MŰEGYETEM 1782

Numerical Computation of Parallel and Central Projections

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Natural Sciences

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Motivation

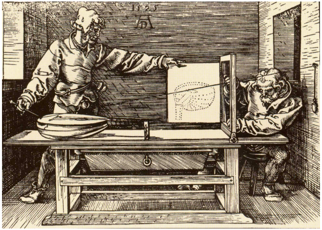
Aim of art and architecture:

model the space in the plane such that it can be reconstructed from the image/images

↓

philosophers, artists, architectures and mathematicians were constructing such descriptive systems

- axonometry – parallel projection keeps parallel lines
- perspectivity – central projection gives the illusion of space from one point of view

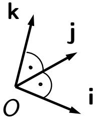


Coordinate systems

World/Object coordinate system $\{O, x, y, z\}$ - locate the points in 3d

Descartes (orthogonal) coordinate system:

- given origin – starting point of basis vectors
- basis: orthogonal unit vectors $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$



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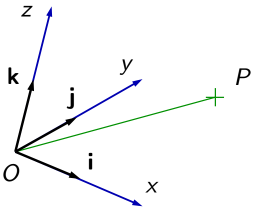
$\overrightarrow{OP}$ vector can be written as the linear combination of $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$

$$\overrightarrow{OP} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$



Coordinates of point

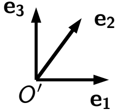
$$P = (x, y, z)$$

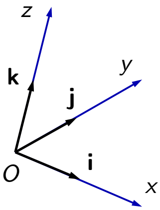
Coordinate systems

World/Object coordinate system $\{O, x, y, z\}$ - locate the points in 3d

Image/Display coordinate system $\{O', \xi, \eta, \zeta\}$ - locate the image of points

- new origin – starting point of new basis vectors
- basis: orthogonal unit vectors $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$





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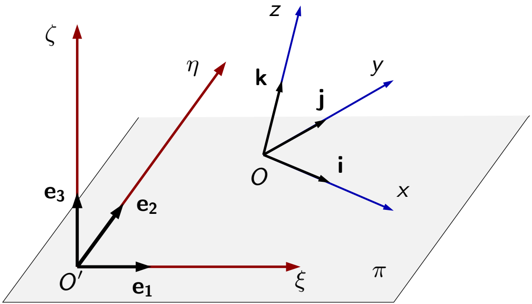
π: *projection plane*

- the plane of the vectors $\mathbf{e_1}$ and $\mathbf{e_2}$
- the plane, where we draw the 2d image of the object



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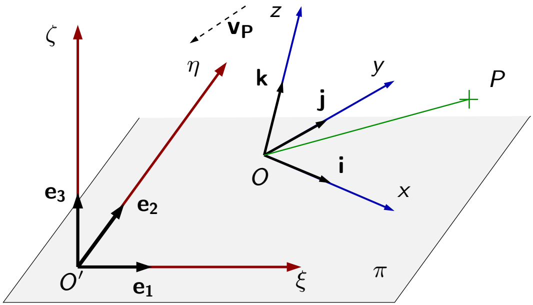
$\mathbf{v_P}$: direction vector of projection in point P



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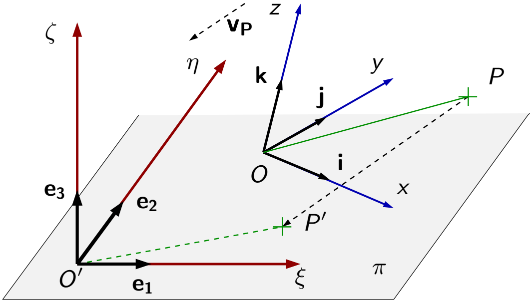
P' : the projection of point P to the plane π in the direction of the vector $\mathbf{v_P}$



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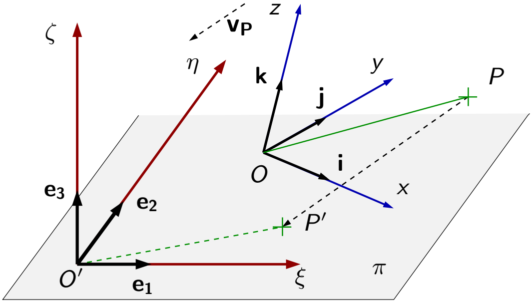
$\mathbf{v_P}$: direction vector of projection in point P

P' : the projection of point P

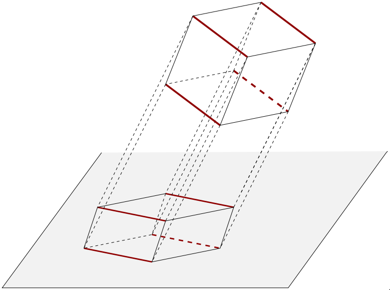
$$\overrightarrow{O'P'} = \xi'\mathbf{e_1} + \eta'\mathbf{e_2} + 0\mathbf{e_3}$$

$$P' = (\xi', \eta', 0)$$

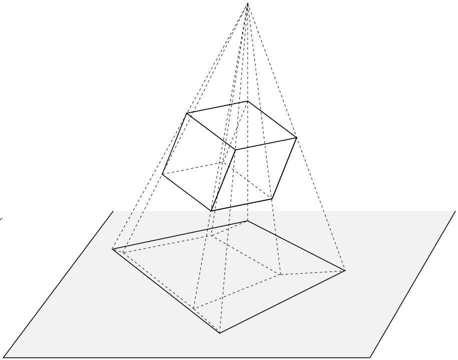


Two Types of Projection

Parallel Projection
(Axonometry)



