,g) := \max\left\{|c_f - c_g|, \|O_f - O_g\|, \|t_f - t_g\|\right\} \tag{Eq. 1.19}$$

where a similarity $h : \mathbb{R}^d \to \mathbb{R}^d$ *has the form* $h(x) = c_h O_h x + t_h$*, where* $c_h \in \mathbb{R}^+, O_h$ *is an orthogonal matrix, and* $t_h \in \mathbb{R}^d$*.*

Given the WSC we can state the following assertions, but before that, define the Moran cut-set for a self-similar IFS $\{S_i\}$ and $\delta \in (0,1)$ as

$$M_\delta = \left\{S_{i_1} \circ \cdots \circ S_{i_k} \;\middle|\; |(S_{i_1} \circ \cdots \circ S_{i_k})(\Lambda)| \le \delta < |(S_{i_1} \circ \cdots \circ S_{i_{k-1}})(\Lambda)|\right\} \tag{Eq. 1.20}$$

where we denote a set, E's diameter by $|E|$.

Lemma 1.4 (Zerner, [19]) *A self-similar IFS* $\{S_i\}_{i\in\Sigma}$ *satisfies the WSC if and only if*

$$\exists x \in \Lambda \quad \exists \varepsilon > 0 \quad \forall \delta > 0 \quad \forall S_{\mathtt{i}}, S_{\mathtt{j}} \in M_\delta : \text{ if } (S_{\mathtt{i}}^{-1} S_{\mathtt{j}})(x) \ne x \implies |(S_{\mathtt{i}}^{-1} S_{\mathtt{j}})(x) - x| > \varepsilon. \tag{Eq. 1.21}$$

Theorem 1.5 (Zerner, [19]) *Given a self-similar IFS* $\{S_i\}_{i\in\Sigma}$ *satisfying the WSC, we have that*

$$\dim_{\mathrm{H}}(\Lambda) = \lim_{\delta\to 0} \frac{\log \#\{M_\delta\}}{-\log \delta}. \tag{Eq. 1.22}$$

Lemma 1.6 (Zerner, [19]) *Given a self-similar IFS* $\{S_i\}_{i\in\Sigma}$ *satisfying the WSC, we have that* $\exists n < \infty\ \forall x \in \mathbb{R}^d\ \forall \delta > 0$ *we have that:*

$$\#\left\{S_{\mathtt{j}} \;\middle|\; S_{\mathtt{j}} \in M_{\delta/|\Lambda|} \text{ and } S_{\mathtt{j}}(\Lambda) \cap B(x,\delta) \ne \emptyset\right\} \le n. \tag{Eq. 1.23}$$

The WSC is only defined for self-similar sets, since an exact overlap and some rotation could provide a sufficient series to the identity map. So in this paper we will use the assumption, that the projections to the coordinate axes, who are one-dimensional, and hence self-similar, attain the WSC.

2 The setup and results

Let

$$\mathfrak{F} := \left\{S_i(x_1,x_2) := \begin{bmatrix} r_{i,1} & 0 \\ 0 & r_{i,2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} t_{i,1} \\ t_{i,2} \end{bmatrix}\right\}_{i=1}^{\mathtt{m}} \tag{Eq. 2.1}$$

be an IFS, Λ it's attractor. Define the projection IFS-s by

$$\begin{aligned} \underline{\mathfrak{F}} &:= \left\{\underline{S}_i(x_1) := r_{i,1}x_1 + t_{i,1}\right\}_{i=1}^{\mathtt{m}} \\ |\mathfrak{F} &:= \left\{|S_i(x_2) := r_{i,2}x_2 + t_{i,2}\right\}_{i=1}^{\mathtt{m}}, \end{aligned} \tag{Eq. 2.2}$$

denote their attractor by $\underline{\Lambda}$, $|\Lambda$ respectively. In some cases for convenience we write $\mathrm{proj}_j(\ldots)$ instead of $\underline{\ldots}$ and $|\ldots$. The case when $j = 1$ is the projection to the x axis, so proj_1 refers to $\underline{\ldots}$. Writing $\mathrm{conv}(\ldots)$ for the convex hull, we assume that $[0,1]^2 = \mathrm{conv}(\underline{\Lambda}) \times \mathrm{conv}(|\Lambda)$, this can be done without the loss of any generality because if $\mathrm{conv}(\underline{\Lambda}) \times \mathrm{conv}(|\Lambda) =: [a_1,b_b] \times [a_2,b_2]$ then define a diagonal affinity A moving $[a_1,b_b] \times [a_2,b_2]$ into $[0,1]^2$, then $A(\Lambda)$ will have self-affine structure with generating IFS: $\mathfrak{F}' := \{A \circ S_i \circ A^{-1}\}_{i=1}^{\mathfrak{m}}$, and will have the same Hausdorff dimension, Box-counting dimension, ... We might call this assumption the property that the attractor or the IFS fills $[0,1]^2$. Denote

$$\begin{aligned} r_{\max,1} &:= \max\left(\{|r_{i,1}|\}_{i=1}^{\mathfrak{m}}\right) \quad r_{\min,1} := \min\left(\{|r_{i,1}|\}_{i=1}^{\mathfrak{m}}\right) \\ r_{\max,2} &:= \max\left(\{|r_{i,2}|\}_{i=1}^{\mathfrak{m}}\right) \quad r_{\min,2} := \min\left(\{|r_{i,2}|\}_{i=1}^{\mathfrak{m}}\right). \end{aligned} \tag{Eq. 2.3}$$

Definition 2.1 (Symbolic space) *Let Σ denote the set $\{1,\ldots,\mathfrak{m}\}$, for $k \in \mathbb{N}$ let Σ^k the set $\{1,\ldots,\mathfrak{m}\}^k$, Σ^∞ the set $\Sigma^{\mathbb{N}}$, and Σ^* the set $\bigcup_{k\in\mathbb{N}} \Sigma^k$. The elements of Σ^∞ are the infinite words, who are denoted by boldface letters such as $\mathbf{i},\mathbf{j},\mathbf{k}$. While finite words in Σ^* should be denoted by the style $\mathtt{i},\mathtt{j},\mathtt{k}$. In particular, if we refer to an element in Σ, then we will use i,j,k. The length of a word is denoted by $|\mathtt{i}|$.*

For sufficiently large finite word $\mathtt{i} = (i_1, i_2, \ldots, i_{|\mathtt{i}|})$ or for any infinite one, we use the notations:

$$\begin{aligned} \mathtt{i}|_{(k,\ell]} &:= (i_{k+1}, \ldots, i_\ell) \\ \mathtt{i}- &:= (i_1, \ldots, i_{|\mathtt{i}|-1}). \end{aligned} \tag{Eq. 2.4}$$

One can say that $\mathtt{i}$ is a prefix of $\mathtt{j}$ or $\mathbf{j}$, iff for some ℓ:

$$\mathtt{i} = \mathtt{j}|_{(0,\ell]} \quad \text{or} \quad \mathtt{i} = \mathbf{j}|_{(0,\ell]}. \tag{Eq. 2.5}$$

For $\mathtt{i} = (i_1, i_2, \ldots, i_{|\mathtt{i}|}) \in \Sigma^*$, then write

$$r_{\mathtt{i},1} := \prod_{\ell=1}^{|\mathtt{i}|} r_{i_\ell,1} \qquad r_{\mathtt{i},2} := \prod_{\ell=1}^{|\mathtt{i}|} r_{i_\ell,2}, \tag{Eq. 2.6}$$

similarly define

$$S_{\mathtt{i}} := S_{i_1} \circ \cdots \circ S_{i_{|\mathtt{i}|}}. \tag{Eq. 2.7}$$

For $\mathbf{i} \in \Sigma^\infty$ the pointwise limit of

$$S_{i_1} \circ \cdots \circ S_{i_n} \tag{Eq. 2.8}$$

is, by the contracting property, a function mapping the whole $\mathbb{R}^2$ to a single point in it, hence one can define the map π which maps to an infinite word $\mathbf{i}$ to the point in $\mathbb{R}^2$ where the whole plane is deformation retracting.

$$\forall \mathbf{i} \in \Sigma^\infty \ \exists! \pi(\mathbf{i}) \in \mathbb{R}^2 \text{ such that } S_{\mathbf{i}} : \mathbb{R}^2 \to \pi(\mathbf{i}). \tag{Eq. 2.9}$$

We say that two functions, S_i and S_j are cylinder-disjoint, if

$$S_i\big((0,1)^2\big) \cap S_j\big((0,1)^2\big) = \emptyset. \tag{Eq. 2.10}$$

Otherwise call them cylinder-intersecting.

We define various sets generated by the functions in the IFS:

$$\begin{aligned} &\triangle_\delta := \big\{\mathtt{i} \in \Sigma^* \mid \min\{|r_{\mathtt{i},1}|, |r_{\mathtt{i},2}|\} \le \delta < \min\{|r_{\mathtt{i}-,1}|, |r_{\mathtt{i}-,2}|\}\big\}, \\ &\triangle_\delta^1 := \big\{\mathtt{i} \in \triangle_\delta \bigm| |r_{\mathtt{i},1}| > |r_{\mathtt{i},2}|\big\}, \qquad \triangle_\delta^2 := \big\{\mathtt{i} \in \triangle_\delta \bigm| |r_{\mathtt{i},1}| \le |r_{\mathtt{i},2}|\big\}, \\ &\mathbb{S}_\delta^1 := \big\{S_{\mathtt{i}} \mid \mathtt{i} \in \triangle_\delta^1\big\}, \qquad \mathbb{S}_\delta^2 := \big\{S_{\mathtt{i}} \mid \mathtt{i} \in \triangle_\delta^2\big\}, \end{aligned} \tag{Eq. 2.11}$$

$$\begin{aligned} &\mathbb{G}_n := \big\{S_{\mathtt{i}} \bigm| \mathtt{i} \in \Sigma^n\big\}, \\ &|\mathbb{G}_n := \big\{|S_{\mathtt{i}} \bigm| \mathtt{i} \in \Sigma^n\big\}, \qquad \underline{\mathbb{G}}_n := \big\{\underline{S}_{\mathtt{i}} \bigm| \mathtt{i} \in \Sigma^n\big\}, \\ &\mathbb{G}_n^1 := \big\{S_{\mathtt{i}} \in \mathbb{G}_n \bigm| |r_{\mathtt{i},1}| > |r_{\mathtt{i},2}|\big\}, \qquad \mathbb{G}_n^2 := \big\{S_{\mathtt{i}} \in \mathbb{G}_n \bigm| |r_{\mathtt{i},1}| \le |r_{\mathtt{i},2}|\big\}. \end{aligned} \tag{Eq. 2.12}$$

2.1 Main results

The characterisation of the upper box-counting dimension is not usual. We consider this, since we expect that the formulas below to explain the box-counting dimension as well, and we expect the currently assumed condition for the homogeneity of the projection IFSs to be only a technical one. On the other hand, for the Hausdorff dimension, where we also obtain results with the homogeneity assumption, we expect the general formulas to be more involved.

We start with the result for the upper box-counting dimension, which compares to the results of Feng and Wang (compare Eq. 2.15, Eq. 2.16 to Eq. 1.13).

Theorem 2.1 (Upper Box-counting Dimension) *Let*

$$\mathfrak{F} := \left\{ S_i(x_1, x_2) := \begin{bmatrix} r_{i,1} & 0 \\ 0 & r_{i,2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} t_{i,1} \\ t_{i,2} \end{bmatrix} \right\}_{i=1}^{\mathtt{m}} \tag{Eq. 2.13}$$

be a diagonal self-affine IFS, with attractor filling $[0,1]^2$ *such that the principal projection IFSs,* $\underline{\mathfrak{F}} := \left\{ \underline{S}_i(x_1) := r_{i,1}x_1 + t_{i,1} \right\}_{i=1}^{\mathtt{m}}$ *and* $|\mathfrak{F} := \left\{ |S_i(x_2) := r_{i,2}x_2 + t_{i,2} \right\}_{i=1}^{\mathtt{m}}$ *satisfy the weak separation condition. Then*

1. *we have that*

$$\begin{aligned} \overline{\dim}_{\mathrm{B}}\Lambda = \max_{j=1,2} \limsup_{\delta \to 0} \Bigg(& \frac{\log\left(\#\mathbb{S}_\delta^j\right)}{-\log\delta} \\ & + \dim_{\mathrm{B}} \mathrm{proj}_j(\Lambda) \left(1 + \frac{\log\left(M_{\dim_{\mathrm{B}} \mathrm{proj}_j(\Lambda)} \{ |r_{\mathtt{i},j}| \mid S_{\mathtt{i}} \in \mathbb{S}_\delta^j \} \right)}{-\log\delta} \right) \Bigg) \end{aligned} \tag{Eq. 2.14}$$

   where $M_p(x_1, \dots, x_n) := \left(\frac{1}{n} \sum_{i=1}^n x_i^p \right)^{\frac{1}{p}}$ *is the power mean.*

2. *We also have that* $\overline{\dim}_{\mathrm{B}}\Lambda = \max\{d_*^1, d_*^2\}$ *where*

$$\begin{aligned} 1 &= \sum_{S_{\mathtt{i}} \in \mathbb{S}_\delta^1} \left(\frac{|r_{\mathtt{i},1}|}{|r_{\mathtt{i},2}|} \right)^{\dim_{\mathrm{B}} \mathrm{proj}_1(\Lambda)} |r_{\mathtt{i},2}|^{d_\delta^1}, \quad d_*^1 := \limsup_{\delta \to 0} d_\delta^1 \\ 1 &= \sum_{S_{\mathtt{i}} \in \mathbb{S}_\delta^2} \left(\frac{|r_{\mathtt{i},2}|}{|r_{\mathtt{i},1}|} \right)^{\dim_{\mathrm{B}} \mathrm{proj}_2(\Lambda)} |r_{\mathtt{i},1}|^{d_\delta^2}, \quad d_*^2 := \limsup_{\delta \to 0} d_\delta^2. \end{aligned} \tag{Eq. 2.15}$$

3. *Finally:* $\overline{\dim}_{\mathrm{B}}\Lambda = \max\{\overline{d}_*^1, \overline{d}_*^2\}$ *where*

$$\begin{aligned} 1 &= \sum_{S_{\mathtt{i}} \in \mathbb{G}_n^1} \left(\frac{|r_{\mathtt{i},1}|}{|r_{\mathtt{i},2}|} \right)^{\dim_{\mathrm{B}} \mathrm{proj}_1(\Lambda)} |r_{\mathtt{i},2}|^{\overline{d}_n^1}, \quad \overline{d}_*^1 := \limsup_{n \to \infty} \overline{d}_n^1 \\ 1 &= \sum_{S_{\mathtt{i}} \in \mathbb{G}_n^2} \left(\frac{|r_{\mathtt{i},2}|}{|r_{\mathtt{i},1}|} \right)^{\dim_{\mathrm{B}} \mathrm{proj}_2(\Lambda)} |r_{\mathtt{i},1}|^{\overline{d}_n^2}, \quad \overline{d}_*^2 := \limsup_{n \to \infty} \overline{d}_n^2. \end{aligned} \tag{Eq. 2.16}$$

Remark: As mentioned above, both $\dim_{\mathrm{B}} \mathrm{proj}_1(\Lambda) = \dim_{\mathrm{B}}(\underline{\Lambda})$ and $\dim_{\mathrm{B}} \mathrm{proj}_2(\Lambda) = \dim_{\mathrm{B}}(|\Lambda)$ can be calculated by the theorem of Zerner.

Assume that the generating functions only differ by the translations, and not by the contracting ratios among the x-, and y-axis. Then without the loss of any generality we will assume $|r_1| < |r_2|$. Looking at the expression Gatzouras and Lalley obtained, we expect the general case for the Hausdorff dimension to be not straight forward from the proof, what we will present, although many ideas generalise well.