Central Projection
(Perspectivity)



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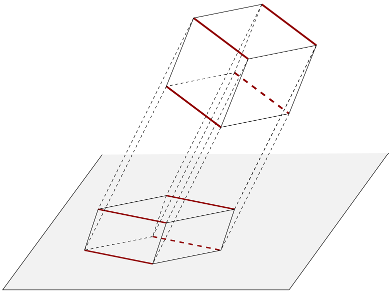
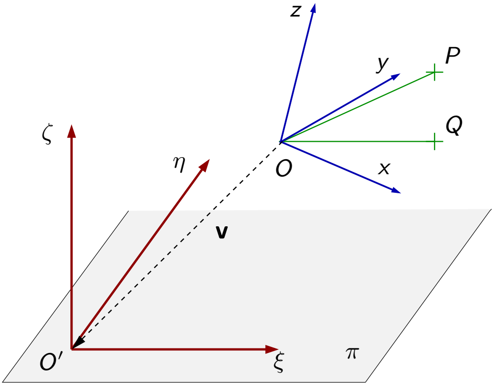


The Parallel Projection

- the direction of the projection

$$\mathbf{v} = \overrightarrow{OO'}$$



The Parallel Projection

- the direction of the projection

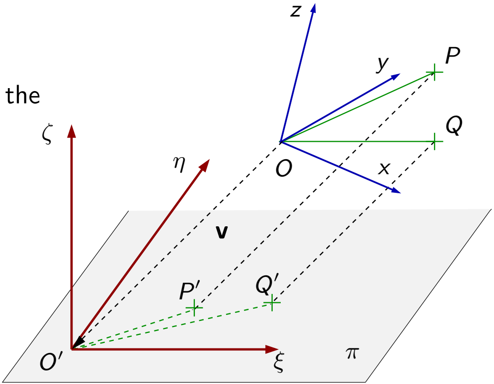
$$\mathbf{v} = \overrightarrow{OO'}$$

- the direction of projection is the same in each point P or Q

$$\downarrow$$

projection lines are parallel:

$$\overrightarrow{PP'} \parallel \overrightarrow{OO'}$$



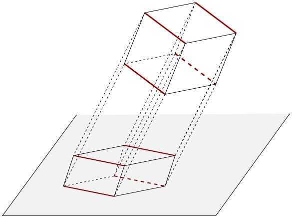
Properties

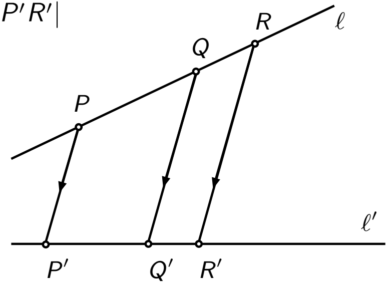
Keeps the parallelity:

$$\overline{PQ} \| \overline{RS} \quad \rightarrow \quad \overline{P'Q'} \| \overline{R'S'}$$

Therefore it keeps the ratio of the length of line segments along a line:

$$\frac{|\overline{PQ}|}{|\overline{PR}|} = \frac{|\overline{P'Q'}|}{|\overline{P'R'}|}$$

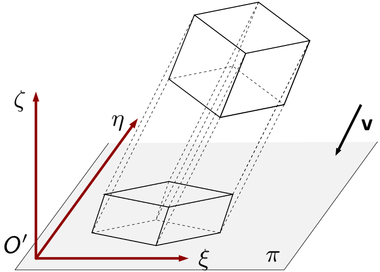




Special Case

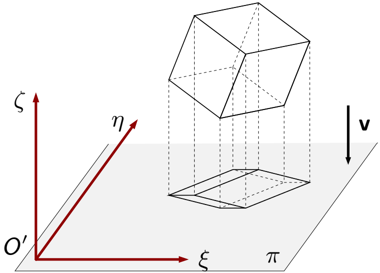
Oblique (general case)

- the direction of projection is arbitrary



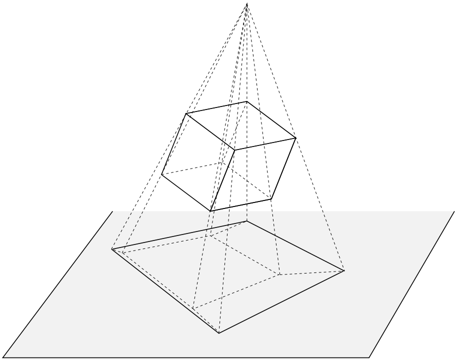
Orthogonal (special case)

- the direction of projection is perpendicular to the projection plane
- the length of a line segment decreases in the projection



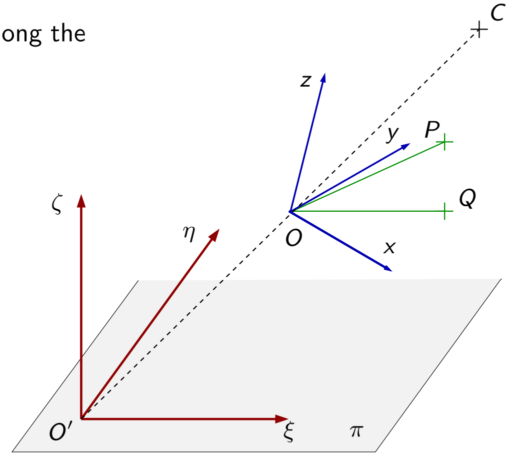
Two Types of Projection

Central Projection
(Perspectivity)



The Central Projection

- given a center point C along the line of $\overline{OO'}$



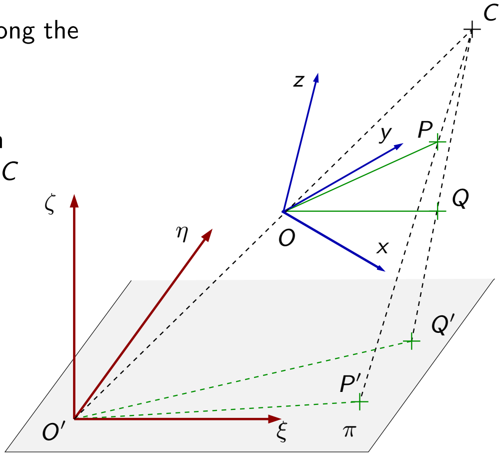
The Central Projection

- given a center point C along the line of $\overline{OO'}$
- the direction of projection passes through the point C

$$\downarrow$$

direction of projection:

$$\mathbf{v_P} = \overrightarrow{CP}$$

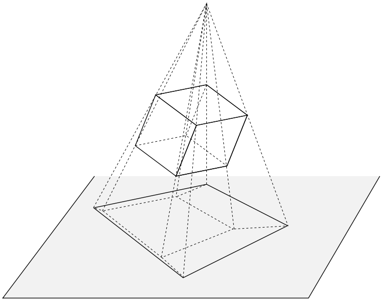


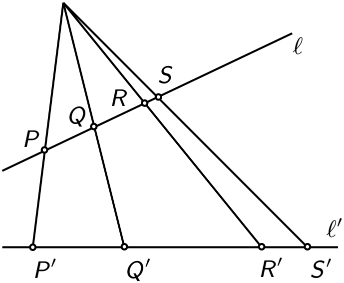
Properties

Keeps the cross ratio:

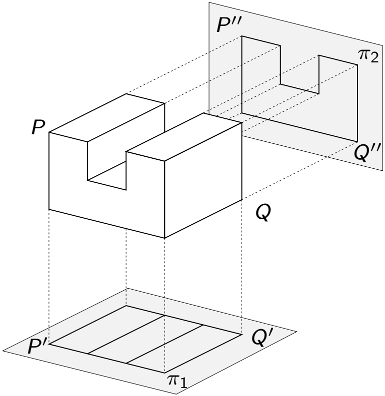
$$[PQRS] = [P'Q'R'S']$$

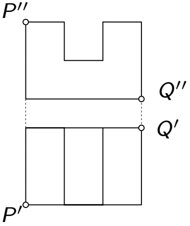
$$\frac{|\overline{PQ}|}{|\overline{RQ}|}\frac{|\overline{QS}|}{|\overline{PS}|} = \frac{|\overline{P'Q'}|}{|\overline{R'Q'}|}\frac{|\overline{Q'S'}|}{|\overline{P'S'}|}$$





Monge's System: orthogonal projections on mutually orthogonal image planes





The two image planes, π_1 and π_2 are usually the *xy* and *yz* coordinate planes of the world coordinate system.

The projection of a point is constructed with the help of the projection of its coordinate lines in the world coordinate system.

Projection Method

- 1. step: Basis transformation

- 2. step: Projection

Step 1: Basis Transformation

Projections are computed in the image coordinate system.

Aim: find the coordinates of the point P given in the world coordinate system $\{O, x, y, z\}$ in the image coordinate system $\{O', \xi, \eta, \zeta\}$

- *translation* of the world coordinate system $\{O, x, y, z\}$

$$O \quad \rightarrow \quad O'$$

compute $\overrightarrow{O'P}$ in the world coordinate system

- *transformation* of basis vectors

$$\{\mathbf{i}, \mathbf{j}, \mathbf{k}\} \quad \rightarrow \quad \{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$$

compute the coordinates of $\overrightarrow{O'P}$ in the image coordinate system

- *translation*

$$\overrightarrow{O'P} = \overrightarrow{OP} - \overrightarrow{OO'}$$

- *transformation*

  If the world coordinates of $P_{world} = (x, y, z)$ in the translated system, then

$$\overrightarrow{O'P} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

  so we have to find the coordinates of the basis vectors $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ in the new basis $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$:

$$\begin{aligned} \mathbf{i} &= b_{11}\mathbf{e_1} + b_{12}\mathbf{e_2} + b_{13}\mathbf{e_3} \\ \mathbf{j} &= b_{21}\mathbf{e_1} + b_{22}\mathbf{e_2} + b_{23}\mathbf{e_3} \\ \mathbf{k} &= b_{31}\mathbf{e_1} + b_{32}\mathbf{e_2} + b_{33}\mathbf{e_3}. \end{aligned}$$