Theorem 2.2 (Box-counting Dimension) *Let*

$$\mathfrak{F} := \left\{ S_i(x_1, x_2) := \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} t_{i,1} \\ t_{i,2} \end{bmatrix} \right\}_{i=1}^{\mathfrak{m}} \qquad \text{(Eq. 2.17)}$$

be a diagonal self-affine IFS, with attractor filling $[0,1]^2$ *such that the principal projection IFSs,* $\underline{\mathfrak{F}} := \left\{ \underline{S}_i(x_1) := r_1 x_1 + t_{i,1} \right\}_{i=1}^{\mathfrak{m}}$ *and* $|\mathfrak{F} := \left\{ |S_i(x_2) := r_2 x_2 + t_{i,2} \right\}_{i=1}^{\mathfrak{m}}$ *satisfy the weak separation condition. Then the box-counting dimension exists. Furthermore, W.L.O.G. assuming* $|r_1| < |r_2|$

$$\begin{aligned} \dim_{\mathrm{B}}(\Lambda) &= -\lim_{n\to\infty} \frac{1}{n} \left\{ \frac{\log\left(\#|\mathbb{G}_n\right)}{\log|r_2|} \Big(1 - \frac{\log|r_2|}{\log|r_1|} \Big) + \frac{\log\left(\#\mathbb{G}_n\right)}{\log|r_1|} \right\} \\ &= \dim_{\mathrm{B}}(|\Lambda) \Big(1 - \frac{\log|r_2|}{\log|r_1|} \Big) - \lim_{n\to\infty} \frac{1}{n} \frac{\log\left(\mathbb{G}_n^*\right)}{\log|r_1|}. \end{aligned} \qquad \text{(Eq. 2.18)}$$

For the statement concerning about the Hausdorff dimension, classify the elements of $\mathbb{G}_n$ with respect to their projections into $|\mathbb{G}_n$:

$$\mathbb{G}_n = \bigcup_{|S_{\mathtt{i}} \in |\mathbb{G}_n} R_{|S_{\mathtt{i}}} \quad \text{, where} \quad R_{|S_{\mathtt{i}}} := \left\{ S_{\mathtt{j}} \in \mathbb{G}_n \;\middle|\; |S_{\mathtt{j}} = |S_{\mathtt{i}} \right\}. \qquad \text{(Eq. 2.19)}$$

Theorem 2.3 (Hausdorff Dimension) *Let*

$$\mathfrak{F} := \left\{ S_i(x_1, x_2) := \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} t_{i,1} \\ t_{i,2} \end{bmatrix} \right\}_{i=1}^{\mathfrak{m}} \qquad \text{(Eq. 2.20)}$$

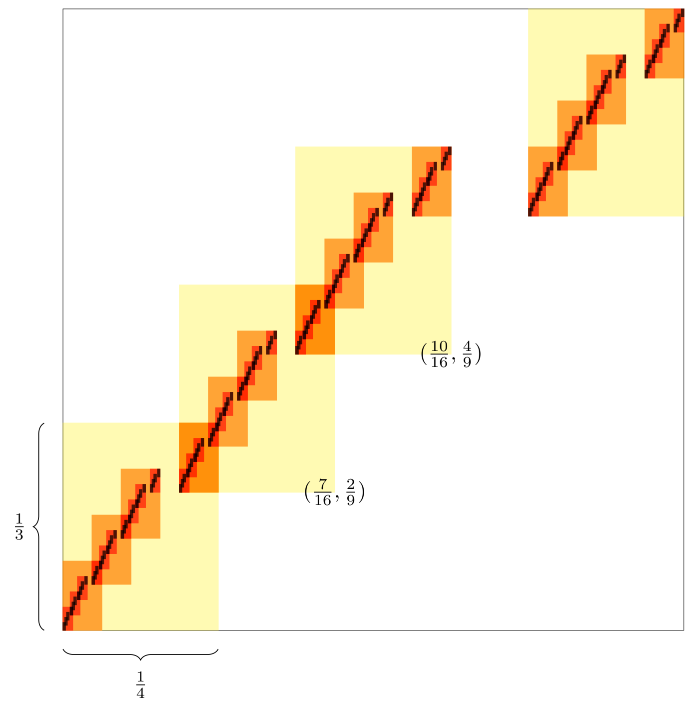
be a diagonal self-affine IFS, with attractor filling $[0,1]^2$ *such that the principal projection IFSs,* $\underline{\mathfrak{F}} := \left\{ \underline{S}_i(x_1) := r_1 x_1 + t_{i,1} \right\}_{i=1}^{\mathfrak{m}}$ *and* $|\mathfrak{F} := \left\{ |S_i(x_2) := r_2 x_2 + t_{i,2} \right\}_{i=1}^{\mathfrak{m}}$ *satisfy the weak separation condition. W.L.O.G. suppose that* $|r_1| < |r_2|$*. Then*

$$\dim_{\mathrm{H}}(\Lambda) = \lim_{n\to\infty} \left\{ -\frac{1}{n} \log_{|r_2|} \left\{ \sum_{|S_{\mathtt{i}} \in |\mathbb{G}_n} \left(\#R_{|S_{\mathtt{i}}}\right)^{\log_{|r_1|} |r_2|} \right\} \right\}. \qquad \text{(Eq. 2.21)}$$

In the next section we present two examples, and then we present the proofs, which take many ideas from the mentioned paper of Gatzouras and Lalley ([15]), as well as adopts some ideas of Zerner ([19]).

2.2 Examples

In general, the presented formulas can be challenging to compute, but with some ingenuity, many man made examples can be solved. We present one example dedicated to the box-counting dimension, and one to the Hausdorff.



$$\mathfrak{F} := \{S_1, S_2, S_3, S_4\},$$

$$S_1 := \begin{bmatrix} 1/4 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

$$S_2 := \begin{bmatrix} 1/4 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 3/16 \\ 2/9 \end{bmatrix},$$

$$S_3 := \begin{bmatrix} 1/4 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 6/16 \\ 4/9 \end{bmatrix},$$

$$S_4 := \begin{bmatrix} 1/4 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 3/4 \\ 2/3 \end{bmatrix},$$

$$|\mathfrak{F} := \{|S_1, |S_2, |S_3, |S_4\},$$

$$|S_1 := x/3,$$
$$|S_2 := x/3 + 2/9,$$
$$|S_3 := x/3 + 4/9,$$
$$|S_4 := x/3 + 2/3.$$

Figure 6: The first example.

2.2.1 First example

Consider the IFS defined in figure 6.

Remark 1 *The dimension defining structure of* Λ *is bipartite:*

- *It has exact overlaps on all levels after 1, but these are generated only by 2 equalities:* $S_{14} = S_{21}$ *and* $S_{24} = S_{31}$*.*
- $|\Lambda$ *has additional exact overlaps, generated by the equality:* $|S_{34} = |S_{41}$*.*

For proving that the projections satisfy the WSC either use Lemma 4.5.2 and Theorem 4.4.10 from [3] or remembering Lemma 1.4 observe the following: $|\mathbb{G}_n$s are the Moran cut-sets of $|\Lambda$ by the homogeneous contractions. If we let $n \in \mathbb{N}^+, |S_{\mathtt{i}}, |S_{\mathtt{j}} \in |\mathbb{G}_n$ be such that $|S_{\mathtt{i}}(x) \neq |S_{\mathtt{j}}(x)$, then

$$\left|(|S_{\mathtt{i}}^{-1} \circ |S_{\mathtt{j}})(x) - x\right| > \varepsilon \iff \left||S_{\mathtt{j}}(x) - |S_{\mathtt{i}}(x)\right| > \varepsilon \cdot \left(\frac{1}{3}\right)^n \tag{Eq. 2.22}$$

holds with $x = 0$ and any $\varepsilon < \frac{2}{3}$, since cylinders at the same level either completely overlap, or overlap with the intersection being $\frac{1}{3}$ of the whole length of the cylinders. Precisely

$$\left||S_{\mathtt{j}}(0) - |S_{\mathtt{i}}(0)\right| = \left|\sum_{k=0}^{n-1} \frac{a_k}{3^k} - \sum_{\ell=0}^{n-1} \frac{b_\ell}{3^\ell}\right| \geq \left|\frac{a_p}{3^p} - \frac{b_p}{3^p}\right| \geq \frac{2}{3} \cdot \left(\frac{1}{3}\right)^n, \tag{Eq. 2.23}$$

where $a_k, b_\ell \in \{0, \frac{2}{9}, \frac{4}{9}, \frac{2}{3}\}$, and $p = \sup\left\{k \in \{1, \ldots, n\} \mid a_k \neq b_k\right\}$ exists since $|S_{\mathtt{i}}(0) \neq |S_{\mathtt{j}}(0)$.

The same obviously holds for $\underline{\Lambda}$ with x again being 0 and $\varepsilon < \frac{3}{4}$. Now for Theorem 2.2 it is enough to compute the quantities

$$\frac{\log\big(\#|\mathbb{G}_n\big)}{n}, \quad \frac{\log\big(\#\mathbb{G}_n\big)}{n}. \tag{Eq. 2.24}$$

For the growth rate of $\mathbb{G}_n$ and $|\mathbb{G}_n$ we notice is that the cylinders form two type of objects a disjoint cylinder square, and the 3 overlapping one building some kind of stair. Call these two constellations type 1 and type 2. Now we can see that a type produces after one iteration exactly one type 1 and a type 2, while a type 2 gives rise in the next level to a type 1 and 3 type 2. Hence,

$$\#\mathbb{G}_n = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & 3\end{bmatrix}^n\begin{bmatrix}1\\ 3\end{bmatrix}. \tag{Eq. 2.25}$$

where the left vector is for the initial square, and the right vector finally decomposes the constellations into not entirely overlapping n-th level cylinders. Therefore by the Perron-Frobenius Theorem $\frac{\log\left(\#\mathbb{G}_n\right)}{n}$ is $\log(\lambda)$, where λ is the largest eigenvalue, $2+\sqrt{2}$ of the matrix in the middle. For $|\mathbb{G}_n$ recognise that

$$\#|\mathbb{G}_n = 3\#|\mathbb{G}_{n-1} + 1,\ |\mathbb{G}_0 = 1 \quad\Longrightarrow\quad |\mathbb{G}_n = \sum_{i=0}^{n} 3^i \tag{Eq. 2.26}$$

and hence $\frac{\log\left(\#|\mathbb{G}_n\right)}{n} = \log 3$ (which agrees with the observation that $|\Lambda = [0,1]$, and hence $\dim_{\mathrm{B}}(|\Lambda) = 1$, and $r_2 = 1/3$). We conclude that

$$\begin{aligned}\dim_{\mathrm{B}}(\Lambda) &= -\lim_{n\to\infty}\frac{1}{n}\left\{\frac{\log\big(\#|\mathbb{G}_n\big)}{\log|r_2|}\Big(1-\frac{\log|r_2|}{\log|r_1|}\Big) + \frac{\log\big(\#\mathbb{G}_n\big)}{\log|r_1|}\right\}\\ &= \frac{\log 3}{\log 3}\Big(1-\frac{\log 3}{\log 4}\Big) + \frac{\log(2+\sqrt{2})}{\log 4} = 1.093295401221\ldots\end{aligned} \tag{Eq. 2.27}$$

2.2.2 Second example

Consider the example presented in figure 7. With a similar method as used in the first example, one can prove that the x and y projections attain the WSC.
For the Hausdorff dimension we use

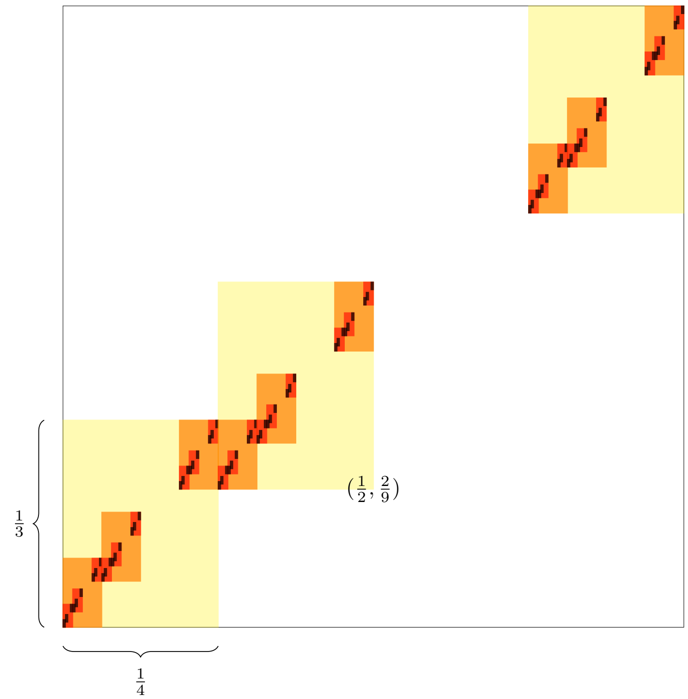
$$\dim_{\mathrm{H}}(\Lambda) = \lim_{n\to\infty}\left\{\frac{1}{n}\log_3\left\{\sum_{|S_{\mathtt{i}}\in|\mathbb{G}_n}(\#R_{|S_{\mathtt{i}}})^{\log_4 3}\right\}\right\}. \tag{Eq. 2.28}$$

Using the symbolic notation, and Remark 1 with equation (4.62) from [3] we have the bijection

$$|\mathbb{G}_n \longleftrightarrow \Gamma_n := \Big\{\mathtt{i} = (i_1 i_2 \ldots i_n) \in \{1,2,3\}^n \;\Big|\; \forall k = 1,\ldots,n-1 : i_k i_{k+1} \neq 13\Big\}. \tag{Eq. 2.29}$$

For $|S \in |\mathbb{G}_n$, and $|S \longleftrightarrow \mathtt{i}$, then let $(\#R_{|S_{\mathtt{i}}})^{\log_4 3}$ be denoted by $R_{\mathtt{i}}$. Then

$$\begin{aligned}\sum_{|S_{\mathtt{i}}\in|\mathbb{G}_n}(\#R_{|S_{\mathtt{i}}})^{\log_4 3} = \sum_{\mathtt{i}\in\Gamma_n} R_{\mathtt{i}} &= \sum_{\substack{\mathtt{i}\in\Gamma_n\\ i_n=1}} R_{\mathtt{i}} + \sum_{\substack{\mathtt{i}\in\Gamma_n\\ i_n=3}} R_{\mathtt{i}} + \sum_{k=1}^{n-1}\sum_{\substack{\mathtt{i}\in\Gamma_n\\ i_n=2\\ i_{n-1}=2\\ \cdots\\ i_{n-k+2}=2\\ i_{n-k+2}\neq 2}} R_{\mathtt{i}}\\ &=: a_0^{(n)} + a_1^{(n)} + \sum_{k=1}^{n-1} a_{k+1}^{(n)}\end{aligned} \tag{Eq. 2.30}$$



$$\mathfrak{F} := \{S_1, S_2, S_3\},$$
$$S_1 := \begin{bmatrix} 1/4 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$
$$S_2 := \begin{bmatrix} 1/4 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1/4 \\ 2/9 \end{bmatrix},$$
$$S_3 := \begin{bmatrix} 1/4 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 3/4 \\ 2/3 \end{bmatrix},$$
$$|\mathfrak{F} := \{|S_1, |S_2, |S_3\},$$
$$|S_1 := x/3,$$
$$|S_2 := x/3 + 2/9,$$
$$|S_3 := x/3 + 2/3.$$

Figure 7: The second example.

where the last line defined $a_i^{(n)}$ $i \in \{0, 1, \dots, n\}$ in order. Let $a_i^{(n)} := 0$ for $i > n$, and denote $a^{(n)} := \big(a_0^{(n)}, a_1^{(n)}, a_2^{(n)}, \dots\big) \in \mathbb{R}^{\mathbb{N}}$. Let $\alpha = \log_4 3$. Notice that

$$\dim_{\mathrm{H}}(\Lambda) = \lim_{n\to\infty} \left\{ \frac{1}{n} \log_3 \|a^{(n)}\|_1 \right\}, \tag{Eq. 2.31}$$

where $\|.\|_1$ is the usual 1-norm of real sequences. The decomposition of $a^{(n)}$ may seem ad hoc, now we show what it represents: if $\mathtt{i} \in \Gamma_n$ ends with 1 or 3, then the restriction, that 13 cannot occur, means that for any $j \in \{1, 2, 3\}$ we have $R_{\mathtt{i}j} = R_{\mathtt{i}}$. On the other hand, if $\mathtt{i}$ ends with exactly ℓ 2s, then $R_{\mathtt{i}2} = R_{\mathtt{i}3} = R_{\mathtt{i}}$, but $R_{\mathtt{i}1} = (\ell+1)^\alpha \cdot R_{\mathtt{i}}$, since

$$|S_{\mathtt{i}22\dots221} = |S_{\mathtt{i}22\dots213} = |S_{\mathtt{i}22\dots133} = \dots = |S_{\mathtt{i}13\dots333} \tag{Eq. 2.32}$$

while $S_{\mathtt{i}22\dots221}, S_{\mathtt{i}22\dots213}, S_{\mathtt{i}22\dots133}, \dots, S_{\mathtt{i}13\dots333}$ are $\ell+1$ different functions, if in $\mathtt{i}22\dots221$ the 1 in the end was preceded by exactly ℓ 2s. Therefore

$$\begin{aligned} a_0^{(n)} = \sum_{\substack{\mathtt{i}\in\Gamma_n \\ i_n=1}} R_{\mathtt{i}} &= \sum_{\substack{\mathtt{i}\in\Gamma_n \\ i_n=1 \\ i_{n-1}=1}} R_{\mathtt{i}} + \sum_{\substack{\mathtt{i}\in\Gamma_n \\ i_n=1 \\ i_{n-1}=3}} R_{\mathtt{i}} + \sum_{k=1}^{n-1} \sum_{\substack{\mathtt{i}\in\Gamma_n \\ i_n=1 \\ i_{n-1}=2 \\ \dots \\ i_{n-k+1}=2 \\ i_{n-k}\neq 2}} R_{\mathtt{i}} \\ &= a_0^{(n-1)} + a_1^{(n-1)} + \sum_{k=1}^{n-1} (k+1)^\alpha a_{k+1}^{(n-1)} \Big(+ 0 \text{ disguised as } \sum_{k=n}^{\infty} (k+1)^\alpha a_{k+1}^{(n-1)} \Big). \end{aligned} \tag{Eq. 2.33}$$

Similarly

$$a_1^{(n)} = \sum_{\substack{\mathtt{i}\in\Gamma_n \\ i_n=3}} R_{\mathtt{i}} = \sum_{\substack{\mathtt{i}\in\Gamma_n \\ i_n=3 \\ i_{n-1}=1}} R_{\mathtt{i}} + \sum_{\substack{\mathtt{i}\in\Gamma_n \\ i_n=3 \\ i_{n-1}=3}} R_{\mathtt{i}} + \sum_{k=1}^{n-1} \sum_{\substack{\mathtt{i}\in\Gamma_n \\ i_n=3 \\ i_{n-1}=2 \\ \cdots \\ i_{n-k+1}=2 \\ i_{n-k}\neq 2}} R_{\mathtt{i}}$$

$$= 0 + a_1^{(n-1)} + \sum_{k=1}^{n-1} a_{k+1}^{(n-1)} \Big(+ 0 \text{ disguised as } \sum_{k=n}^{\infty} a_{k+1}^{(n-1)} \Big). \tag{Eq. 2.34}$$

Next

$$a_2^{(n)} = \sum_{\substack{\mathtt{i}\in\Gamma_n \\ i_n=2 \\ i_{n-1}\neq 2}} R_{\mathtt{i}} = a_0^{(n-1)} + a_1^{(n-1)}, \tag{Eq. 2.35}$$

while for $j \in \{3, \dots, n\}$:

$$a_j^{(n)} = \sum_{\substack{\mathtt{i}\in\Gamma_n \\ i_n=2 \\ i_{n-1}=2 \\ \cdots \\ i_{n-j+2}=2 \\ i_{n-j+1}\neq 2}} R_{\mathtt{i}} = \sum_{\substack{\mathtt{i}\in\Gamma_{n-1} \\ i_{n-1}=2 \\ i_{n-2}=2 \\ \cdots \\ i_{n-j+2}=2 \\ i_{n-j+1}\neq 2}} R_{\mathtt{i}} = a_{j-1}^{(n-1)}. \tag{Eq. 2.36}$$