  The basis transformation matrix: $\mathbf{B} = \{b_{ij}\}_{i=1,2,3}^{j=1,2,3}$

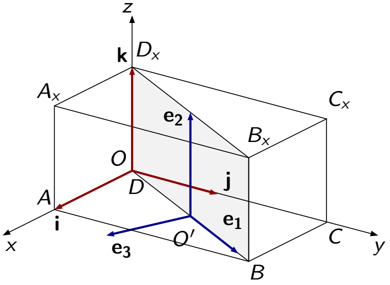
  New coordinates in the image coordinate system:

$$P_{img} = (\xi, \eta, \zeta) = \mathbf{B} \cdot (x, y, z)^{\mathsf{T}}.$$

Example

The image coordinate system lies in the diagonal plane of the prism defined by $(0,0,0)$ and $(1,2,2)$

World coord.sys.: $\begin{cases} O(0,0,0) \\ \mathbf{i}(1,0,0) \\ \mathbf{j}(0,1,0) \\ \mathbf{k}(0,0,1) \end{cases}$ Image coord.sys.: $\begin{cases} O'(0.5,1,0) \\ \mathbf{e_1}(1/\sqrt{5},2/\sqrt{5},0) \\ \mathbf{e_2}(0,0,1) \\ \mathbf{e_3}(2/\sqrt{5},-1/\sqrt{5},0) \end{cases}$

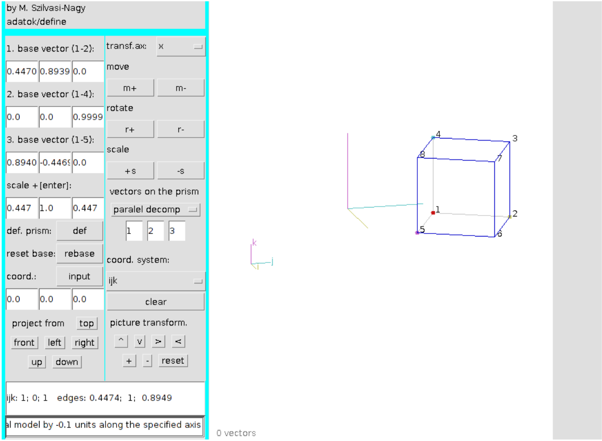


$$\begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \end{pmatrix} \begin{pmatrix} x-0.5 \\ y-1 \\ z \end{pmatrix}$$

x	1	1	1	1	0	0	0	0
y	0	0	2	2	2	2	0	0
z	0	1	0	1	0	1	0	1
	A	A_x	B	B_x	C	C_x	D	D_x
ξ	$\frac{-3}{2\sqrt{5}}$	$\frac{-3}{2\sqrt{5}}$	$\frac{5}{2\sqrt{5}}$	$\frac{5}{2\sqrt{5}}$	$\frac{3}{2\sqrt{5}}$	$\frac{3}{2\sqrt{5}}$	$\frac{-5}{2\sqrt{5}}$	$\frac{-5}{2\sqrt{5}}$
η	0	1	0	1	0	1	0	1
ζ	$\frac{2}{\sqrt{5}}$	$\frac{2}{\sqrt{5}}$	0	0	$\frac{-2}{\sqrt{5}}$	$\frac{-2}{\sqrt{5}}$	0	0

PrismVec: Java Applet

http://www.math.bme.hu/~szilvasi/prism/prism.html



Step 2: Projection on the Image Plane – Parallel Projection

After the transformation of the basis the coordinates of the point P are given in the image coordinate system $\{O', \xi, \eta, \zeta\}$.

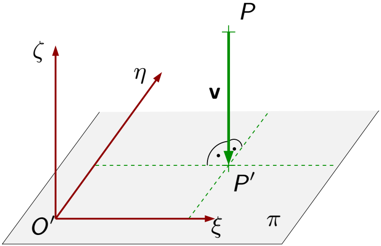
Orthogonal Projection

$$\overrightarrow{O'P} = \xi \mathbf{e_1} + \eta \mathbf{e_2} + \zeta \mathbf{e_3} : \quad P(\xi, \eta, \zeta)$$

$$\overrightarrow{O'P'} = \xi' \mathbf{e_1} + \eta' \mathbf{e_2} + 0 \cdot \mathbf{e_3} : \quad P'(\xi, \eta, 0)$$

The matrix of the projection:

$$\begin{pmatrix} \mathbf{e_1} \\ \mathbf{e_2} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e_1} \\ \mathbf{e_2} \\ \mathbf{e_3} \end{pmatrix}$$



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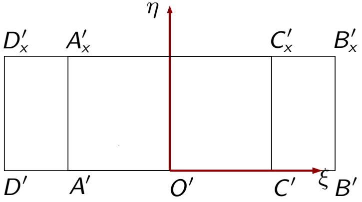
Orthogonal Projection: Example

	A	A_x	B	B_x	C	C_x	D	D_x
ξ	$\frac{-3}{2\sqrt{5}}$	$\frac{-3}{2\sqrt{5}}$	$\frac{5}{2\sqrt{5}}$	$\frac{5}{2\sqrt{5}}$	$\frac{3}{2\sqrt{5}}$	$\frac{3}{2\sqrt{5}}$	$\frac{-5}{2\sqrt{5}}$	$\frac{-5}{2\sqrt{5}}$
η	0	1	0	1	0	1	0	1
ζ	$\frac{2}{\sqrt{5}}$	$\frac{2}{\sqrt{5}}$	0	0	$\frac{-2}{\sqrt{5}}$	$\frac{-2}{\sqrt{5}}$	0	0
	A'	A'_x	B'	B'_x	C'	C'_x	D'	D'_x
ξ'	$\frac{-3}{2\sqrt{5}}$	$\frac{-3}{2\sqrt{5}}$	$\frac{5}{2\sqrt{5}}$	$\frac{5}{2\sqrt{5}}$	$\frac{3}{2\sqrt{5}}$	$\frac{3}{2\sqrt{5}}$	$\frac{-5}{2\sqrt{5}}$	$\frac{-5}{2\sqrt{5}}$
η'	0	1	0	1	0	1	0	1
ζ'	0	0	0	0	0	0	0	0



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Oblique Projection – Input data: $\mathbf{v}(v_1, v_2, v_3)_{\text{img}}$ (direction of projection)

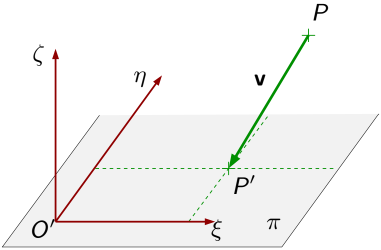
$$\overrightarrow{O'P'} = \overrightarrow{O'P} + \lambda \mathbf{v}$$

$$P' = \begin{pmatrix} \xi' \\ \eta' \\ 0 \end{pmatrix} = \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} + \lambda \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$\lambda = \frac{-\zeta}{v_3} \quad \Rightarrow \quad \xi' = \xi - \zeta \frac{v_1}{v_3};\ \eta' = \eta - \zeta \frac{v_2}{v_3}$$

The matrix of the projection:

$$\begin{pmatrix} \xi' \\ \eta' \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\frac{v_1}{v_3} \\ 0 & 1 & -\frac{v_2}{v_3} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}$$



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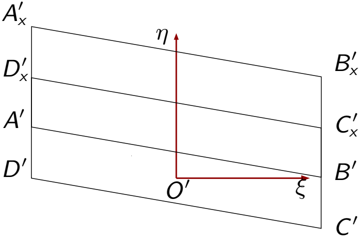
After the transformation of the basis the coordinates of the point P are given in the image coordinate system $\{O', \xi, \eta, \zeta\}$.

Oblique Projection: Example – $\mathbf{v}(2, 0, -1)_{\text{world}} = \mathbf{v}(\frac{2}{\sqrt{5}}, -1, \frac{4}{\sqrt{5}})_{\text{img}}$

	A	A_x	B	B_x	C	C_x	D	D_x
ξ	$\frac{-3}{2\sqrt{5}}$	$\frac{-3}{2\sqrt{5}}$	$\frac{5}{2\sqrt{5}}$	$\frac{5}{2\sqrt{5}}$	$\frac{3}{2\sqrt{5}}$	$\frac{3}{2\sqrt{5}}$	$\frac{-5}{2\sqrt{5}}$	$\frac{-5}{2\sqrt{5}}$
η	0	1	0	1	0	1	0	1
ζ	$\frac{2}{\sqrt{5}}$	$\frac{2}{\sqrt{5}}$	0	0	$\frac{-2}{\sqrt{5}}$	$\frac{-2}{\sqrt{5}}$	0	0

	A'	A'_x	B'	B'_x	C'	C'_x	D'	D'_x
ξ'	$\frac{-5}{2\sqrt{5}}$	$\frac{-5}{2\sqrt{5}}$	$\frac{5}{2\sqrt{5}}$	$\frac{5}{2\sqrt{5}}$	$\frac{5}{2\sqrt{5}}$	$\frac{5}{2\sqrt{5}}$	$\frac{-5}{2\sqrt{5}}$	$\frac{-5}{2\sqrt{5}}$
η'	$\frac{1}{2}$	$\frac{3}{2}$	0	1	0	$\frac{-1}{2}$	$\frac{1}{2}$	1
ζ'	0	0	0	0	0	0	0	0



The matrix of the projection:

$$\begin{pmatrix} \xi' \\ \eta' \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \frac{-1}{2} \\ 0 & 1 & \frac{\sqrt{5}}{4} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}$$

Step 2: Projection on the Image Plane – Axonometric Projection

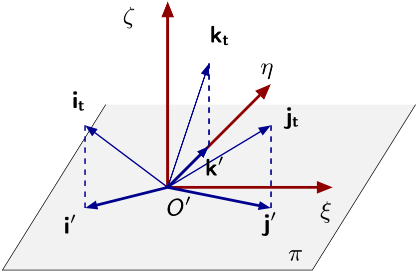
After the transformation of the basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ the basis vectors $\{\mathbf{i_t}, \mathbf{j_t}, \mathbf{k_t}\}$ are given in the image coordinate system.