From these we conclude that

$$a^{(n)} = \mathbf{L}a^{(n-1)} = \cdots = \mathbf{L}^n(1,1,1,0,0,\dots) = \mathbf{L}^{n+1}(0,1,0,\dots) \tag{Eq. 2.37}$$

where we define the operator $\mathbf{L} : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ as

$$\mathbf{L} = \begin{bmatrix} 1 & 1 & 2^\alpha & 3^\alpha & 4^\alpha & 5^\alpha & \cdots \\ 0 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{Eq. 2.38}$$

Lemma 2.4 *There exists a unique $\infty > \lambda^* > 1$ such that there is $a \in \mathbb{R}^{\mathbb{N}}$ with positive entries, $a_0, a_1, a_2 \geq 1$ and with $\mathbf{L}a = \lambda^* a$. Furthermore*

$$\lambda^* = \frac{1}{\lambda^* - 1} \sum_{k=2}^{\infty} k^\alpha (\lambda^*)^{2-k} + \frac{(\lambda^*)^2}{(\lambda^* - 1)^3}. \tag{Eq. 2.39}$$

Lemma 2.5 $\frac{1}{n} \log \|a^{(n)}\|_1 \to \log \lambda^*$*, and hence*

$$\dim_{\mathrm{H}}(\Lambda) = \frac{\log \lambda^*}{\log 3}. \tag{Eq. 2.40}$$

Proof of Lemma 2.4:

Let $a \in \mathbb{R}^{\mathbb{N}}$ be such as in the statement, then we will arrive to an equation unequally solved by a λ^*. We assume

$$\sum_{k=2}^{\infty} k^{\alpha} a_k < \infty \qquad \text{(Eq. 2.41)}$$

so that $\lambda^* < \infty$. Suppose $\mathbf{L}a = \lambda a$, $\lambda \in (1, \infty)$, then

$$\forall k \geq 3 : \quad \lambda a_k = \mathbf{L}a_k = a_{k-1} \implies a_k = \lambda^{2-k} a_2 \qquad \text{(Eq. 2.42)}$$

$$\lambda a_1 = \mathbf{L}a_1 = a_1 + \sum_{k=2}^{\infty} a_k = a_1 + \sum_{k=2}^{\infty} \lambda^{2-k} a_2 = a_1 + a_2 \frac{\lambda}{\lambda - 1} \implies a_1 = a_2 \frac{\lambda}{(\lambda-1)^2}. \qquad \text{(Eq. 2.43)}$$

Finally

$$\begin{aligned} \lambda a_0 = \mathbf{L}a_0 = a_0 + a_1 + \sum_{k=2}^{\infty} k^{\alpha} a_k = a_0 + a_2 \frac{\lambda}{(\lambda-1)^2} + \sum_{k=2}^{\infty} k^{\alpha} \lambda^{2-k} a_2 \\ \implies a_0 = \frac{1}{\lambda - 1}\Big(\frac{\lambda}{(\lambda-1)^2} + \sum_{k=2}^{\infty} k^{\alpha} \lambda^{2-k}\Big) a_2 \end{aligned} \qquad \text{(Eq. 2.44)}$$

and

$$\lambda a_2 = \mathbf{L}a_2 = a_0 + a_1, \quad a_1 = a_2 \frac{\lambda}{(\lambda-1)^2} \qquad \text{(Eq. 2.45)}$$

implies that

$$\lambda = \frac{1}{\lambda - 1}\Big(\frac{\lambda}{(\lambda-1)^2} + \sum_{k=2}^{\infty} k^{\alpha} \lambda^{2-k}\Big) + \frac{\lambda}{(\lambda-1)^2} = \frac{1}{\lambda-1} \sum_{k=2}^{\infty} k^{\alpha} \lambda^{2-k} + \frac{\lambda^2}{(\lambda-1)^3}. \qquad \text{(Eq. 2.46)}$$

On $\lambda \in (1, \infty)$ the left-hand side of Eq. 2.46 strictly increases from 1 to ∞ continuously, while the right-hand size decreases continuously from ∞ to 0, proving that Eq. 2.46 is solved by a unique λ^* on $(1, \infty)$.

Proof of Lemma 2.5:

Let $M \in \mathbb{N}^+$, define $\mathbf{L}_M : \mathbb{R}^M \to \mathbb{R}^M$ as $(\mathbf{L} \circ \mathrm{proj}_M)|_M$, where proj_M is the projection of $\mathbb{R}^{\mathbb{N}}$ to the subspace spanned by the first M coordinates. Then $\mathbf{L}_M$ can be represented as the non-negative, irreducible aperiodic, M by M matrix:

$$\mathbf{L}_M = \begin{bmatrix} 1 & 1 & 2^{\alpha} & 3^{\alpha} & \cdots & (M-2)^{\alpha} & (M-1)^{\alpha} \\ 0 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}. \qquad \text{(Eq. 2.47)}$$

By the Perron-Frobenius Theorem $\exists!\ \lambda_M > 0$ s.t. $\lim_{n\to\infty} \frac{1}{n} \log \|\mathbf{L}_M^n v\|_1 = \log \lambda_M$ for any $0 \neq v \in \mathbb{R}^M$ with non-negative entries, and there is a $v^* \in \mathbb{R}^M$ with positive entries such that $\mathbf{L}_M v^* = \lambda_M v^*$. Therefore, with computations like Eq. 2.42–Eq. 2.46 we have:

$$\forall k \in [3, M-1] : \quad \lambda_M v_k^* = v_{k-1}^* \implies v_k^* = \lambda_M^{2-k} v_2^* \qquad \text{(Eq. 2.48)}$$

$$\lambda_M v_1^* = v_1^* + \sum_{k=2}^{M-1} v_k^* = v_1^* + \sum_{k=2}^{M-1} \lambda_M^{2-k} v_2^* \implies v_1^* = v_2^* \frac{1}{\lambda_M - 1} \sum_{k=2}^{M-1} \lambda_M^{2-k}. \qquad \text{(Eq. 2.49)}$$

Finally

$$\lambda_M v_0^* = v_0^* + v_1^* + \sum_{k=2}^{M-1} k^\alpha v_k^* \implies v_0^* = \Big(\frac{1}{(\lambda_M - 1)^2} \sum_{k=2}^{M-1} \lambda_M^{2-k} + \frac{1}{\lambda_M - 1} \sum_{k=2}^{M-1} k^\alpha \lambda_M^{2-k} \Big) v_2^* \qquad \text{(Eq. 2.50)}$$

and

$$\lambda_M v_2^* = v_0^* + v_1^*, \qquad v_1^* = v_2^* \frac{1}{\lambda_M - 1} \sum_{k=2}^{M-1} \lambda_M^{2-k} \qquad \text{(Eq. 2.51)}$$

implies that

$$\lambda_M = \frac{1}{v_2^*}(v_0^* + v_1^*) = \frac{\lambda_M}{(\lambda_M - 1)^2} \sum_{k=2}^{M-1} \lambda_M^{2-k} + \frac{1}{\lambda_M - 1} \sum_{k=2}^{M-1} k^\alpha \lambda_M^{2-k}. \qquad \text{(Eq. 2.52)}$$

From Eq. 2.46 and Eq. 2.52 we have that $\lim_{M \to \infty} \lambda_M = \lambda^*$, and $\|\mathbf{L}^{n-1} a^{(1)}\|_1 \geq \|\mathbf{L}_M^{n-1} a^{(1)}|_M\|_1$ follows inductively on n, remembering that all entries of $a^{(1)}$ are non-negative. Whence

$$\liminf_{n \to \infty} \frac{1}{n} \log \|\mathbf{L}^{n-1} a^{(1)}\|_1 \geq \log \lambda_M \longrightarrow \log \lambda^*. \qquad \text{(Eq. 2.53)}$$

Finally let $a \in \mathbb{R}^{\mathbb{N}}$ be such that $\mathbf{L}a = \lambda^* a$, $a_0, a_1, a_2 \geq 1$ and the rest of the entries of a are positive. Then

$$\|\mathbf{L}^{n-1} a^{(1)}\|_1 \leq \|\mathbf{L}^{n-1} a\|_1 = (\lambda^*)^{n-1} \|a\|_1 \qquad \text{(Eq. 2.54)}$$

hence

$$\limsup_{n \to \infty} \frac{1}{n} \log \|\mathbf{L}^{n-1} a^{(1)}\|_1 \leq \log \lambda^* \qquad \text{(Eq. 2.55)}$$

which with Eq. 2.53 proves the statement. Finally Eq. 2.46 lets us to numerically express some digits of the Hausdorff dimension:

$$\begin{aligned} \dim_{\mathrm{H}}(\Lambda) &= \log_3 2.896001351588652942659618472486268180831798155970197529858\ldots \\ &= 0.967885533595539319438037445903385862252724017052009287837\ldots . \end{aligned} \qquad \text{(Eq. 2.56)}$$

3 Upper box-counting dimension in the general case (thm 2.1)

Recall the definitions:

$$\begin{aligned} \triangle_\delta &:= \big\{ \mathtt{i} \in \Sigma^* \mid \min\{|r_{\mathtt{i},1}|, |r_{\mathtt{i},2}|\} \leq \delta < \min\{|r_{\mathtt{i}-,1}|, |r_{\mathtt{i}-,2}|\} \big\} \\ \mathbb{S}_\delta &:= \big\{ S_{\mathtt{i}} \bigm| \mathtt{i} \in \triangle_\delta \big\} \\ \triangle_\delta^1 &:= \big\{ \mathtt{i} \in \triangle_\delta \bigm| |r_{\mathtt{i},1}| > |r_{\mathtt{i},2}| \big\} \\ \triangle_\delta^2 &:= \big\{ \mathtt{i} \in \triangle_\delta \bigm| |r_{\mathtt{i},2}| \geq |r_{\mathtt{i},1}| \big\} \end{aligned} \qquad \text{(Eq. 3.1)}$$

where for $\mathtt{j} = (j_1, \ldots, j_{k-1}, j_k)$ we have defined $\mathtt{j}-$ as $(j_1, \ldots, j_{k-1})$. Now

$$N_\delta(\Lambda) = N_\delta(\bigcup_{\mathtt{i}\in\triangle_\delta} S_{\mathtt{i}}(\Lambda)) \tag{Eq. 3.2}$$

and even

$$N_\delta(\Lambda) = N_\delta(\bigcup_{S_{\mathtt{i}}\in\mathbb{S}_\delta} S_{\mathtt{i}}(\Lambda)) \tag{Eq. 3.3}$$

where for each function we choose one representative $\mathtt{i} \in \Sigma^*$. This holds by the fact that $\triangle_\delta$ is a cut-set, meaning that for any $\mathbf{i} \in \Sigma^\infty$ has a unique prefix $\mathtt{i}$ in $\triangle_\delta$. Therefore the set in $N_\delta()$ remains the same. But notice that the number of elements in $\mathbb{S}_\delta$ may be lower than in $\triangle_\delta$, which in the limit will be the enforcer for the dimension to drop from the values the formula of Bedford and McMullen would assign. Next define

$$\begin{aligned} \mathbb{S}^1_\delta &:= \left\{ S_{\mathtt{i}} \mid \mathtt{i} \in \triangle^1_\delta \right\} \\ \mathbb{S}^2_\delta &:= \left\{ S_{\mathtt{i}} \mid \mathtt{i} \in \triangle^2_\delta \right\} \end{aligned} \tag{Eq. 3.4}$$

and

$$\begin{aligned} |\mathbb{S}^2_\delta &:= \left\{ |S_{\mathtt{i}} \in |\mathfrak{F} \;\middle|\; S_{\mathtt{i}} \in \mathbb{S}^2_\delta \right\} \\ \underline{\mathbb{S}}^2_\delta &:= \left\{ \underline{S}_{\mathtt{i}} \in \underline{\mathfrak{F}} \;\middle|\; S_{\mathtt{i}} \in \mathbb{S}^2_\delta \right\}. \end{aligned} \tag{Eq. 3.5}$$

Then

$$N_\delta(\bigcup_{S_{\mathtt{i}}\in\mathbb{S}_\delta} S_{\mathtt{i}}(\Lambda)) = N_\delta(\bigcup_{j\in\{1,2\}} \bigcup_{S_{\mathtt{i}}\in\mathbb{S}^j_\delta} S_{\mathtt{i}}(\Lambda)). \tag{Eq. 3.6}$$

Whence we may conclude the (finite) stability of the upper box-counting dimension:

$$\begin{aligned} \max_{j\in\{1,2\}} \left\{ N_\delta(\bigcup_{S_{\mathtt{i}}\in\mathbb{S}^j_\delta} S_{\mathtt{i}}(\Lambda)) \right\} &\le N_\delta(\bigcup_{j\in\{1,2\}} \bigcup_{S_{\mathtt{i}}\in\mathbb{S}^j_\delta} S_{\mathtt{i}}(\Lambda)) \le \\ &\le N_\delta(\bigcup_{S_{\mathtt{i}}\in\mathbb{S}^1_\delta} S_{\mathtt{i}}(\Lambda)) + N_\delta(\bigcup_{S_{\mathtt{i}}\in\mathbb{S}^2_\delta} S_{\mathtt{i}}(\Lambda)) \le 2 \cdot \max_{j\in\{1,2\}} \left\{ N_\delta(\bigcup_{S_{\mathtt{i}}\in\mathbb{S}^j_\delta} S_{\mathtt{i}}(\Lambda)) \right\}. \end{aligned} \tag{Eq. 3.7}$$

Therefore

$$\begin{aligned} \overline{\dim}_{\mathrm{B}}\Lambda = \limsup_{\delta\to 0} \frac{\log N_\delta(\Lambda)}{-\log\delta} &= \limsup_{\delta\to 0} \frac{\log N_\delta(\bigcup_{j\in\{1,2\}} \bigcup_{S_{\mathtt{i}}\in\mathbb{S}^j_\delta} S_{\mathtt{i}}(\Lambda))}{-\log\delta} \\ &= \limsup_{\delta\to 0} \frac{\log \max_{j\in\{1,2\}} \left\{ N_\delta(\bigcup_{S_{\mathtt{i}}\in\mathbb{S}^j_\delta} S_{\mathtt{i}}(\Lambda)) \right\}}{-\log\delta} \\ &= \max_{j\in\{1,2\}} \left\{ \limsup_{\delta\to 0} \frac{\log N_\delta(\bigcup_{S_{\mathtt{i}}\in\mathbb{S}^j_\delta} S_{\mathtt{i}}(\Lambda))}{-\log\delta} \right\}, \end{aligned} \tag{Eq. 3.8}$$

where we used that

$$\forall i:\ \limsup_\delta a^{(i)}_\delta \le \limsup_\delta \max_i a^{(i)}_\delta \implies \max_i \limsup_\delta a^{(i)}_\delta \le \limsup_\delta \max_i a^{(i)}_\delta \tag{Eq. 3.9}$$

and we can take a sufficient subsequence of $\max_i a^{(i)}_\delta$, say δ_k such that the lim sup realises over it, then there is an index j such that $\max_i a^{(i)}_{\delta_k} = a^{(j)}_{\delta_k}$ for infinitely many k, and hence

$$\exists j:\ \limsup_\delta \max_i a^{(i)}_\delta = \limsup_k \max_i a^{(i)}_{\delta_k} = \limsup_k a^{(j)}_{\delta_k} \le \max_i \limsup_\delta a^{(i)}_\delta. \tag{Eq. 3.10}$$

Now we proved

Lemma 3.1

$$\overline{\dim}_{\mathrm{B}}\Lambda = \max_{j\in\{1,2\}}\left\{\limsup_{\delta\to 0}\frac{\log N_\delta(\bigcup_{S_{\mathtt{i}}\in\mathbb{S}^j_\delta} S_{\mathtt{i}}(\Lambda))}{-\log\delta}\right\}, \tag{Eq. 3.11}$$

Eq. 3.11 lets us focus on $N_\delta(\bigcup_{S_{\mathtt{i}}\in\mathbb{S}^1_\delta} S_{\mathtt{i}}(\Lambda))$ and $N_\delta(\bigcup_{S_{\mathtt{i}}\in\mathbb{S}^2_\delta} S_{\mathtt{i}}(\Lambda))$ individually. Notice that $\mathbb{S}^2_\delta$ is defined to be slightly more general, and hence we massage this one while constantly monitoring that for $\mathbb{S}^1_\delta$ the same would work as well.

3.1 Upper and lower bounds

We start with an easy upper bound:

$$N_\delta(\bigcup_{S_{\mathtt{i}}\in\mathbb{S}^2_\delta} S_{\mathtt{i}}(\Lambda)) \le \sum_{S_{\mathtt{i}}\in\mathbb{S}^2_\delta} N_\delta(S_{\mathtt{i}}(\Lambda)), \tag{Eq. 3.12}$$

which will work.
Towards the lower bound we construct:

$$\begin{aligned}
\widetilde{\mathcal{M}}^1_\delta &:= \left\{\mathtt{j}\in\Sigma^* \;\middle|\; \left|\underline{S}_{\mathtt{j}}(\underline{\Lambda})\right| \le \delta < \left|\underline{S}_{\mathtt{j}-}(\underline{\Lambda})\right|\right\} = \left\{\mathtt{j}\in\Sigma^* \;\middle|\; |r_{\mathtt{j},1}| \le \delta < |r_{\mathtt{j}-,1}|\right\}\\
\mathcal{M}^1_\delta &:= \left\{S_{\mathtt{j}} = S_{j_1}\circ\ldots\circ S_{j_k} \;\middle|\; \mathtt{j} = (j_1,\ldots,j_k)\in\widetilde{\mathcal{M}}^1_\delta\right\}\\
\widetilde{\mathcal{M}}^2_\delta &:= \left\{\mathtt{j}\in\Sigma^* \;\middle|\; \left||S_{\mathtt{j}}(|\Lambda)\right| \le \delta < \left||S_{\mathtt{j}-}(|\Lambda)\right|\right\} = \left\{\mathtt{j}\in\Sigma^* \;\middle|\; |r_{\mathtt{j},2}| \le \delta < |r_{\mathtt{j}-,2}|\right\}\\
\mathcal{M}^2_\delta &:= \left\{S_{\mathtt{j}} = S_{j_1}\circ\ldots\circ S_{j_k} \;\middle|\; \mathtt{j} = (j_1,\ldots,j_k)\in\widetilde{\mathcal{M}}^2_\delta\right\}.
\end{aligned} \tag{Eq. 3.13}$$

Then let

$$\begin{aligned}
\underline{\mathcal{M}}^1_\delta &:= \left\{\underline{S} \;\middle|\; S\in\mathcal{M}^1_\delta\right\}, \qquad & \underline{\mathcal{M}}^2_\delta &:= \left\{\underline{S} \;\middle|\; S\in\mathcal{M}^2_\delta\right\}\\
|\mathcal{M}^1_\delta &:= \left\{|S \;\middle|\; S\in\mathcal{M}^1_\delta\right\}, \qquad & |\mathcal{M}^2_\delta &:= \left\{|S \;\middle|\; S\in\mathcal{M}^2_\delta\right\}.
\end{aligned} \tag{Eq. 3.14}$$

Here $\underline{\mathcal{M}}^1_\delta$ and $|\mathcal{M}^2_\delta$ are subsets of the usual Moran covers of the projection IFSs, but the rest aren't. Fortunately, they can be covered by not too many Moran covers. Let's study $\triangle^2_\delta$:

$$\mathtt{i}\in\triangle^2_\delta \implies |r_{\mathtt{i},1}| \le \delta < |r_{\mathtt{i}-,1}| \implies S_{\mathtt{i}}\in\mathcal{M}^1_\delta \implies \underline{\mathbb{S}}^2_\delta \subseteq \underline{\mathcal{M}}^1_\delta. \tag{Eq. 3.15}$$

Here we can use Lemma 1.6, since $\underline{\mathfrak{F}}$ is assumed to satisfy the WSP and hence $\exists n<\infty\ \forall x\in\mathbb{R}\ \forall\delta>0:$

$$\#\left\{\underline{S}_{\mathtt{i}}\in\underline{\mathcal{M}}^1_\delta \;\middle|\; \underline{S}_{\mathtt{i}}(\underline{\Lambda})\cap[x-\delta,x+\delta]\ne\emptyset\right\} \le n \tag{Eq. 3.16}$$

from where

$$\#\left\{\underline{S}_{\mathtt{i}}\in\underline{\mathcal{S}}^2_\delta \;\middle|\; \underline{S}_{\mathtt{i}}(\underline{\Lambda})\cap[x-\delta,x+\delta]\ne\emptyset\right\} \le \#\left\{\underline{S}_{\mathtt{i}}\in\underline{\mathcal{M}}^1_\delta \;\middle|\; \underline{S}_{\mathtt{i}}(\underline{\Lambda})\cap[x-\delta,x+\delta]\ne\emptyset\right\} \le n. \tag{Eq. 3.17}$$

In other words: for any interval $[x-\delta,x+\delta]$ only n many $\underline{S}_{\mathtt{i}}(\underline{\Lambda})$ can intersect it, where $\underline{S}_{\mathtt{i}}\in\underline{\mathcal{S}}^2_\delta$. This implies that any δ length interval may only cover n many cylinders at most. Whence

$$n\cdot N_\delta\big(\bigcup_{\underline{S}_{\mathtt{i}}\in\underline{\mathbb{S}}^2_\delta}\underline{S}_{\mathtt{i}}(\underline{\Lambda})\big) \ge \sum_{\underline{S}_{\mathtt{i}}\in\underline{\mathbb{S}}^2_\delta} N_\delta\big(\underline{S}_{\mathtt{i}}(\underline{\Lambda})\big). \tag{Eq. 3.18}$$

For the other projection we only have

$$\begin{aligned}
\mathtt{i}\in\triangle^2_\delta \implies \delta < |r_{\mathtt{i}-,2}|,\ |r_{\mathtt{i},1}| \le |r_{\mathtt{i},2}| &\implies \max\{|r_{\mathtt{i},1}|, \delta r_{\min,2}\} \le |r_{\mathtt{i},2}|\\
&\implies \delta\cdot\max\{r_{\min,1}, r_{\min,2}\} \le |r_{\mathtt{i},2}|
\end{aligned} \tag{Eq. 3.19}$$

$$\left.\begin{array}{l}|r_{\mathtt{i},2}| \leq (r_{\max,2})^{|\mathtt{i}|} \\ (r_{\min,1})^{|\mathtt{i}|} \leq |r_{\mathtt{i},1}| \leq \delta\end{array}\right\} \implies |r_{\mathtt{i},2}| \leq (r_{\max,2})^{\frac{\log(\delta)}{\log(r_{\min,1})}} = \delta^{\frac{\log(r_{\max,2})}{\log(r_{\min,1})}}. \qquad \text{(Eq. 3.20)}$$

Hence we have

$$\mathtt{i} \in \triangle^2_\delta \implies S_{\mathtt{i}} \in \bigcup_{k=h_1}^{h_2} \mathcal{M}^2_{(r_{\max,2})^k} \implies |\mathbb{S}^2_\delta \subseteq \bigcup_{k=h_1}^{h_2} |\mathcal{M}^2_{(r_{\max,2})^k} \qquad \text{(Eq. 3.21)}$$

where

$$h_1 := \left\lfloor \frac{\log(\delta)}{\log(r_{\min,1})} \right\rfloor, \quad h_2 := \left\lceil \frac{\log(\delta \cdot \max\{r_{\min,1}, r_{\min,2}\})}{\log(r_{\max,2})} \right\rceil = \left\lceil \frac{\log(\delta)}{\log(r_{\max,2})} + \frac{\log(\max\{r_{\min,1}, r_{\min,2}\})}{\log(r_{\max,2})} \right\rceil. \qquad \text{(Eq. 3.22)}$$

Now Lemma 1.6 gives the system $|\mathfrak{F}$ an $m < \infty$ such that for each any $x \in \mathbb{R}$ for any k: $[x - (r_{\max,2})^k, x + (r_{\max,2})^k]$ might only cut into at most m-many $|S_{\mathtt{i}}(|\Lambda)$, where $|S_{\mathtt{i}} \in |\mathcal{M}^2_{(r_{\max,2})^k}$. Therefore at most $(m(h_2 - h_1))$-many $|S_{\mathtt{i}}(|\Lambda)$ from $\left\{|S_{\mathtt{i}}(|\Lambda) \mid |S_{\mathtt{i}}|\mathbb{S}^2_\delta(|\Lambda)\right\} \subseteq \left\{|S_{\mathtt{i}}(|\Lambda) \mid |S_{\mathtt{i}} \in \bigcup_{k=h_1}^{h_2} |\mathcal{M}^2_{(r_{\max,2})^k}\right\}$ can cut into $[x - (r_{\max,2})^{h_2}, x + (r_{\max,2})^{h_2}]$. This means that any length δ interval can only cover at most $(m(h_2 - h_1)\frac{\delta}{2(r_{\max,2})^{h_2}})$-many $|S_{\mathtt{i}}(|\Lambda)$, where $|S_{\mathtt{i}}$ is from $|\mathbb{S}^2_\delta$. Then

$$m(h_2 - h_1)\frac{\delta}{2(r_{\max,2})^{h_2}} \cdot N_\delta\Big(\bigcup_{|S_{\mathtt{i}} \in |\mathbb{S}^2_\delta} |S_{\mathtt{i}}(|\Lambda)\Big) \geq \sum_{|S_{\mathtt{i}} \in |\mathbb{S}^2_\delta} N_\delta(|S_{\mathtt{i}}(|\Lambda)). \qquad \text{(Eq. 3.23)}$$

Now using the two projections:

$$N_\delta\Big(\bigcup_{S_{\mathtt{i}} \in \mathbb{S}^2_\delta} S_{\mathtt{i}}(\Lambda)\Big) \geq \frac{1}{n} \cdot \frac{2(r_{\max,2})^{h_2}}{\delta m(h_2 - h_1)} \cdot \sum_{S_{\mathtt{i}} \in \mathbb{S}^2_\delta} N_\delta(S_{\mathtt{i}}(\Lambda)). \qquad \text{(Eq. 3.24)}$$

Substituting the values h_1 and h_2

$$\begin{aligned}
N_\delta\Big(\bigcup_{S_{\mathtt{i}} \in \mathbb{S}^2_\delta} S_{\mathtt{i}}(\Lambda)\Big) &\geq \frac{2}{nm} \cdot \frac{(r_{\max,2})^{\left\lceil \frac{\log(\delta \cdot \max\{r_{\min,1}, r_{\min,2}\})}{\log(r_{\max,2})} \right\rceil}}{\left(\left\lceil \frac{\log(\delta \cdot \max\{r_{\min,1}, r_{\min,2}\})}{\log(r_{\max,2})} \right\rceil - \left\lfloor \frac{\log(\delta)}{\log(r_{\min,1})} \right\rfloor\right)\delta} \cdot \sum_{S_{\mathtt{i}} \in \mathbb{S}^2_\delta} N_\delta(S_{\mathtt{i}}(\Lambda)) \\
&\geq \frac{2}{nm} \cdot \frac{(r_{\max,2})^{\frac{\log(\delta \cdot \max\{r_{\min,1}, r_{\min,2}\})}{\log(r_{\max,2})} + 1}}{\left(\frac{\log(\delta \cdot \max\{r_{\min,1}, r_{\min,2}\})}{\log(r_{\max,2})} - \frac{\log(\delta)}{\log(r_{\min,1})} + 2\right)\delta} \cdot \sum_{S_{\mathtt{i}} \in \mathbb{S}^2_\delta} N_\delta(S_{\mathtt{i}}(\Lambda)) \\
&= \frac{2}{nm} \cdot \frac{(r_{\max,2}) \cdot \delta \cdot \max\{r_{\min,1}, r_{\min,2}\}}{\left(\frac{\log(\delta \cdot \max\{r_{\min,1}, r_{\min,2}\})}{\log(r_{\max,2})} - \frac{\log(\delta)}{\log(r_{\min,1})} + 2\right)\delta} \cdot \sum_{S_{\mathtt{i}} \in \mathbb{S}^2_\delta} N_\delta(S_{\mathtt{i}}(\Lambda)) \\
&= \frac{2}{nm} \cdot \frac{(r_{\max,2}) \cdot \max\{r_{\min,1}, r_{\min,2}\}}{\frac{\log(\delta)}{\log(r_{\max,2})} + \frac{\log(\max\{r_{\min,1}, r_{\min,2}\})}{\log(r_{\max,2})} - \frac{\log(\delta)}{\log(r_{\min,1})} + 2} \cdot \sum_{S_{\mathtt{i}} \in \mathbb{S}^2_\delta} N_\delta(S_{\mathtt{i}}(\Lambda)).
\end{aligned} \qquad \text{(Eq. 3.25)}$$

Using once L'Hopital's rule, and elementary analysis one can derive that

$$\lim_{\delta \to 0} \frac{\log\left(\frac{2}{nm} \cdot \frac{(r_{\max,2}) \cdot \max\{r_{\min,1}, r_{\min,2}\}}{\frac{\log(\delta)}{\log(r_{\max,2})} + \frac{\log(\max\{r_{\min,1}, r_{\min,2}\})}{\log(r_{\max,2})} - \frac{\log(\delta)}{\log(r_{\min,1})} + 2}\right)}{-\log \delta} = 0 \qquad \text{(Eq. 3.26)}$$

and then from Eq. 3.12 and Eq. 3.25 we conclude the statement

Lemma 3.2

$$\limsup_{\delta\to 0}\frac{\log N_\delta(\bigcup_{S_{\mathtt{i}}\in\mathbb{S}^2_\delta}S_{\mathtt{i}}(\Lambda))}{-\log\delta}=\limsup_{\delta\to 0}\frac{\log\sum_{S_{\mathtt{i}}\in\mathbb{S}^2_\delta}N_\delta(S_{\mathtt{i}}(\Lambda))}{-\log\delta}. \qquad \text{(Eq. 3.27)}$$

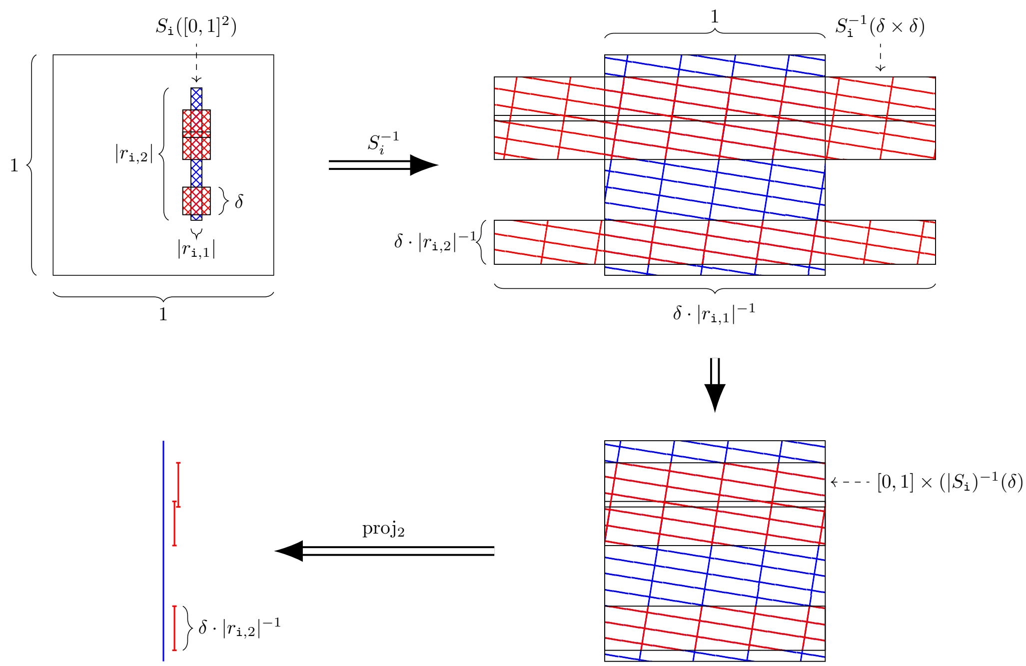
3.2 To the atomic level!

Now we need to estimate the covering of one $S_{\mathtt{i}}(\Lambda)$ for $\mathtt{i}\in\triangle^2_\delta$. We certainly have $|r_{\mathtt{i},1}|\leq\delta$, whence:

$$\begin{aligned}N_\delta(S_{\mathtt{i}}(\Lambda)) &=: N_{\delta\times\delta}(S_{\mathtt{i}}(\Lambda))\\ &= N_{S_{\mathtt{i}}^{-1}(\delta\times\delta)}(\Lambda)\\ &= N_{S_{\mathtt{i}}^{-1}(\delta\times\delta)}([0,1]\times|S_{\mathtt{i}}(|\Lambda)) \\ &= N_{\delta\cdot|r_{\mathtt{i},1}|^{-1}\times\delta\cdot|r_{\mathtt{i},2}|^{-1}}([0,1]\times|S_{\mathtt{i}}(|\Lambda))\\ &= N_{\delta\cdot|r_{\mathtt{i},2}|^{-1}}(|\Lambda)\end{aligned} \qquad \text{(Eq. 3.28)}$$

where we abuse the notation $N_a(b)$ to mean the optimal covering of the set b wherever it is by copies of a (we use this only here and nowhere else).

Figure 8: visualizing the argument for a fixed cylinder and three δ by δ square transforming



For self-similar sets, and in particular for $|\Lambda$ and, $\underline{\Lambda}$ the box-counting dimension exists ([5]). Therefore

$$\dim_{\mathrm{B}}(|\Lambda)=\lim_{\delta\to 0}\frac{\log N_\delta(|\Lambda)}{-\log\delta} \qquad \text{(Eq. 3.29)}$$

meaning that $\forall\varepsilon\ \exists\Gamma=\Gamma(\varepsilon)>0$ such that $\forall\delta\leq\Gamma$ we have:

$$\begin{aligned}\frac{\log N_\delta(|\Lambda)}{-\log\delta}&\in\Big(\dim_{\mathrm{B}}(|\Lambda)-\varepsilon,\dim_{\mathrm{B}}(|\Lambda)+\varepsilon\Big)\\ N_\delta(|\Lambda)&\in\Big(\delta^{-\dim_{\mathrm{B}}(|\Lambda)+\varepsilon},\delta^{-\dim_{\mathrm{B}}(|\Lambda)-\varepsilon}\Big).\end{aligned}\tag{Eq. 3.30}$$

This alone wouldn't be enough because as δ approaches 0, the value of $\delta\cdot|r_{\mathtt{i},2}|^{-1}$ may not converge, but rather cycle back every so often. Luckily, we have that for $\mathtt{i}\in\triangle^2_\delta$: $\delta\cdot|r_{\mathtt{i},2}|^{-1}\leq\delta\cdot(\delta\cdot\max\{r_{\min,1},r_{\min,2}\})^{-1}=\min\{r^{-1}_{\min,1},r^{-1}_{\min,2}\}$. Therefore

Lemma 3.3 *For any $\varepsilon>0$, there is a $c=c(\varepsilon)>0$ such that for any $\mathtt{i}\in\triangle^2_\delta$ we have*

$$N_\delta(S_{\mathtt{i}}(\Lambda))\in\Big(c^{-1}\cdot\Big(\frac{\delta}{|r_{\mathtt{i},2}|}\Big)^{-\dim_{\mathrm{B}}(|\Lambda)+\varepsilon},c\cdot\Big(\frac{\delta}{|r_{\mathtt{i},2}|}\Big)^{-\dim_{\mathrm{B}}(|\Lambda)-\varepsilon}\Big).\tag{Eq. 3.31}$$