$$\overrightarrow{O'P} = \overrightarrow{OP} - \overrightarrow{OO'} = (x_t, y_t, z_t)$$

$\{\mathbf{i}', \mathbf{j}', \mathbf{k}'\}$: the projection of the basis

The length of these new vectors (distorsion factors):

$$q_x = |\mathbf{i}'|, q_y = |\mathbf{j}'|, q_y = |\mathbf{k}'|$$



Unit vectors of the projected basis:

$$\mathbf{i}'_0 = \frac{\mathbf{i}'}{|\mathbf{i}'|},\ \mathbf{j}'_0 = \frac{\mathbf{j}'}{|\mathbf{j}'|},\ \mathbf{k}'_0 = \frac{\mathbf{k}'}{|\mathbf{k}'|}$$

Thus we can compute:

$$\overrightarrow{O'P'} = x_t\mathbf{i}' + y_t\mathbf{j}' + z_t\mathbf{k}' = x_t q_x \mathbf{i}'_0 + y_t q_y \mathbf{j}'_0 + z_t q_z \mathbf{k}'_0 = \xi'\mathbf{e_1} + \eta'\mathbf{e_2}.$$

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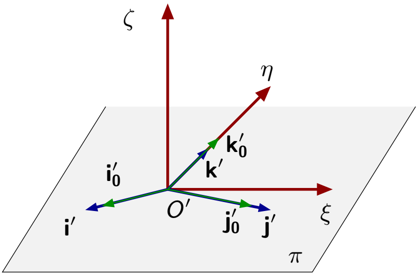
$$q_x = |\mathbf{i}'|, q_y = |\mathbf{j}'|, q_y = |\mathbf{k}'|$$

Unit vectors of the projected basis:

$$\mathbf{i}_0' = \frac{\mathbf{i}'}{|\mathbf{i}'|},\ \mathbf{j}_0' = \frac{\mathbf{j}'}{|\mathbf{j}'|},\ \mathbf{k}_0' = \frac{\mathbf{k}'}{|\mathbf{k}'|}$$



Thus we can compute:

$$\overrightarrow{O'P'} = x_t\mathbf{i}' + y_t\mathbf{j}' + z_t\mathbf{k}' = x_t q_x \mathbf{i}_0' + y_t q_y \mathbf{j}_0' + z_t q_z \mathbf{k}_0' = \xi' \mathbf{e_1} + \eta' \mathbf{e_2}.$$

Step 2: Projection on the Image Plane – Axonometric Projection

After the transformation of the basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ the basis vectors $\{\mathbf{i_t}, \mathbf{j_t}, \mathbf{k_t}\}$ are given in the image coordinate system.

$$\overrightarrow{O'P} = \overrightarrow{OP} - \overrightarrow{OO'} = (x_t, y_t, z_t)$$

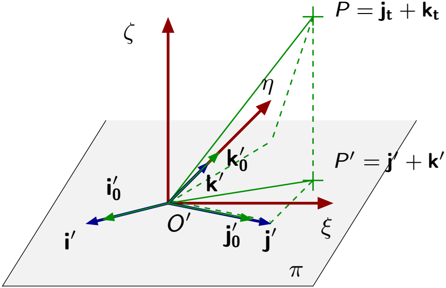
$\{\mathbf{i}', \mathbf{j}', \mathbf{k}'\}$: the projection of the basis

The length of these new vectors (distorsion factors):

$$q_x = |\mathbf{i}'|, q_y = |\mathbf{j}'|, q_y = |\mathbf{k}'|$$

Unit vectors of the projected basis:

$$\mathbf{i}_0' = \frac{\mathbf{i}'}{|\mathbf{i}'|},\ \mathbf{j}_0' = \frac{\mathbf{j}'}{|\mathbf{j}'|},\ \mathbf{k}_0' = \frac{\mathbf{k}'}{|\mathbf{k}'|}$$



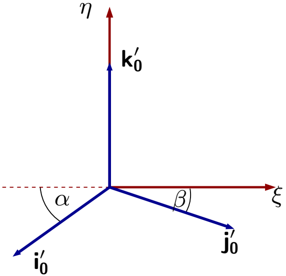
Thus we can compute:

$$\overrightarrow{O'P'} = x_t\mathbf{i}' + y_t\mathbf{j}' + z_t\mathbf{k}' = x_t q_x \mathbf{i}_0' + y_t q_y \mathbf{j}_0' + z_t q_z \mathbf{k}_0' = \xi'\mathbf{e_1} + \eta'\mathbf{e_2}.$$

Characterization of Axonometric Projection

The projection of the basis vectors are given by

- the distorsion factors: $q_x = |\mathbf{i}'|, q_y = |\mathbf{j}'|, q_y = |\mathbf{k}'|$
- angles compared to the ξ-axis: α, β



$$\begin{aligned} \mathbf{i}_0' &= -\cos\alpha\,\mathbf{e_1} - \sin\alpha\,\mathbf{e_2} \\ \mathbf{j}_0' &= \cos\beta\,\mathbf{e_1} - \sin\beta\,\mathbf{e_2} \\ \mathbf{k}_0' &= \mathbf{e_2} \end{aligned}$$