To see this we only need to derive an appropriate c:

$$\begin{aligned}\sup_{\mathtt{i}\in\triangle^2_\delta}\left\{\frac{N_{\delta\cdot|r_{\mathtt{i},2}|^{-1}}(|\Lambda)}{\big(\frac{\delta}{|r_{\mathtt{i},2}|}\big)^{-\dim_{\mathrm{B}}(|\Lambda)-\varepsilon}}\right\}&=\max\left\{1,\sup_{\mathtt{i}\in\triangle^2_\delta,\delta\cdot|r_{\mathtt{i},2}|^{-1}>\Gamma}\left\{\frac{N_{\delta\cdot|r_{\mathtt{i},2}|^{-1}}(|\Lambda)}{\big(\frac{\delta}{|r_{\mathtt{i},2}|}\big)^{-\dim_{\mathrm{B}}(|\Lambda)-\varepsilon}}\right\}\right\}\\ &=\max\left\{1,\sup_{\mathtt{i}\in\triangle^2_\delta,\delta\cdot|r_{\mathtt{i},2}|^{-1}>\Gamma}\left\{N_{\delta\cdot|r_{\mathtt{i},2}|^{-1}}(|\Lambda)\big(\frac{\delta}{|r_{\mathtt{i},2}|}\big)^{\dim_{\mathrm{B}}(|\Lambda)+\varepsilon}\right\}\right\}\\ &\leq\max\left\{1,N_\Gamma(|\Lambda)\cdot(\min\{r^{-1}_{\min,1},r^{-1}_{\min,2}\})^{\dim_{\mathrm{B}}(|\Lambda)+\varepsilon}\right\}=:c_1\end{aligned}\tag{Eq. 3.32}$$

$$\begin{aligned}\inf_{\mathtt{i}\in\triangle^2_\delta}\left\{\frac{N_{\delta\cdot|r_{\mathtt{i},2}|^{-1}}(|\Lambda)}{\big(\frac{\delta}{|r_{\mathtt{i},2}|}\big)^{-\dim_{\mathrm{B}}(|\Lambda)+\varepsilon}}\right\}&=\min\left\{1,\inf_{\mathtt{i}\in\triangle^2_\delta,\delta\cdot|r_{\mathtt{i},2}|^{-1}>\Gamma}\left\{\frac{N_{\delta\cdot|r_{\mathtt{i},2}|^{-1}}(|\Lambda)}{\big(\frac{\delta}{|r_{\mathtt{i},2}|}\big)^{-\dim_{\mathrm{B}}(|\Lambda)+\varepsilon}}\right\}\right\}\\ &=\min\left\{1,\inf_{\mathtt{i}\in\triangle^2_\delta,\delta\cdot|r_{\mathtt{i},2}|^{-1}>\Gamma}\left\{N_{\delta\cdot|r_{\mathtt{i},2}|^{-1}}(|\Lambda)\big(\frac{\delta}{|r_{\mathtt{i},2}|}\big)^{\dim_{\mathrm{B}}(|\Lambda)-\varepsilon}\right\}\right\}\\ &\geq\min\left\{1,N_{\min\{r^{-1}_{\min,1},r^{-1}_{\min,2}\}}(|\Lambda)\cdot(\Gamma)^{\dim_{\mathrm{B}}(|\Lambda)+\varepsilon}\right\}=:c_2\end{aligned}\tag{Eq. 3.33}$$

$$c:=\max\{c_1,c_2^{-1}\}.\tag{Eq. 3.34}$$

3.3 Bounds arriving at once

Remembering Eq. 3.27, we use Eq. 3.31 and Eq. 3.20:

$$\begin{aligned}\limsup_{\delta\to0}\frac{\log\Big(\sum_{S_{\mathtt{i}}\in\mathbb{S}^2_\delta}N_\delta(S_{\mathtt{i}}(\Lambda))\Big)}{-\log\delta}&\leq\limsup_{\delta\to0}\frac{\log\Big(\sum_{S_{\mathtt{i}}\in\mathbb{S}^2_\delta}c\cdot\big(\frac{\delta}{|r_{\mathtt{i},2}|}\big)^{-\dim_{\mathrm{B}}(|\Lambda)-\varepsilon}\Big)}{-\log\delta}\\ &\leq\limsup_{\delta\to0}\frac{\log c}{-\log\delta}+\limsup_{\delta\to0}\frac{\log\Big(\sum_{S_{\mathtt{i}}\in\mathbb{S}^2_\delta}\big(\frac{\delta}{|r_{\mathtt{i},2}|}\big)^{-\dim_{\mathrm{B}}(|\Lambda)}\cdot\delta^{\varepsilon(\frac{\log(r_{\max,2})}{\log(r_{\min,1})}-1)}\Big)}{-\log\delta}\\ &\leq\limsup_{\delta\to0}\frac{\log\Big(\sum_{S_{\mathtt{i}}\in\mathbb{S}^2_\delta}\big(\frac{\delta}{|r_{\mathtt{i},2}|}\big)^{-\dim_{\mathrm{B}}(|\Lambda)}\Big)}{-\log\delta}+\varepsilon\Big(1-\frac{\log(r_{\max,2})}{\log(r_{\min,1})}\Big)\end{aligned}\tag{Eq. 3.35}$$

similarly

$$\limsup_{\delta\to 0}\frac{\log\Big(\sum_{S_{\mathtt{i}}\in\mathbb{S}^2_\delta} N_\delta(S_{\mathtt{i}}(\Lambda))\Big)}{-\log\delta} \geq \limsup_{\delta\to 0}\frac{\log\Big(\sum_{S_{\mathtt{i}}\in\mathbb{S}^2_\delta} c^{-1}\cdot\big(\frac{\delta}{|r_{\mathtt{i},2}|}\big)^{-\dim_{\mathrm{B}}(|\Lambda)+\varepsilon}\Big)}{-\log\delta}$$

$$\geq \liminf_{\delta\to 0}\frac{\log c^{-1}}{-\log\delta} + \limsup_{\delta\to 0}\frac{\log\Big(\sum_{S_{\mathtt{i}}\in\mathbb{S}^2_\delta}\big(\frac{\delta}{|r_{\mathtt{i},2}|}\big)^{-\dim_{\mathrm{B}}(|\Lambda)}\cdot\delta^{-\varepsilon(\frac{\log(r_{\max,2})}{\log(r_{\min,1})}-1)}\Big)}{-\log\delta} \quad \text{(Eq. 3.36)}$$

$$\geq \limsup_{\delta\to 0}\frac{\log\Big(\sum_{S_{\mathtt{i}}\in\mathbb{S}^2_\delta}\big(\frac{\delta}{|r_{\mathtt{i},2}|}\big)^{-\dim_{\mathrm{B}}(|\Lambda)}\Big)}{-\log\delta} - \varepsilon\left(1-\frac{\log(r_{\max,2})}{\log(r_{\min,1})}\right).$$

Observe that ε was arbitrary, hence:

Lemma 3.4

$$\limsup_{\delta\to 0}\frac{\log\Big(\sum_{S_{\mathtt{i}}\in\mathbb{S}^2_\delta} N_\delta(S_{\mathtt{i}}(\Lambda))\Big)}{-\log\delta} = \limsup_{\delta\to 0}\frac{\log\Big(\sum_{S_{\mathtt{i}}\in\mathbb{S}^2_\delta}\big(\frac{\delta}{|r_{\mathtt{i},2}|}\big)^{-\dim_{\mathrm{B}}(|\Lambda)}\Big)}{-\log\delta}. \quad \text{(Eq. 3.37)}$$

This can be reorganised to:

$$\frac{\log\Big(\sum_{S_{\mathtt{i}}\in\mathbb{S}^2_\delta}\big(\frac{\delta}{|r_{\mathtt{i},2}|}\big)^{-\dim_{\mathrm{B}}(|\Lambda)}\Big)}{-\log\delta} = \frac{\log\#\mathbb{S}^2_\delta}{-\log\delta} + \frac{\log\Big(\frac{1}{\#\mathbb{S}^2_\delta}\sum_{S_{\mathtt{i}}\in\mathbb{S}^2_\delta}\big(\frac{|r_{\mathtt{i},2}|}{\delta}\big)^{\dim_{\mathrm{B}}(|\Lambda)}\Big)}{-\log\delta}$$

$$= \frac{\log\#\mathbb{S}^2_\delta}{-\log\delta} + \frac{\log\Big(\delta^{-\dim_{\mathrm{B}}(|\Lambda)}\cdot\frac{1}{\#\mathbb{S}^2_\delta}\sum_{S_{\mathtt{i}}\in\mathbb{S}^2_\delta}|r_{\mathtt{i},2}|^{\dim_{\mathrm{B}}(|\Lambda)}\Big)}{-\log\delta} \quad \text{(Eq. 3.38)}$$

$$= \frac{\log\#\mathbb{S}^2_\delta}{-\log\delta} + \dim_{\mathrm{B}}(|\Lambda)\left(1+\frac{\log M_p\{|r_{\mathtt{i},2}| \mid S_{\mathtt{i}}\in\mathbb{S}^2_\delta\}}{-\log\delta}\right)$$

where we have the power mean

$$M_p(x_1,\ldots,x_n) := \Big(\frac{1}{n}\sum_{i=1}^n x_i^p\Big)^{\frac{1}{p}} \quad \text{(Eq. 3.39)}$$

with exponent $p=\dim_{\mathrm{B}}(|\Lambda)\in[0,1]$ and with $n=\#\mathbb{S}^2_\delta$. Using Eq. 3.11, Eq. 3.27 and Eq. 3.37 (in order) conclude

$$\overline{\dim}_{\mathrm{B}}\Lambda = \limsup_{\delta\to 0}\frac{\log N_\delta(\Lambda)}{-\log\delta} = \max_{j\in\{1,2\}}\left\{\limsup_{\delta\to 0}\frac{\log N_\delta(\bigcup_{S_{\mathtt{i}}\in\mathbb{S}^j_\delta} S_{\mathtt{i}}(\Lambda))}{-\log\delta}\right\}$$

$$= \max_{j\in\{1,2\}}\left\{\limsup_{\delta\to 0}\left(\frac{\log\#\mathbb{S}^j_\delta}{-\log\delta} + \dim_{\mathrm{B}}\mathrm{proj}_j(\Lambda)\left(1+\frac{\log M_{\dim_{\mathrm{B}}\mathrm{proj}_j(\Lambda)}\{|r_{\mathtt{i},j}| \mid S_{\mathtt{i}}\in\mathbb{S}^j_\delta\}}{-\log\delta}\right)\right)\right\} \quad \text{(Eq. 3.40)}$$

which is the first part of Theorem 2.1.

3.4 Secondary formula

Now we can achieve a formula complimenting that Feng and Wang obtained (Eq. 1.13). For $\delta > 0$ define $d^1_\delta, d^2_\delta, d^1_*, d^2_*$ with the following equations:

$$
\begin{aligned}
1 &= \sum_{S_{\mathtt{i}} \in \mathbb{S}^1_\delta} \left(\frac{|r_{\mathtt{i},1}|}{|r_{\mathtt{i},2}|}\right)^{\dim_{\mathrm{B}}(\underline{\Lambda})} |r_{\mathtt{i},2}|^{d^1_\delta}, \quad d^1_* := \limsup_{\delta\to 0} d^1_\delta \\
1 &= \sum_{S_{\mathtt{i}} \in \mathbb{S}^2_\delta} \left(\frac{|r_{\mathtt{i},2}|}{|r_{\mathtt{i},1}|}\right)^{\dim_{\mathrm{B}}(|\Lambda)} |r_{\mathtt{i},1}|^{d^2_\delta}, \quad d^2_* := \limsup_{\delta\to 0} d^2_\delta.
\end{aligned}
\tag{Eq. 3.41}
$$

Then

$$
\begin{aligned}
1 = \sum_{S_{\mathtt{i}} \in \mathbb{S}^1_\delta} \left(\frac{|r_{\mathtt{i},1}|}{|r_{\mathtt{i},2}|}\right)^{\dim_{\mathrm{B}}(\underline{\Lambda})} |r_{\mathtt{i},2}|^{d^1_\delta} &\leq \sum_{S_{\mathtt{i}} \in \mathbb{S}^1_\delta} |r_{\mathtt{i},1}|^{\dim_{\mathrm{B}}(\underline{\Lambda})} \cdot (\delta r_{\min,2})^{-\dim_{\mathrm{B}}(\underline{\Lambda})} \cdot \delta^{d^1_\delta} \\
&\leq \sum_{S_{\mathtt{i}} \in \mathbb{S}^1_\delta} \left(\frac{\delta}{|r_{\mathtt{i},1}|}\right)^{-\dim_{\mathrm{B}}(\underline{\Lambda})} \cdot r_{\min,2}^{-\dim_{\mathrm{B}}(\underline{\Lambda})} \cdot \delta^{d^1_\delta}
\end{aligned}
\tag{Eq. 3.42}
$$

hence taking the $\log_{\delta^{-1}}$ of both sides, using techniques from Eq. 3.38

$$
d^1_\delta \leq \frac{\log \#\mathbb{S}^1_\delta}{-\log \delta} + \dim_{\mathrm{B}}(\underline{\Lambda})\left(1 + \frac{\log M_{\dim_{\mathrm{B}}(\underline{\Lambda})}\{|r_{\mathtt{i},1}| \mid S_{\mathtt{i}} \in \mathbb{S}^1_\delta\}}{-\log \delta}\right) + \dim_{\mathrm{B}}(\underline{\Lambda})\frac{\log r_{\min,2}}{\log \delta} \tag{Eq. 3.43}
$$

and similarly

$$
\begin{aligned}
1 = \sum_{S_{\mathtt{i}} \in \mathbb{S}^1_\delta} \left(\frac{|r_{\mathtt{i},1}|}{|r_{\mathtt{i},2}|}\right)^{\dim_{\mathrm{B}}(\underline{\Lambda})} |r_{\mathtt{i},2}|^{d^1_\delta} &\geq \sum_{S_{\mathtt{i}} \in \mathbb{S}^1_\delta} |r_{\mathtt{i},1}|^{\dim_{\mathrm{B}}(\underline{\Lambda})} \cdot \delta^{-\dim_{\mathrm{B}}(\underline{\Lambda})} \cdot (\delta r_{\min,2})^{d^1_\delta} \\
&\geq \sum_{S_{\mathtt{i}} \in \mathbb{S}^1_\delta} \left(\frac{\delta}{|r_{\mathtt{i},1}|}\right)^{-\dim_{\mathrm{B}}(\underline{\Lambda})} \cdot r_{\min,2}^{d^1_\delta} \cdot \delta^{d^1_\delta}
\end{aligned}
\tag{Eq. 3.44}
$$

which implies

$$
d^1_\delta \geq \frac{\log \#\mathbb{S}^1_\delta}{-\log \delta} + \dim_{\mathrm{B}}(\underline{\Lambda})\left(1 + \frac{\log M_{\dim_{\mathrm{B}}(\underline{\Lambda})}\{|r_{\mathtt{i},1}| \mid S_{\mathtt{i}} \in \mathbb{S}^1_\delta\}}{-\log \delta}\right) - d^1_\delta\frac{\log r_{\min,2}}{\log \delta} \tag{Eq. 3.45}
$$

so

$$
d^1_\delta\left(1 + \frac{\log r_{\min,2}}{\log \delta}\right) \geq \frac{\log \#\mathbb{S}^1_\delta}{-\log \delta} + \dim_{\mathrm{B}}(\underline{\Lambda})\left(1 + \frac{\log M_{\dim_{\mathrm{B}}(\underline{\Lambda})}\{|r_{\mathtt{i},1}| \mid S_{\mathtt{i}} \in \mathbb{S}^1_\delta\}}{-\log \delta}\right). \tag{Eq. 3.46}
$$

For $\mathbb{S}^2_\delta$ and d^2_δ the same can be done. Both $\frac{\log r_{\min,2}}{\log \delta}$ and $\frac{\log r_{\min,1}}{\log \delta}$ goes to 0 as δ approaches 0, hence remembering Eq. 3.40 we got:

$$
\overline{\dim}_{\mathrm{B}}(\Lambda) = \max\{d^1_*, d^2_*\}. \tag{Eq. 3.47}
$$

3.5 Tertiary formula

The sets $\mathbb{S}_\delta^1$ and $\mathbb{S}_\delta^2$ are uncomfortable to compute, but we will show that a similar formula is true with the level-n functions instead. Recall

$$\begin{aligned} \mathbb{G}_n &:= \left\{ S_{\mathtt{i}} \mid \mathtt{i} \in \Sigma^n \right\} \\ \mathbb{G}_n^1 &:= \left\{ S_{\mathtt{i}} \in \mathbb{G}_n \mid |r_{\mathtt{i},1}| \geq |r_{\mathtt{i},2}| \right\} \\ \mathbb{G}_n^2 &:= \left\{ S_{\mathtt{i}} \in \mathbb{G}_n \mid |r_{\mathtt{i},1}| < |r_{\mathtt{i},2}| \right\}. \end{aligned} \tag{Eq. 3.48}$$

Then define

$$\begin{aligned} 1 &= \sum_{S_{\mathtt{i}} \in \mathbb{G}_n^1} \left(\frac{|r_{\mathtt{i},1}|}{|r_{\mathtt{i},2}|} \right)^{\dim_B(\underline{\Lambda})} |r_{\mathtt{i},2}|^{\overline{d}_n^1}, \quad \overline{d}_*^1 := \limsup_{n\to\infty} \overline{d}_n^1 \\ 1 &= \sum_{S_{\mathtt{i}} \in \mathbb{G}_n^2} \left(\frac{|r_{\mathtt{i},2}|}{|r_{\mathtt{i},1}|} \right)^{\dim_B(|\Lambda)} |r_{\mathtt{i},1}|^{\overline{d}_n^2}, \quad \overline{d}_*^2 := \limsup_{n\to\infty} \overline{d}_n^2. \end{aligned} \tag{Eq. 3.49}$$

To see that they also give the Box-dimension, for any $\eta \in [r_{\max,2}, 1)$ we have:

$$d_*^1 = \inf \left\{ \alpha > 0 \;\middle|\; \limsup_{k\to\infty} \left(\sum_{S_{\mathtt{i}} \in \mathbb{S}_{\eta^k}^1} \left(\frac{|r_{\mathtt{i},1}|}{|r_{\mathtt{i},2}|} \right)^{\dim_B(\underline{\Lambda})} |r_{\mathtt{i},2}|^{\alpha} \right)^{1/k} < 1 \right\} \tag{Eq. 3.50}$$

and

$$\overline{d}_*^1 = \inf \left\{ \alpha > 0 \;\middle|\; \limsup_{n\to\infty} \left(\sum_{S_{\mathtt{i}} \in \mathbb{G}_n^1} \left(\frac{|r_{\mathtt{i},1}|}{|r_{\mathtt{i},2}|} \right)^{\dim_B(\underline{\Lambda})} |r_{\mathtt{i},2}|^{\alpha} \right)^{1/n} < 1 \right\}. \tag{Eq. 3.51}$$

Indeed, the first equality holds by the following observations: For any $\alpha > d_*^1$ there exists $\varepsilon > 0$, there is a $k_0 > 0$ for which any $k > k_0$ satisfies $\alpha - \varepsilon > d_{\eta^k}^1$ (we will have the hierarchy: $\alpha > \alpha - \varepsilon > d_{\eta^k}^1$), and therefore by the definition of $d_{\eta^k}^1$, and using: $|r_{\mathtt{i},2}| \leq \eta^k$ we will have

$$\begin{aligned} &\limsup_{k\to\infty} \left(\sum_{S_{\mathtt{i}} \in \mathbb{S}_{\eta^k}^1} \left(\frac{|r_{\mathtt{i},1}|}{|r_{\mathtt{i},2}|} \right)^{\dim_B \operatorname{proj}_1(\Lambda)} |r_{\mathtt{i},2}|^{\alpha} \right)^{1/k} \\ &\quad < \eta^{-\varepsilon} \cdot \limsup_{k\to\infty} \left(\sum_{S_{\mathtt{i}} \in \mathbb{S}_{\eta^k}^1} \left(\frac{|r_{\mathtt{i},1}|}{|r_{\mathtt{i},2}|} \right)^{\dim_B \operatorname{proj}_1(\Lambda)} |r_{\mathtt{i},2}|^{\alpha} \right)^{1/k} \\ &\quad = \limsup_{k\to\infty} \left(\sum_{S_{\mathtt{i}} \in \mathbb{S}_{\eta^k}^1} \left(\frac{|r_{\mathtt{i},1}|}{|r_{\mathtt{i},2}|} \right)^{\dim_B \operatorname{proj}_1(\Lambda)} |r_{\mathtt{i},2}|^{\alpha} \eta^{-\varepsilon k} \right)^{1/k} \\ &\quad \leq \limsup_{k\to\infty} \left(\sum_{S_{\mathtt{i}} \in \mathbb{S}_{\eta^k}^1} \left(\frac{|r_{\mathtt{i},1}|}{|r_{\mathtt{i},2}|} \right)^{\dim_B \operatorname{proj}_1(\Lambda)} |r_{\mathtt{i},2}|^{\alpha-\varepsilon} \right)^{1/k} \\ &\quad \leq \limsup_{k\to\infty} \left(\sum_{S_{\mathtt{i}} \in \mathbb{S}_{\eta^k}^1} \left(\frac{|r_{\mathtt{i},1}|}{|r_{\mathtt{i},2}|} \right)^{\dim_B \operatorname{proj}_1(\Lambda)} |r_{\mathtt{i},2}|^{d_{\eta^k}^1} \right)^{1/k} = 1. \end{aligned} \tag{Eq. 3.52}$$

For the other way, let $\forall\alpha \le d_*^1$ we have that $\forall k_0 > 0\ \exists k > k_0$ such that $\alpha \le d^1_{\eta^k}$ and now

$$\begin{aligned}\Bigg(\sum_{S_{\mathtt{i}}\in\mathbb{S}^1_{\eta^k}} \Big(\frac{|r_{\mathtt{i},1}|}{|r_{\mathtt{i},2}|}\Big)^{\dim_{\mathrm{B}}\mathrm{proj}_1(\Lambda)}|r_{\mathtt{i},2}|^{\alpha}\Bigg)^{1/k} \\ \ge \Bigg(\sum_{S_{\mathtt{i}}\in\mathbb{S}^1_{\eta^k}} \Big(\frac{|r_{\mathtt{i},1}|}{|r_{\mathtt{i},2}|}\Big)^{\dim_{\mathrm{B}}\mathrm{proj}_1(\Lambda)}|r_{\mathtt{i},2}|^{d^1_{\eta^k}}\Bigg)^{1/k} = 1.\end{aligned} \tag{Eq. 3.53}$$

In the cases of $\alpha > \overline{d}_*^1$, and $\alpha < \overline{d}_*^1$ one can do the same. Next, we show that these two sums differ only by a subexponential amount. Observe

$$\begin{aligned}S_{\mathtt{i}}\in\mathbb{S}^1_{\eta^k} &\implies r_{\min,2}^{|\mathtt{i}|} \le |r_{\mathtt{i},2}| \le \eta^k < |r_{\mathtt{i}-,2}| \le r_{\max,2}^{|\mathtt{i}|-1} \\ &\implies |\mathtt{i}|\cdot\frac{\log r_{\min,2}}{\log\eta} \ge k, \quad (|\mathtt{i}|-1)\cdot\frac{\log r_{\max,2}}{\log\eta}\le k \\ &\implies |\mathtt{i}| \ge k\cdot\frac{\log\eta}{\log r_{\min,2}}, \quad |\mathtt{i}|\le k\cdot\frac{\log\eta}{\log r_{\max,2}}+1.\end{aligned} \tag{Eq. 3.54}$$

Therefore for any d

$$\begin{aligned}\sum_{S_{\mathtt{i}}\in\mathbb{S}^1_{\eta^k}} \Big(\frac{|r_{\mathtt{i},1}|}{|r_{\mathtt{i},2}|}\Big)^{\dim_{\mathrm{B}}\mathrm{proj}_1(\Lambda)}|r_{\mathtt{i},2}|^{d} \le \sum_{S_{\mathtt{i}}\in\mathbb{G}^1_{k\frac{\log\eta}{\log r_{\min,2}}}} \Big(\frac{|r_{\mathtt{i},1}|}{|r_{\mathtt{i},2}|}\Big)^{\dim_{\mathrm{B}}\mathrm{proj}_1(\Lambda)}|r_{\mathtt{i},2}|^{d} + \\ \cdots + \sum_{S_{\mathtt{i}}\in\mathbb{G}^1_{k\frac{\log\eta}{\log r_{\max,2}}+1}} \Big(\frac{|r_{\mathtt{i},1}|}{|r_{\mathtt{i},2}|}\Big)^{\dim_{\mathrm{B}}\mathrm{proj}_1(\Lambda)}|r_{\mathtt{i},2}|^{d}\end{aligned} \tag{Eq. 3.55}$$

now

$$\begin{aligned}&\limsup_{k\to\infty}\Bigg(\sum_{S_{\mathtt{i}}\in\mathbb{S}^1_{\eta^k}} \Big(\frac{|r_{\mathtt{i},1}|}{|r_{\mathtt{i},2}|}\Big)^{\dim_{\mathrm{B}}\mathrm{proj}_1(\Lambda)}|r_{\mathtt{i},2}|^{d}\Bigg)^{1/k} \\ &\le \limsup_{k\to\infty}\Bigg(\sum_{n=k\frac{\log\eta}{\log r_{\min,2}}}^{k\frac{\log\eta}{\log r_{\max,2}}+1}\ \sum_{S_{\mathtt{i}}\in\mathbb{G}^1_n} \Big(\frac{|r_{\mathtt{i},1}|}{|r_{\mathtt{i},2}|}\Big)^{\dim_{\mathrm{B}}\mathrm{proj}_1(\Lambda)}|r_{\mathtt{i},2}|^{d}\Bigg)^{1/k} \\ &\le \limsup_{k\to\infty}\Bigg(\Big(-k\frac{\log\eta}{\log r_{\min,2}}+k\frac{\log\eta}{\log r_{\max,2}}+1\Big)\max_{n=k\frac{\log\eta}{\log r_{\min,2}}}^{k\frac{\log\eta}{\log r_{\max,2}}+1}\ \sum_{S_{\mathtt{i}}\in\mathbb{G}^1_n} \Big(\frac{|r_{\mathtt{i},1}|}{|r_{\mathtt{i},2}|}\Big)^{\dim_{\mathrm{B}}\mathrm{proj}_1(\Lambda)}|r_{\mathtt{i},2}|^{d}\Bigg)^{1/k}\end{aligned} \tag{Eq. 3.56}$$

remembering the definition of $\overline{d}_*^1$, and using that $\lim_{k\to\infty} k^{1/k} = 1$ we can observe that the above equation is not greater than one, for any $d > \overline{d}_*^1$. Therefore,

$$\overline{d}_*^1 \ge d_*^1. \tag{Eq. 3.57}$$

For the opposite direction, we proceed similarly. Observe

$$\begin{aligned}S_{\mathtt{i}}\in\mathbb{G}^1_n &\implies \eta^k r_{\min,2} < |r_{\mathtt{i},2}| \le r^n_{\max,2}, \quad r^n_{\min,2}\le |r_{\mathtt{i},2}|\le\eta^k \\ &\implies k < n\cdot\frac{\log r_{\max,2}}{\log\eta} - \frac{\log r_{\min,2}}{\log\eta}, \quad k\ge n\cdot\frac{\log r_{\min,2}}{\log\eta}.\end{aligned} \tag{Eq. 3.58}$$

Then for any d

$$\sum_{S_{\mathtt{i}}\in\mathbb{G}_n}\left(\frac{|r_{\mathtt{i},1}|}{|r_{\mathtt{i},2}|}\right)^{\dim_{\mathrm{B}}\mathrm{proj}_1(\Lambda)}|r_{\mathtt{i},2}|^d \leq \sum_{k=\left\lceil n\frac{\log r_{\min,2}}{\log\eta}\right\rceil}^{\eta^{\left\lfloor n\frac{\log r_{\max,2}}{\log\eta}-\frac{\log r_{\min,2}}{\log\eta}\right\rfloor}}\sum_{S_{\mathtt{i}}\in\mathbb{S}_{\eta^k}}\left(\frac{|r_{\mathtt{i},1}|}{|r_{\mathtt{i},2}|}\right)^{\dim_{\mathrm{B}}\mathrm{proj}_1(\Lambda)}|r_{\mathtt{i},2}|^d \quad \text{(Eq. 3.59)}$$

therefore

$$\begin{aligned}&\limsup_{n\to\infty}\left(\sum_{S_{\mathtt{i}}\in\mathbb{G}_n}\left(\frac{|r_{\mathtt{i},1}|}{|r_{\mathtt{i},2}|}\right)^{\dim_{\mathrm{B}}\mathrm{proj}_1(\Lambda)}|r_{\mathtt{i},2}|^d\right)^{1/n}\\ &\leq\limsup_{n\to\infty}\left(\sum_{k=\left\lceil n\frac{\log r_{\min,2}}{\log\eta}\right\rceil}^{\left\lfloor n\frac{\log r_{\max,2}}{\log\eta}-\frac{\log r_{\min,2}}{\log\eta}\right\rfloor}\sum_{S_{\mathtt{i}}\in\mathbb{S}_{\eta^k}}\left(\frac{|r_{\mathtt{i},1}|}{|r_{\mathtt{i},2}|}\right)^{\dim_{\mathrm{B}}\mathrm{proj}_1(\Lambda)}|r_{\mathtt{i},2}|^d\right)^{1/n}\\ &\leq\limsup_{n\to\infty}\left(a_n\cdot\max_{k=\left\lceil n\frac{\log r_{\min,2}}{\log\eta}\right\rceil}^{\left\lfloor n\frac{\log r_{\max,2}}{\log\eta}-\frac{\log r_{\min,2}}{\log\eta}\right\rfloor}\sum_{S_{\mathtt{i}}\in\mathbb{S}_{\eta^k}}\left(\frac{|r_{\mathtt{i},1}|}{|r_{\mathtt{i},2}|}\right)^{\dim_{\mathrm{B}}\mathrm{proj}_1(\Lambda)}|r_{\mathtt{i},2}|^d\right)^{1/n}\end{aligned} \quad \text{(Eq. 3.60)}$$

where $a_n := \left\lfloor n\frac{\log r_{\max,2}}{\log\eta}-\frac{\log r_{\min,2}}{\log\eta}\right\rfloor-\left\lceil n\frac{\log r_{\min,2}}{\log\eta}\right\rceil$, but still $\limsup_{n\to\infty}a_n^{1/n}=1$. Again assigning any $d>d_*^1$, we achieve that by the definition of d_*^1 the above equation is bounded by 1, and hence

$$\overline{d}_*^1\geq d_*^1 \quad \text{(Eq. 3.61)}$$

whence

$$\overline{d}_*^1=d_*^1 \quad \text{(Eq. 3.62)}$$

as stated. A mirror computation shows the $\overline{d}_*^2=d_*^2$ case.

4 Box-counting dimension in the homogeneous case (thm 2.2)

Assume, as in the theorem, that for any $i\in\Sigma$: $r_{i,1}=r_1, r_{i,2}=r_2$. Suppose that $|r_1|<|r_2|$. Now from 2.1 with the use of $|r_{\mathtt{i},1}|\leq\delta<|r_{\mathtt{i}-,1}|\leq|r_{\mathtt{i},1}|\cdot r_{\min,1}^{-1}$ we can conclude

$$\begin{aligned}\overline{\dim}_{\mathrm{B}}(\Lambda)&=\limsup_{n\to\infty}\frac{\log\left(\sum_{S_{\mathtt{i}}\in\mathbb{G}_n}\left(\frac{|r_1|}{|r_2|}\right)^{-n\dim_{\mathrm{B}}(|\Lambda)}\right)}{-n\log|r_1|}\\ &=\limsup_{n\to\infty}\frac{\log\left(\left(\frac{|r_1|}{|r_2|}\right)^{-n\dim_{\mathrm{B}}(|\Lambda)}\#\mathbb{G}_n\right)}{-n\log|r_1|}\\ &=\frac{\log\left(\left(\frac{|r_1|}{|r_2|}\right)^{-\dim_{\mathrm{B}}(|\Lambda)}\right)}{-\log|r_1|}+\limsup_{n\to\infty}\frac{\log\left(\#\mathbb{G}_n\right)}{-n\log|r_1|}\\ &=\dim_{\mathrm{B}}(|\Lambda)\Big(1-\frac{\log|r_2|}{\log|r_1|}\Big)+\limsup_{n\to\infty}\frac{\log\left(\#\mathbb{G}_n\right)}{-n\log|r_1|}.\end{aligned} \quad \text{(Eq. 4.1)}$$

For the $\liminf$ we will consider subsystems, define one *n-th level strongly separated sub-self-affine set* (nothing stops us from abbreviating it by SSSSAS), G_n^* with the following algorithm:

- Factorise $\mathbb{G}_n$:
$$\mathbb{G}_n = \bigcup_{|S_\mathtt{i} \in |\mathbb{G}_n} R_{|S_\mathtt{i}} \quad \text{,where} \quad R_{|S_\mathtt{i}} := \big\{ S_\mathtt{j} \in \mathbb{G}_n \;\big|\; |S_\mathtt{j} = |S_\mathtt{i} \big\}. \tag{Eq. 4.2}$$

- For each $R_{|S_\mathtt{i}}$ let $R^*_{|S_\mathtt{i}}$ be a maximal populous subset such that functions in it are cylinder-disjoint.

- Now define $|G_n^* \subset |\mathbb{G}_n$ iteratively: Order the elements of $|\mathbb{G}_n$ with the values $\#R^*_{|S_\mathtt{i}}$, then continuously remove maximal elements (whose $\#R^*_{|S_\mathtt{i}}$ are maximal) from $|\mathbb{G}_n$, adding them to $|G_n^*$ iff their respective $|S_\mathtt{i}$ is cylinder-disjoint from those who correspond to the ones added before to $|G_n^*$.

- Finally
$$G_n^* := \bigcup_{|S_\mathtt{i} \in |G_n^*} R^*_{|S_\mathtt{i}}. \tag{Eq. 4.3}$$

G_n^* is another diagonal IFS, we let Λ_n denote its attractor. Trivially (since $G_n^* \subset \mathbb{G}_n$) $\Lambda_n \subset \Lambda$ for any n, consequently
$$\underline{\dim}_\mathrm{B}(\Lambda) \geq \underline{\dim}_\mathrm{B}(\Lambda_n) \tag{Eq. 4.4}$$
for any $n \in \mathbb{N} \setminus \{0\}$. We have that G_n^* is sufficiently large, since using twice Lemma 1.6 we have
$$\begin{aligned}\#G_n^* \leq \#\mathbb{G}_n = \#\Big\{ \bigcup_{|S \in |\mathbb{G}_n} R_{|S} \Big\} &\leq \underline{C} \cdot \#\Big\{ \bigcup_{|S \in |G_n} R^*_{|S} \Big\} \\ &\leq |C \cdot \underline{C} \cdot \#\Big\{ \bigcup_{|S \in |G_n^*} R^*_{|S} \Big\} = |C \cdot \underline{C} \cdot \#G_n^*,\end{aligned} \tag{Eq. 4.5}$$
where the constants $|C$ and $\underline{C}$ depend only on $|\Lambda$ and on $\underline{\Lambda}$. One particular reason why we needed to construct G_n^* is that this IFS satisfies the ROSC. Now using Theorem 1.1, we get that
$$\dim_\mathrm{B}(\Lambda_n) = \max\{d_1, d_2\} \quad \text{where} \quad \sum_{S_i \in G_n^*} |r_1|^{n \dim_\mathrm{B}(\underline{\Lambda})} |r_2|^{n d_1 - n \dim_\mathrm{B}(\underline{\Lambda})} = 1 \tag{Eq. 4.6}$$
$$\sum_{S_i \in G_n^*} |r_1|^{n d_2 - n \dim_\mathrm{B}(|\Lambda)} |r_2|^{n \dim_\mathrm{B}(|\Lambda)} = 1. \tag{Eq. 4.7}$$

Observe that we in fact do not need to figure out whether d_1 or d_2 is larger, since from Eq. 4.7 we obtain:
$$d_2 = d_2(n) = \dim_\mathrm{B}(|\Lambda)\Big(1 - \frac{\log|r_2|}{\log|r_1|}\Big) - \frac{1}{n}\frac{\log\left(G_n^*\right)}{\log|r_1|}, \tag{Eq. 4.8}$$
and now circling back with Eq. 4.4 and Eq. 4.5:
$$\overline{\dim}_\mathrm{B}(\Lambda) \geq \underline{\dim}_\mathrm{B}(\Lambda) \geq \underline{\dim}_\mathrm{B}(\Lambda_n) = \dim_\mathrm{B}(\Lambda_n) \geq d_2(n) \longrightarrow \overline{\dim}_\mathrm{B}(\Lambda). \tag{Eq. 4.9}$$
Finally utilizing Theorem 1.5 and Theorem 1.3
$$\dim_\mathrm{B}(|\Lambda) = \dim_\mathrm{H}(|\Lambda) = -\lim_{n\to\infty} \frac{1}{n} \frac{\log\left(\#|\mathbb{G}_n\right)}{\log|r_2|}, \tag{Eq. 4.10}$$
from where one can conclude the statement of the theorem.