The matrix of the projection:

$$\begin{pmatrix} \xi' \\ \eta' \\ 0 \end{pmatrix} = \begin{pmatrix} -q_x\cos\alpha & q_y\cos\beta & 0 \\ -q_x\sin\beta & -q_y\sin\beta & q_z \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix}$$

Characterization of Axonometric Projection

The projection of the basis vectors are given by

- the distorsion factors: $q_x = |\mathbf{i}'|, q_y = |\mathbf{j}'|, q_y = |\mathbf{k}'|$
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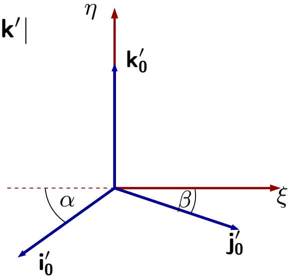


determination of the axonometric projection

oblique	orthogonal
given by $\alpha, \beta, q_x, q_y, q_z$	by $0^\circ < \alpha < 90^\circ$ and $0^\circ < \beta < 90^\circ$

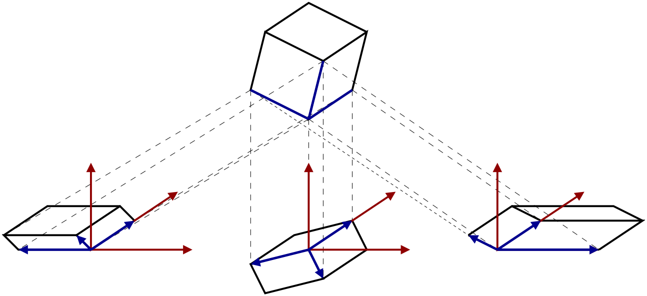
special cases:

- $q_x = q_y = q_z$: isometric axonometry
- $q_x = q_y$ or $q_x = q_z$ or $q_y = q_z$: dimetric axonometry

Oblique Axonometry and Parallel Projection

Theorem (Pohlke's Theorem)

There exists a direction of projection such that the axonometric projection of the unit cube is similar to the parallel projection of the unit cube projected from this direction.



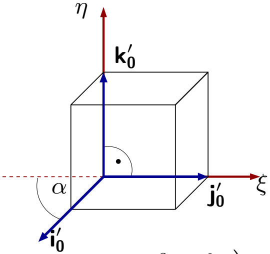
Oblique Axonometry – Special Cases

Cavalier

$\beta = 0;\ q_y = q_z = 1$

$\alpha = 30^\circ, 45^\circ, 60^\circ;$

$q_x = \frac{1}{2}, \frac{2}{3}, 1$



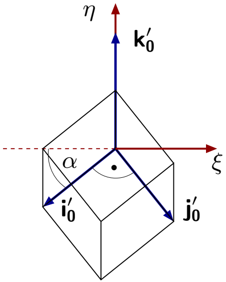
$$\begin{pmatrix} -q_x \cos\alpha & q_y \cos\beta & 0 \\ -q_x \sin\beta & -q_y \sin\beta & q_z \\ 0 & 0 & 0 \end{pmatrix}$$

Military

$\beta = 90^\circ - \alpha;\ q_x = q_y = 1$

$\alpha = 30^\circ, 45^\circ, 60^\circ;$

$q_z = \frac{1}{2}, 1, \frac{1}{100}, \frac{1}{1000} \ldots$

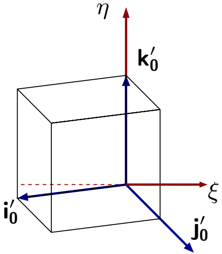


$$\begin{pmatrix} -q_x \cos\alpha & 1 & 0 \\ -q_x \sin\beta & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

"most realistic"

$\alpha = 7.5^\circ;\ \beta = 45^\circ$

$q_x = q_z = 1;\ q_y = 1/2$



$$\begin{pmatrix} -\cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & -\cos\alpha & q_z \\ 0 & 0 & 0 \end{pmatrix}$$

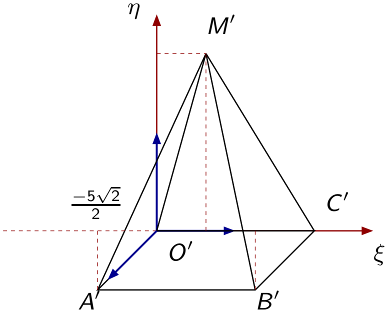
Examlpe

Four-sided regular pyramid projected by the conditions:
$\alpha = 45°, \beta = 0°, q_x = 1/2, q_y = q_z = 1$.
The matrix of the projection:

$$\begin{pmatrix} \xi' \\ \eta' \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{-\sqrt{2}}{4} & 1 & 0 \\ \frac{-\sqrt{2}}{4} & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}$$

	A	B	C	O	M
ξ	10	10	0	0	5
η	0	10	10	0	5
ζ	0	0	0	0	20

	A'	B'	C'	O'	M'
ξ'	$\frac{-5\sqrt{2}}{2}$	$10 - \frac{5\sqrt{2}}{2}$	10	0	$5 - \frac{5\sqrt{2}}{2}$
η'	$\frac{-5\sqrt{2}}{2}$	$\frac{-5\sqrt{2}}{2}$	0	0	$20 - \frac{5\sqrt{2}}{2}$
ζ'	0	0	0	0	0