5 Hausdorff dimension in the homogeneous case (thm 2.3)

We again assume that for any $i \in \Sigma : r_{i,1} = r_1,\ r_{i,2} = r_2$. Suppose $|r_1| < |r_2|$ without loss of generality. We will use the following pair of statements:

Theorem 5.1 (L.S. Young, Theorem 1.4.20 in [3]) *Let A be measurable with $\mu(A) > 0$. Suppose that*

$$\forall x \in A : \qquad a \leq \liminf_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} \leq b. \tag{Eq. 5.1}$$

Then

$$a \leq \dim_{\mathrm{H}}(A) \leq b. \tag{Eq. 5.2}$$

Lemma 5.2 (Volume Lemma, Theorem 1.9.5 in [3]) *Let μ be a Borel, probability measure in $\mathbb{R}^d$, such that*

$$\mu(\forall)\ x : \qquad a \leq \lim_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} \leq b \tag{Eq. 5.3}$$

then

$$a \leq \dim_{\mathrm{H}}(\mu) \leq b, \tag{Eq. 5.4}$$

where $\mu(\forall)\ x$ means: for μ-almost all x. In particular, if $\mu(\forall)\ x \in \Lambda :\ \lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} = d$, then $\dim_{\mathrm{H}}(\mu) = d$.

The plan is the following:

- We will construct a series of measures supported inside the attractor, of whose Hausdorff dimensions can be lower bounded by the Volume Lemma.
- We will use Young's theorem to get an upper bound for the Hausdorff dimension of the attractor.
- Finally, we will observe that the achieved series of lower bounds converge to the upper bound with the use of the assumed weak separation condition.

5.0.1 Approximate squares

The value $\liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}$ is called local dimension, and it suggests to investigate how for an arbitrary $x \in \Lambda$ a small neighbourhood surrounding it looks like. For this, we define, for $\delta > 0$, the *symbolic approximate square*: approximate, because we let k, ℓ be such that $|r_1|^k \leq \delta < |r_1|^{k-1}$ and $|r_2|^\ell \leq \delta < |r_2|^{\ell-1}$. Explicitly $k = k(\delta) := \lceil \log_{|r_1|} \delta \rceil$ and $\ell = \ell(\delta) := \lceil \log_{|r_2|} \delta \rceil$. Then

$$\begin{aligned} B_\delta(\mathbf{i}) :=& \left\{ \mathbf{j} \in \Sigma^\infty \;\middle|\; \mathbf{i}|_{(0,k]} = \mathbf{j}|_{(0,k]} \right\} \bigcap \left\{ \mathbf{j} \in \Sigma^\infty \;\middle|\; |S_{\mathbf{j}|_{(0,\ell]}}([0,1]) = |S_{\mathbf{i}|_{(0,\ell]}}([0,1]) \right\} \\ =& \left\{ \mathbf{j} \;\middle|\; \mathbf{i} \text{ and } \mathbf{j} \text{ have the same letters untill } k \text{ steps} \right\} \\ & \bigcap \left\{ \mathbf{j} \;\middle|\; S_{\mathbf{i}}([0,1]^2), S_{\mathbf{j}}([0,1]^2) \text{ are in the same row in the } \ell\text{-th step} \right\} \end{aligned} \tag{Eq. 5.5}$$

is called a symbolic approximate square, its image trough π is to be called an *approximate square*. The second set in the intersection can be characterized as the set of $\mathbf{j}$s such that $\sum_{i=k+1}^{\ell} t_{\mathbf{i}_i,2} \cdot |r_2|^i = \sum_{i=k+1}^{\ell} t_{\mathbf{j}_i,2} \cdot |r_2|^i$, which means that from the $k+1$-th to the ℓ-th level their y coordinate wise translation agrees with the one for $\mathbf{i}$, we denote this by $t_{\mathbf{i}|_{(k,\ell]},2} = t_{\mathbf{j}|_{(k,\ell]},2}$.

One similarly can define the *n-symbolic approximate squares*, $B^n_\delta(\mathbf{i})$, who take k and ℓ to be multiples of n $(k(\delta) := n\lceil \log_{r_1}(\delta)/n \rceil, \ell(\delta) := n\lceil \log_{r_2}(\delta)/n \rceil)$.

5.1 Lower bound

The measures who will provide the lower bound will arise as a self-affine measures of a large enough separated subsystem. For this let G_n^* be an *n-th level strongly separated sub-self-affine set* (defined in Eq. 4.3).

Define a *self-affine measure* on it: Let p_n be a probability vector of length $\mathfrak{m}^n$, weighting the elements of Σ^n in a way that if $S_{\mathtt{i}} \notin G_n^*$, then $p_{\mathtt{i}} := 0$, and if $S_{\mathtt{i}} = S_{\mathtt{j}}$ then only the lexicographically smallest can have weight. Define ν_{p_n} on $\Sigma^{n*} := \bigcup_{m\in\mathbb{N}} \left(\{1,\dots,\mathfrak{m}\}^n\right)^m$: for $\mathtt{i} \in \Sigma^{n*}$ let $\nu_p(\mathtt{i}) := \prod_{i=1}^{|\mathtt{i}|} p_{\mathtt{i}_i}$, then by Kolmogorov's existence theorem we can extend this to a measure on Σ_n^∞. Let μ_n be the push-forward measure of this extended measure: $\mu_n = \mu_{p_n} = \pi_* \nu_{p_n}$.

We have that $\dim_{\mathrm{H}}(\Lambda_n) \geq \dim_{\mathrm{H}}(\mu_n)$ for any particular choice of p_n, so we will use the Volume Lemma (5.2) to lower bound $\dim_{\mathrm{H}}(\mu_n)$, and then maximize the given formula in p_n to achieve the sufficiently large lower bound.

To relate the measure of a Euclidean ball and of an approximate square we have the following lemma:

Lemma 5.3 *Assuming μ_n is not supported in a line segment, we have that*

$$\mu_n\left(\pi(\mathbf{i}) \;\middle|\; B\left(\pi(\mathbf{i}), \delta |r_1|^{n\sqrt{\log_{|r_1|^n}\delta}}\right) \cap \Lambda_n \subset \pi\left(B_\delta^n(\mathbf{i})\right) \text{ for all } \delta \text{ sufficiently small}\right) = 1. \quad \text{(Eq. 5.6)}$$

The proof can be found in [15] with the address Lemma 3.2. The assumption restricting μ_n from being supported in a line segment will be satisfied with the general assumption that the original IFS fills $[0,1]^2$, and with the choice of p_n we will achieve from the maximization at the end of this section.

In essence, the Lemma lets us to cover a Euclidean ball's intersection with the attractor by an approximate square, with only experiencing a subexponential price, which gets eliminated as we take $\lim_{r\to 0}$. Precisely: given that μ_n is not supported only on a line segment, by the Lemma 5.3 we have that for r sufficiently small (i.e. $\nu_n(\forall)\mathbf{i}\ \exists r_o = r_o(\mathbf{i})\ \forall r < r_0$)

$$\mu_n\left(B\left(\pi(\mathbf{i}), r|r_1|^{n\sqrt{\log_{|r_1|^n} r}}\right)\right) \leq \mu_n\left(\pi\left(B_r^n(\mathbf{i})\right)\right) \quad \text{(Eq. 5.7)}$$

for $\mu_n(\forall)$ $\mathbf{i}$, and by the separation of the cylinder rectangles in the constructed Λ_ns we have that

$$\mu_n\left(\pi\left(B_r^n(\mathbf{i})\right)\right) = \nu_{p_n}\left(B_r^n(\mathbf{i})\right) \quad \text{(Eq. 5.8)}$$

for $\mathbf{i}$ such that $\pi(\mathbf{i}) \in \Lambda_n$ and for any $\delta = |r_2|^N$, $N \in \mathbb{N}$ (the existence of the limit will reason the ability to use a subsequence). Also

$$\lim_{r\to 0} \frac{\log r}{\log\left(r|r_1|^{n\sqrt{\log_{|r_1|^n} r}}\right)} = 1. \quad \text{(Eq. 5.9)}$$

For $\mathbf{i}$ such that $\pi(\mathbf{i}) \in \Lambda_n$, we have an explicit formula for the symbolic measure of a symbolic approximate square

$$\nu_{p_n}\left(B_r^n(\mathbf{i})\right) = \left(\prod_{i=0}^{\frac{k(r)}{n}-1} p_{\mathbf{i}|_{(i\cdot n,(i+1)\cdot n]}}\right)\left(\prod_{i=0}^{\frac{\ell(r)-k(r)}{n}-1} \sum_{\mathtt{j}\in\Sigma^n:\ t_{\mathbf{i}|_{(i\cdot n,(i+1)\cdot n]},2} = t_{\mathtt{j}|_{(0,n]},2}} p_{\mathtt{j}}\right) \quad \text{(Eq. 5.10)}$$

Finally $|r_2|^\ell \le \delta < |r_2|^{\ell-1}$ means that $\ell(r)\log|r_2| \le \log r < \log(|r_2|^{\ell(r)} \cdot \frac{1}{|r_2|})$, and hence

$$\lim_{r\to 0} \frac{\log r}{\ell(r)\log|r_2|} = 1. \tag{Eq. 5.11}$$

Out of all of these we get that

$$\lim_{r\to 0} \frac{\log \mu_n(B(\pi(\mathbf{i}), r))}{\log r} \ge \lim_{r\to 0} \frac{\log(\nu_{p_n}(B_r^n(\mathbf{i})))}{\ell(r)\log|r_2|}. \tag{Eq. 5.12}$$

Now calculate!

$$\begin{aligned}
&\frac{\log(\nu_{p_n}(B_r^n(\mathbf{i})))}{\ell(r)\log|r_2|} = \\
&\quad = \frac{\log\left(\prod_{i=0}^{\frac{k(r)}{n}-1} p_{\mathbf{i}|_{(i\cdot n,(i+1)\cdot n]}}\right)}{\ell(r)\log|r_2|} \\
&\qquad + \frac{\log\left(\prod_{i=0}^{\frac{\ell(r)-k(r)}{n}-1} \sum_{\mathtt{j}\in\Sigma^n:\ t_{\mathbf{i}|_{(i\cdot n,(i+1)\cdot n]},2}=t_{\mathtt{j}|_{(0,n]},2}} p_{\mathtt{j}}\right)}{\ell(r)\log|r_2|} \\
&\quad = \frac{\sum_{i=0}^{\frac{k(r)}{n}-1} \log\left(p_{\mathbf{i}|_{(i\cdot n,(i+1)\cdot n]}}\right)}{\ell(r)\log|r_2|} \\
&\qquad + \frac{\sum_{i=0}^{\frac{\ell(r)-k(r)}{n}-1} \log\left(\sum_{\mathtt{j}\in\Sigma^n:\ t_{\mathbf{i}|_{(i\cdot n,(i+1)\cdot n]},2}=t_{\mathtt{j}|_{(0,n]},2}} p_{\mathtt{j}}\right)}{\ell(r)\log|r_2|} \\
&\quad = \frac{1}{n}\frac{k(r)}{\ell(r)} \frac{\frac{1}{\frac{k(r)}{n}} \sum_{i=0}^{\frac{k(r)}{n}-1} \log\left(p_{\mathbf{i}|_{(i\cdot n,(i+1)\cdot n]}}\right)}{\log|r_2|} \\
&\qquad + \frac{1}{n}\left(1-\frac{k(r)}{\ell(r)}\right) \frac{\frac{1}{\frac{\ell(r)-k(r)}{n}} \sum_{i=0}^{\frac{\ell(r)-k(r)}{n}-1} \log\left(\sum_{\mathtt{j}\in\Sigma^n:\ t_{\mathbf{i}|_{(i\cdot n,(i+1)\cdot n]},2}=t_{\mathtt{j}|_{(0,n]},2}} p_{\mathtt{j}}\right)}{\log|r_2|}
\end{aligned} \tag{Eq. 5.13}$$

Following by the Strong Law of Large Numbers, we have that:

$$\lim_{r\to 0}\left(\frac{1}{\frac{k(r)}{n}} \sum_{i=0}^{\frac{k(r)}{n}-1} \log\left(p_{\mathbf{i}|_{(i\cdot n,(i+1)\cdot n]}}\right)\right) = \sum_{\mathtt{i}\in\Sigma^n} p_{\mathtt{i}}\log p_{\mathtt{i}} =: A \tag{Eq. 5.14}$$

$$\begin{aligned}
&\lim_{r\to 0}\left(\frac{1}{\frac{\ell(r)-k(r)}{n}} \sum_{i=0}^{\frac{\ell(r)-k(r)}{n}-1} \log\left(\sum_{\mathtt{j}\in\Sigma^n:\ t_{\mathbf{i}|_{(i\cdot n,(i+1)\cdot n]},2}=t_{\mathtt{j}|_{(0,n]},2}} p_{\mathtt{j}}\right)\right) \\
&\qquad = \sum_{\mathtt{i}\in\Sigma^n} p_{\mathtt{i}} \log\left(\sum_{\mathtt{j}\in\Sigma^n:\ t_{\mathtt{i},2}=t_{\mathtt{j},2}} p_{\mathtt{j}}\right) =: B
\end{aligned} \tag{Eq. 5.15}$$

also

$$\lim_{r\to 0}\left(\frac{k(r)}{\ell(r)}\right) = \frac{\log|r_2|}{\log|r_1|}. \tag{Eq. 5.16}$$

Hence

$$\lim_{r\to 0}\frac{\log \mu_n(B(\pi(\mathbf{i}),r))}{\log r} \geq \frac{1}{n}\left(\frac{A}{\log|r_1|} + \left(1-\frac{\log|r_2|}{\log|r_1|}\right)\frac{B}{\log|r_2|}\right) \tag{Eq. 5.17}$$

for any $\mathbf{i}$ such that $\pi(\mathbf{i})$ is in Λ_n. This is still dependent on the choices of $p_{\mathtt{i}}$, but can be maximized using for example the Lagrange multiplier method.

For a fix n let $p_{i,j}$ be $p_{\mathtt{i}}$ iff $S_{\mathtt{i}} \in G_n^*$ is the j'th cylinder rectangle in the i'th row in Λ_n, and $\mathtt{i}$ is the lexicographically smallest among those, then

$$\begin{aligned}F(p) := \frac{1}{n}&\frac{\sum_i\sum_j p_{i,j}\log p_{i,j}}{\log|r_1|}\\ &+\frac{1}{n}(\sum_i(\sum_j p_{i,j})\log(\sum_j p_{i,j}))\left(\frac{1}{\log|r_2|}-\frac{1}{\log|r_1|}\right)+\lambda(\sum_i\sum_j p_{i,j}-1)\end{aligned} \tag{Eq. 5.18}$$

is our Lagrangian. Differentiate it to get

$$\begin{aligned}\frac{\partial(F(p))}{\partial p_{k,l}} &= \frac{1}{n}\frac{1}{\log|r_1|}(\log p_{k,l}+1)+\frac{1}{n}\frac{\partial}{\partial p_{k,l}}(\sum_i(\sum_j p_{i,j})\log(\sum_j p_{i,j}))\left(\frac{1}{\log|r_2|}-\frac{1}{\log|r_1|}\right)+\lambda\\ &= \frac{1}{n}\frac{1}{\log|r_1|}(\log p_{k,l}+1)+\frac{1}{n}\left(\frac{1}{\log|r_2|}-\frac{1}{\log|r_1|}\right)\left(\frac{\partial}{\partial p_{k,l}}((\sum_j p_{k,j})\log(\sum_j p_{k,j}))\right)+\lambda\\ &= \frac{1}{n}\frac{1}{\log|r_1|}(\log p_{k,l}+1)+\frac{1}{n}\left(\frac{1}{\log|r_2|}-\frac{1}{\log|r_1|}\right)\left(\frac{\partial}{\partial p_{k,l}}(\sum_j p_{k,j})\log(\sum_j p_{k,j})\right.\\ &\qquad\left.+(\sum_j p_{k,j})\frac{\partial}{\partial p_{k,l}}\log(\sum_j p_{k,j})\right)+\lambda\\ &= \frac{1}{n}\frac{1}{\log|r_1|}(\log p_{k,l}+1)+\frac{1}{n}\left(\frac{1}{\log|r_2|}-\frac{1}{\log|r_1|}\right)\left(\log\sum_j p_{k,j}+1\right)+\lambda\end{aligned} \tag{Eq. 5.19}$$

hence

$$\begin{aligned}\frac{\partial(F(p))}{\partial p_{k,l}} = 0 &\iff \log p_{k,l}+\log_{|r_2|}|r_1|+(\log\sum_j p_{k,j})(\log_{|r_2|}|r_1|-1)+\lambda n\log r_1 = 0\\ &\iff p_{k,l} = \frac{(\sum_j p_{k,j})^{(1-\log_{|r_2|}|r_1|)}}{|r_1|^{\lambda n+1/\log|r_2|}}.\end{aligned} \tag{Eq. 5.20}$$

Now one can deduce that $p_{k,\ell}$ is the same for functions having cylinders rectangles in the same row. Denote the measure on the i'th row by p_i, and the number of cylinder rectangles in the i'th row by n_i, then

$$p_k := \sum_\ell p_{k,\ell} = n_k p_{k,1} = n_k\frac{(\sum_j p_{k,j})^{(1-\log_{|r_2|}|r_1|)}}{|r_1|^{\lambda n+1/\log|r_2|}} = n_k\frac{(p_k)^{(1-\log_{|r_2|}|r_1|)}}{|r_1|^{\lambda n+1/\log|r_2|}}, \tag{Eq. 5.21}$$

using that $\sum_k p_k = 1$

$$p_k = \frac{n_k^{\log_{|r_1|}|r_2|}}{\sum_\ell n_\ell^{\log_{|r_1|}|r_2|}},\qquad p_{k,l} = \frac{1}{n_k}\frac{n_k^{\log_{|r_1|}|r_2|}}{\sum_\ell n_\ell^{\log_{|r_1|}|r_2|}} = \frac{n_k^{\log_{|r_1|}|r_2|-1}}{\sum_\ell n_\ell^{\log_{|r_1|}|r_2|}}. \tag{Eq. 5.22}$$

Using these, and the notation: $K := \sum_\ell n_\ell^{\log_{|r_1|}|r_2|}$

$$\begin{aligned}\lim_{r\to 0}\frac{\log\mu(B(\pi(\mathbf{i}),r))}{\log r} &\geq \frac{1}{n}\left(\frac{\sum_k\sum_\ell p_{k,\ell}\log p_{k,\ell}}{\log|r_1|}+\left(1-\frac{\log|r_2|}{\log|r_1|}\right)\frac{\sum_k p_k\log p_k}{\log|r_2|}\right)\\
&= \frac{1}{n}\Bigg(\frac{\sum_k\sum_\ell p_{k,\ell}\Big((\log_{|r_1|}|r_2|-1)\log n_k-\log K\Big)}{\log|r_1|}\\
&\qquad+\left(1-\frac{\log|r_2|}{\log|r_1|}\right)\frac{\sum_k p_k\Big(\log_{|r_1|}|r_2|\log n_k-\log K\Big)}{\log|r_2|}\Bigg)\\
&= \frac{1}{n}\Bigg(\frac{(\log_{|r_1|}|r_2|-1)\sum_k p_k\log n_k}{\log|r_1|}-\frac{\log K}{\log|r_1|}\\
&\qquad+\left(1-\frac{\log|r_2|}{\log|r_1|}\right)\frac{\sum_k p_k\log_{|r_1|}|r_2|\log n_k}{\log|r_2|}-\left(1-\frac{\log|r_2|}{\log|r_1|}\right)\frac{\log K}{\log|r_2|}\Bigg)=-\frac{1}{n}\frac{\log K}{\log|r_2|}\end{aligned}\tag{Eq. 5.23}$$

and then with the Volume Lemma (5.2) we obtain

Lemma 5.4

$$\forall n:\quad \dim_{\mathrm{H}}(\Lambda)\geq -\frac{1}{n}\log_{|r_2|}\left(\sum_{|S_{\mathtt{i}}\in|G_n^*}\left(\#R^*_{|S_{\mathtt{i}}}\right)^{\log_{|r_1|}|r_2|}\right).\tag{Eq. 5.24}$$

5.2 Upper bound

As mentioned before, we will use Young's theorem to deduce a sufficient series of upper bounds. This requires one to lower bound the measure of arbitrary small balls around ALL the points of the attractor. This means the measures μ_n are insufficient, since they leave many functions, and hence many points in Λ, measureless. To combat this, we only change them a little.