Orthogonal Axonometry

The distorsion is determined by the angles $0^\circ < \alpha, \beta < 90^\circ$:

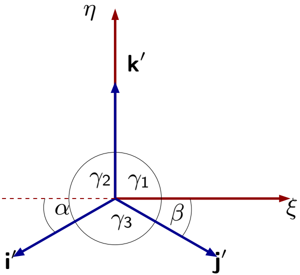
$$\gamma_1 = 90^\circ + \beta; \qquad \gamma_2 = 90^\circ + \alpha; \qquad \gamma_3 = 180^\circ - (\alpha + \beta)$$

$$q_x = \sqrt{\frac{-\cos\gamma_1}{\sin\gamma_2 \sin\gamma_3}}; \quad q_y = \sqrt{\frac{-\cos\gamma_2}{\sin\gamma_1 \sin\gamma_3}}; \quad q_z = \sqrt{\frac{-\cos\gamma_3}{\sin\gamma_1 \sin\gamma_2}}$$

Isometric case:
$\gamma_1 = \gamma_2 = \gamma_3 = 120^\circ$
implies that
$q_x = q_y = q_z = 1$

The matrix of the projection:

$$\begin{pmatrix} \frac{-\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{-1}{2} & \frac{-1}{2} & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

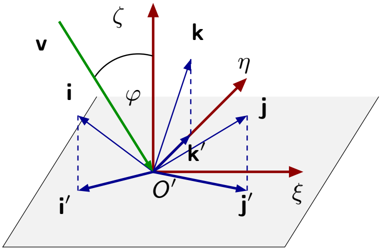


Distortion Factors

Theorem (Distorsion Factors of the Axonometric Projection)

$$q_x^2 + q_y^2 + q_z^2 = 2 + \tan^2 \varphi$$

where q_x, q_y and q_z are the distorsion factors of the projection and φ is the angle between the direction of the projection **v** *and the ζ-axis.*



Special case: orthogonal axonometry, where $\varphi = 0$, thus $q_x^2 + q_y^2 + q_z^2 = 2$.

Step 2: Projection on the Image Plane – Central Projection

After the transformation of the basis the coordinates of the point P are given in the image coordinate system $\{O', \xi, \eta, \zeta\}$.

Central Projection – Input data: $C(0, 0, d)_{\text{img}}$ (center of projection)

$$\overrightarrow{O'P'} = \overrightarrow{O'C} + \mu\overrightarrow{CP'}$$

$$P' = \begin{pmatrix} \xi' \\ \eta' \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ d \end{pmatrix} + \mu \begin{pmatrix} \xi \\ \eta \\ \zeta - d \end{pmatrix}$$

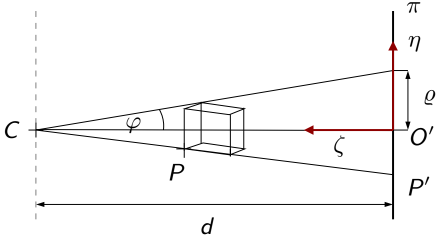
$$\mu = \frac{d}{d-\zeta} \quad \Rightarrow \quad \xi' = \frac{d\xi}{d-\zeta};\ \eta' = \frac{d\eta}{d-\zeta}; \qquad \zeta \neq d$$

Side view of the central projection:

φ : half angle of the viewing cone

ϱ: the radius of the viewing cone

"realistic" picture if $\varrho \approx \frac{d}{2}$



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$$\mu = \frac{d}{d-\zeta} \quad \Rightarrow \quad \xi' = \frac{d\xi}{d-\zeta};\ \eta' = \frac{d\eta}{d-\zeta}; \qquad \zeta \neq d$$

In homogeneous coordinates:

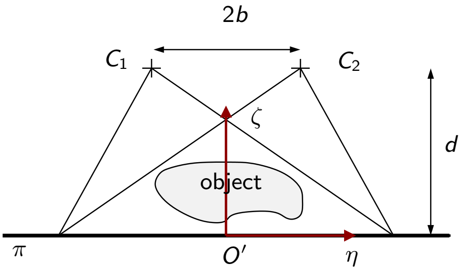
$$\begin{pmatrix} x_1' \\ x_2' \\ 0 \\ x_3' \end{pmatrix} = \begin{pmatrix} d & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$\xi = \frac{x_1}{x_4}, \quad \eta = \frac{x_2}{x_4}, \quad \zeta = \frac{x_3}{x_4}, \quad \xi' = \frac{x_1'}{x_4'}, \quad \eta' = \frac{x_2'}{x_4'}$$

Stereotechnique

Preparing two pictures on a common image plane using central projection with two different centers.

$$C_1(0, b, d), \qquad C_2(0, -b, d)$$



$$P' : \quad \xi' = \frac{d\xi}{d-\zeta}, \quad \eta' = \frac{d\eta \pm b\zeta}{d-\zeta}, \quad \zeta' = 0$$

Using different colors (red, green) gives us the easiest 3d image technique.

Compoly: Java Applet

http://www.math.bme.hu/~geom/compolyk/compe/compolye.html