Let $\eta_n = \eta_{p_n}$ be defined with weight distribution on Σ^n, such that if for $\mathtt{i},\mathtt{j}\in\Sigma^n$ we have that $S_{\mathtt{i}} = S_{\mathtt{j}}$, then only the lexicographically smallest word can have weight, extended it by the Kolmogorov extension theorem to infinite words. This again is a self affine measure respective to Σ^n, which in general could not necessarily be derived from a measure defined on Σ, and then extended to Σ^n as a product measure. Denote $\varrho_n = \varrho_{p_n} = \pi_*\eta_{p_n}$ its push forward. Now we have that for any $\mathbf{j}\in\Sigma^\infty$ there is an $\mathbf{i}\in\Sigma^\infty$ with $\pi(\mathbf{i}) = \pi(\mathbf{j})$ such that

$$\begin{aligned}\varrho_{p_n}(B(\pi(\mathbf{j}),\sqrt{2}r)) &\geq \varrho_{p_n}(B_r^n(\mathbf{i}))\\
&\geq \left(\prod_{i=0}^{\frac{k(r)}{n}-1}p_{\mathbf{i}|_{(i\cdot n,(i+1)\cdot n]}}\right)\left(\prod_{i=0}^{\frac{\ell(r)-k(r)}{n}-1}\sum_{\mathtt{j}\in\Sigma^n:\ t_{\mathbf{i}|_{(i\cdot n,(i+1)\cdot n]},2}=t_{\mathtt{j}|_{(0,n]},2}}p_{\mathtt{j}}\right).\end{aligned}\tag{Eq. 5.25}$$

Now choose the weights exactly as in the lower bound: For a fix n let $p_{i,j}$ be $p_{\mathtt{i}}$ iff $S_{\mathtt{i}}\in\mathbb{G}_n$ is the j'th cylinder rectangle in the i'th row in Λ_n, and $\mathtt{i}$ is the lexicographically smallest among those, then choose

$$p_{k,l} = \frac{1}{n_k}\frac{n_k^{\log_{|r_1|}|r_2|}}{\sum_\ell n_\ell^{\log_{|r_1|}|r_2|}} = \frac{n_k^{\log_{|r_1|}|r_2|-1}}{\sum_\ell n_\ell^{\log_{|r_1|}|r_2|}}\tag{Eq. 5.26}$$

where n_i denotes the number of cylinder rectangles from $\mathbb{G}$ in the i'th row, and where the sum $\sum_\ell n_\ell^{\log_{|r_1|}|r_2|}$ is as in the lower bound. With this we have that

$$\Big(\prod_{i=0}^{\frac{k(r)}{n}-1} p_{\mathbf{i}|_{(i\cdot n,(i+1)\cdot n]}}\Big)\Big(\prod_{i=0}^{\frac{\ell(r)-k(r)}{n}-1} \sum_{\mathtt{j}\in\Sigma^n:\ t_{\mathbf{i}|_{(i\cdot n,(i+1)\cdot n]},2}=t_{\mathtt{j}|_{(0,n]},2}} p_{\mathtt{j}}\Big)$$

$$= \frac{\Big(\prod_{i=0}^{\frac{k(r)}{n}-1} n(\mathbf{i}|_{(i\cdot n,(i+1)\cdot n]})^{\log_{|r_1|}|r_2|-1}\Big)\Big(\prod_{i=\frac{k(r)}{n}}^{\frac{\ell(r)}{n}-1} n(\mathbf{i}|_{(i\cdot n,(i+1)\cdot n]})^{\log_{|r_1|}|r_2|}\Big)}{\Big(\sum_\ell n_\ell^{\log_{|r_1|}|r_2|}\Big)^{\frac{\ell(r)}{n}}} \quad \text{(Eq. 5.27)}$$

where for $\mathtt{j}\in\Sigma^n$ we define $n(\mathtt{j})$ the number of cylinder rectangles in the same row as $S_{\mathtt{j}}$. Then

$$\begin{aligned}
\frac{\log \varrho_{p_n}(B_r^n(\mathbf{j}))}{\ell(r)\log|r_2|} &\le \Big(\frac{\log_{|r_1|}|r_2|-1}{\ell(r)\log|r_2|}\Big)\sum_{i=0}^{\frac{k(r)}{n}-1}\log n(\mathbf{i}|_{(i\cdot n,(i+1)\cdot n]}) + \Big(\frac{\log_{|r_1|}|r_2|}{\ell(r)\log|r_2|}\Big) \\
&\quad\cdot \sum_{i=\frac{k(r)}{n}}^{\frac{\ell(r)}{n}-1}\log n(\mathbf{i}|_{(i\cdot n,(i+1)\cdot n]}) - \Big(\frac{\ell(r)}{n\ell(r)\log|r_2|}\Big)\log\sum_\ell n_\ell^{\log_{|r_1|}|r_2|} \\
&= -\frac{1}{n}\Big(\log_{|r_2|}\sum_\ell n_\ell^{\log_{|r_1|}|r_2|}\Big) + \frac{1}{\ell(r)}\Bigg[\frac{1}{\log|r_1|}\sum_{i=0}^{\frac{\ell(r)}{n}-1}\log n(\mathbf{i}|_{(i\cdot n,(i+1)\cdot n]}) \\
&\quad - \frac{1}{\log|r_2|}\sum_{i=0}^{\frac{k(r)}{n}-1}\log n(\mathbf{i}|_{(i\cdot n,(i+1)\cdot n]})\Bigg] \qquad \text{(Eq. 5.28)} \\
&= -\frac{1}{n}\Big(\log_{|r_2|}\sum_\ell n_\ell^{\log_{|r_1|}|r_2|}\Big) + \frac{1}{\log|r_1|}\Bigg[\frac{1}{\ell(r)}\sum_{i=0}^{\frac{\ell(r)}{n}-1}\log n(\mathbf{i}|_{(i\cdot n,(i+1)\cdot n]}) \\
&\quad - \frac{\log|r_1|}{\log|r_2|}\frac{k(r)}{\ell(r)}\frac{1}{k(r)}\sum_{i=0}^{\frac{k(r)}{n}-1}\log n(\mathbf{i}|_{(i\cdot n,(i+1)\cdot n]})\Bigg]
\end{aligned}$$

finally we have that $k(r)/\ell(r)\to\log|r_1|/\log|r_2|$ and

$$\begin{aligned}
&\limsup_{r\to\infty}\frac{1}{\ell(r)}\sum_{i=0}^{\frac{\ell(r)}{n}-1}\log n(\mathbf{i}|_{(i\cdot n,(i+1)\cdot n]}) - \frac{1}{k(r)}\sum_{i=0}^{\frac{k(r)}{n}-1}\log n(\mathbf{i}|_{(i\cdot n,(i+1)\cdot n]}) \\
&\qquad = \log\limsup_{r\to\infty}\frac{\Big(\prod_{i=0}^{\frac{\ell(r)}{n}-1} n(\mathbf{i}|_{(i\cdot n,(i+1)\cdot n]})\Big)^{1/\ell(r)}}{\Big(\prod_{i=0}^{\frac{k(r)}{n}-1} n(\mathbf{i}|_{(i\cdot n,(i+1)\cdot n]})\Big)^{1/k(r)}} = 0. \qquad \text{(Eq. 5.29)}
\end{aligned}$$

Indeed, suppose that the right above $\limsup < 1$, then use the following lemma:

Lemma 5.5 *Given $\{a_n\}_{n\in\mathbb{N}}$, a sequence of real numbers, and $c>1$ if $\limsup_{n\to\infty} a_{\lceil cn\rceil}/a_n < 1$ then $\liminf a_n = 0$.*

One proves this by letting the fraction sequence be at most $\varepsilon\in(0,1)$ for all n after a sufficiently large N, and then for a large M writing a_M as the product of a_N and an increasing number of elements in the fraction sequence who all can be bound by ε-s. Then with

$$a_{k(r)} := \Big(\prod_{i=0}^{\frac{k(r)}{n}-1} n(\mathbf{i}|_{(i\cdot n,(i+1)\cdot n]}) \Big)^{1/k(r)}, \ c := \log|r_1| / \log|r_2| \tag{Eq. 5.30}$$

the assumption that the $\limsup < 1$ (i.e. in Eq. 5.29 we have < 0) forwarded by the lemma gives that for any $\mathbf{i}$

$$\liminf_{r\to 0} a_{k(r)} = 0 \quad \text{while contrary to that:} \quad a_{k(r)} \geq 1. \tag{Eq. 5.31}$$

Hence the formula in Eq. 5.29 must hold with $= 0$. Therefore

$$\liminf_{r\to 0} \frac{\log \varrho_{p_n}(B_r^n(\mathbf{j}))}{\ell(r)\log|r_2|} \leq -\frac{1}{n}\Big(\log_{|r_2|} \sum_{\ell} n_\ell^{\log_{|r_1|}|r_2|} \Big). \tag{Eq. 5.32}$$

Finally using Young's theorem and reformulating it to match the style of the lower bound:

Lemma 5.6

$$\forall n: \quad \dim_{\mathrm{H}}(\Lambda) \leq -\frac{1}{n}\log_{|r_2|}\Bigg(\sum_{|S_\mathtt{i}\in|\mathbb{G}_n} (\#R_{|S_\mathtt{i}})^{\log_{|r_1|}|r_2|} \Bigg). \tag{Eq. 5.33}$$

The not too tired reader may recognise that this method of just substituting the guessed measure would have worked for the lower bound as well, but building up the proof like that would have left no sign of how the sufficient measure may arise from the setup.

5.3 Assembling the bounds

We restate Lemma 1.6 as

- $\exists\ |C < \infty \quad \forall x \in \mathbb{R} \quad \forall \delta > 0$ *we have that:*

$$\#\Big\{ |S_\mathtt{j} \in |\mathbb{G}_{\ell(\delta)} \ \Big| \ |S_\mathtt{j}(\Lambda) \cap B(x,\delta) \neq \emptyset \Big\} \leq |C, \tag{Eq. 5.34}$$

- $\exists\ \underline{C} < \infty \quad \forall x \in \mathbb{R} \quad \forall \delta > 0$ *we have that:*

$$\#\Big\{ \underline{S}_\mathtt{j} \in \underline{\mathbb{G}}_{k(\delta)} \ \Big| \ \underline{S}_\mathtt{j}(\Lambda) \cap B(x,\delta) \neq \emptyset \Big\} \leq \underline{C}, \tag{Eq. 5.35}$$

in particular

- $\exists\ |C < \infty \quad \forall n \in \mathbb{N}\setminus\{0\} \quad \forall |S_\mathtt{i} \in |\mathbb{G}_n$ *we have that:*

$$\#\Big\{ |S_\mathtt{j} \in |\mathbb{G}_n \ \Big| \ |S_\mathtt{j}([0,1]) \cap |S_\mathtt{i}([0,1]) \neq \emptyset \Big\} \leq |C, \tag{Eq. 5.36}$$

- $\exists\ \underline{C} < \infty \quad \forall n \in \mathbb{N}\setminus\{0\} \quad \forall \underline{S}_\mathtt{i} \in \underline{\mathbb{G}}_n$ *we have that:*

$$\#\Big\{ \underline{S}_\mathtt{j} \in \underline{\mathbb{G}}_n \ \Big| \ \underline{S}_\mathtt{j}([0,1]) \cap \underline{S}_\mathtt{i}([0,1]) \neq \emptyset \Big\} \leq \underline{C}. \tag{Eq. 5.37}$$

Finally using these as in the previous theorems

$$\sum_{|S_{\mathtt{i}}\in|\mathbb{G}_n} \left(\#R_{|S_{\mathtt{i}}}\right)^{\log_{|r_1|}|r_2|} \le \sum_{|S_{\mathtt{i}}\in|\mathbb{G}_n} \left(\underline{C}\cdot\#R^*_{|S_{\mathtt{i}}}\right)^{\log_{|r_1|}|r_2|} \le \underline{C}^{\log_{|r_1|}|r_2|}\cdot|C\cdot \sum_{|S_{\mathtt{i}}\in|G^*_n} \left(\#R^*_{|S_{\mathtt{i}}}\right)^{\log_{|r_1|}|r_2|} \tag{Eq. 5.38}$$

and hence we have

$$\dim_{\mathrm{H}}(\Lambda) = \lim_{n\to\infty} -\frac{1}{n}\log_{|r_2|}\left(\sum_{|S_{\mathtt{i}}\in|\mathbb{G}_n} \left(\#R_{|S_{\mathtt{i}}}\right)^{\log_{|r_1|}|r_2|}\right) \tag{Eq. 5.39}$$

as stated in the theorem.

References

[1] K. Barański. Hausdorff dimension of the limit sets of some planar geometric constructions. *Adv. Math.*, 210(1):215–245, 2007.

[2] B. Bárány. Dimension of the generalized 4-corner set and its projections. *Ergodic Theory Dynam. Systems*, 32(4):1190–1215, 2012.

[3] B. Bárány, K. Simon, and B. Solomyak. *Self-similar and self-affine sets and measures*, volume 276 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, [2023] ©2023.

[4] T. Bedford. *Crinkly curves, Markov partitions and dimension*. PhD thesis, University of Warwick, 1984.

[5] K. Falconer. *Fractal geometry*. John Wiley & Sons, Ltd., Chichester, third edition, 2014. Mathematical foundations and applications.

[6] K. J. Falconer. The Hausdorff dimension of self-affine fractals. *Math. Proc. Cambridge Philos. Soc.*, 103(2):339–350, 1988.

[7] D.-J. Feng and H. Hu. Dimension theory of iterated function systems. *Comm. Pure Appl. Math.*, 62(11):1435–1500, 2009.

[8] D.-J. Feng and Y. Wang. A class of self-affine sets and self-affine measures. *J. Fourier Anal. Appl.*, 11(1):107–124, 2005.

[9] J. M. Fraser and P. Shmerkin. On the dimensions of a family of overlapping self-affine carpets. *Ergodic Theory Dynam. Systems*, 36(8):2463–2481, 2016.

[10] M. Hochman. On self-similar sets with overlaps and inverse theorems for entropy. *Ann. of Math. (2)*, 180(2):773–822, 2014.

[11] J. E. Hutchinson. Fractals and self-similarity. *Indiana Univ. Math. J.*, 30(5):713–747, 1981.

[12] R. Kenyon and Y. Peres. Measures of full dimension on affine-invariant sets. *Ergodic Theory Dynam. Systems*, 16(2):307–323, 1996.

[13] R. Kenyon and Y. Peres. Correction to: "Hausdorff dimensions of sofic affine-invariant sets" [Israel J. Math. **94** (1996), 157–178; MR1394572 (97h:58064)]. *Israel J. Math.*, 97:347, 1997.

[14] I. Kolossváry. Calculating box dimension with the method of types, 2021.

[15] S. P. Lalley and D. Gatzouras. Hausdorff and box dimensions of certain self-affine fractals. *Indiana Univ. Math. J.*, 41(2):533–568, 1992.

[16] F. Ledrappier and L.-S. Young. The metric entropy of diffeomorphisms. I. Characterization of measures satisfying Pesin's entropy formula. *Ann. of Math. (2)*, 122(3):509–539, 1985.

[17] F. Ledrappier and L.-S. Young. The metric entropy of diffeomorphisms: part II: relations between entropy, exponents and dimension. *Annals of Mathematics*, pages 540–574, 1985.

[18] C. McMullen. The Hausdorff dimension of general Sierpiński carpets. *Nagoya Math. J.*, 96:1–9, 1984.

[19] M. P. W. Zerner. Weak separation properties for self-similar sets. *Proc. Amer. Math. Soc.*, 124(11):3529–3539, 1996.