Local inverse spectrum theorems for real and nonnegative matrices

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Abstract

We prove several inverse spectrum theorems for real, nonnegative and positive matrices. The results are of a local character with respect to the topology generated by the matching distance of the spectral lists of matrices. We prove e.g. that the set of spectral lists of positive matrices is an open set in this topology, and extend a result of Minc. A constructive method is used everywhere, which can produce the realizing matrices explicitly.

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1 Introduction

A result of Brauer ([1, Theorem 27], originating, in fact, from Wielandt [28]) has been playing a remarkable role in the solution of some inverse spectrum problems for nonnegative matrices. A good recapitulation of its significance, history and of the recent research situation is given e.g. by Soto, Borobia and Moro in [25], and it is also instructive to see its application in older and in recent papers as in [20], [22], [2], [23], [24]. An other influential paper in these problems is [29], see also later developments in [23] and [6]. Other significant recent contributions to different aspects of the nonnegative inverse problem are [15], [16], [12].

Questions of the variation of the spectrum of a real matrix under real matrix perturbations were considered in the interesting paper by Hinrichsen and Pritchard [9]. The present paper studies problems, loosely speaking, in the converse direction to [9] for both the real and the nonnegative case.

The aim of this paper is to establish local type inverse spectrum theorems for real or nonnegative or positive matrices. We emphasize that the method is *constructive*: in addition to establishing the existence of *real, nonnegative or positive realizations* (i.e. of matrices with the prescribed spectra), it makes also possible to actually write them down.

We want to fix some *terminology and notation.* A vector $v \in \mathbf{C}^n$ (*without* qualification) will denote a *column vector*, and we shall also use *row vectors*,

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say $h \in \mathbf{C}^n$ (always *with* qualification). The complex conjugate of $z \in \mathbf{C}$ will be denoted by $\overline{z}$, and we shall also write $\overline{v}$ for the *componentwise conjugate* of a column or row vector v. The transpose and the conjugate transpose (i.e. adjoint) of a matrix M will be denoted by M^t, M^*, respectively. We shall sometimes omit the identity I, and write $z - M$ rather than $zI - M$.

For a given square matrix A of order n (or for the corresponding linear operator) and its eigenvalue $z \in \mathbf{C}$ we shall consider *Jordan chains*, i.e. finite sequences of vectors $v_j \in \mathbf{C}^n$ $(j = 1, \ldots, k)$ satisfying

$$(A - z)v_j = v_{j-1} \quad (j = 1, \ldots, k \leq n),\ v_0 := 0, \quad v_1 \neq 0.$$

The vector v_1 is clearly an eigenvector, the others are called *generalized eigenvectors*. A basis of the space consisting of (vectors of) Jordan chains is called a *Jordan basis* for A. If the matrix A has only real entries, with the help of suitable Jordan bases for pairs of subspaces corresponding to a pair of conjugate nonreal eigenvalues, we obtain a basis consisting of exclusively real vectors (cf. [8, pp. 369-370]). We shall consider only such *real Jordan bases* in Section 2.

In order to fix terminology, we shall call a finite sequence of complex numbers $L := (c_1, c_2, \ldots, c_n)$ a *list in* $\mathbf{C}^n$. It may be naturally identified with the corresponding point, so we shall write also $L \in \mathbf{C}^n$. Denote the set of all permutations acting on $\mathbf{C}^n$ by $\Pi(n)$. We shall say that the lists $L, M \in \mathbf{C}^n$ are equivalent, and write $L \equiv M$, if there is $P \in \Pi(n)$ such that $M = PL$. We shall call the list $L \in \mathbf{C}^n$ *self-dual*, and write $L \in S_n$, if $L \equiv \overline{L}$. Note that the set S_n is *not* closed with respect to the usual vector addition in $\mathbf{C}^n$. E.g., if r_1, r_2 are distinct real numbers and z is a nonreal complex number, then $(r_1, r_2) + (z, \overline{z}) \notin S_2$, though the summands are.

It is clear that the relation $\equiv$ is an equivalence relation, hence we can define the corresponding quotient space of $\mathbf{C}^n$:

$$\tilde{\mathbf{C}}^n := \mathbf{C}^n / \equiv .$$

We shall denote the corresponding equivalence classes by $\tilde{L}, \tilde{M}$, etc. It is clear that $L \equiv \overline{L}$ if and only if every element in the class $\tilde{L}$ is equivalent to its conjugate. In this case we can and shall say that the class $\tilde{L}$ is a self-conjugate class, and we shall write $\tilde{L} \in \tilde{S}_n$.

For two lists $L, M \in \mathbf{C}^n$ we shall define their *matching distance* (cf. [27, pp. 167-168] and [13, II.5.2, p.108]) m by

$$m(L, M) := \min_{P \in \Pi(n)} ||L - PM||,$$

where $|| \cdot ||$ denotes the l_∞ norm on $\mathbf{C}^n$. Then m is a pseudo-metric (cf. [27, p. 62]), which induces in the standard way a metric $\tilde{m}$ defined on $\tilde{\mathbf{C}}^n$ by

$$\tilde{m}(\tilde{L}, \tilde{M}) := m(L, M).$$

Note that its restriction to $\tilde{S}_n$ is also a metric. Some of our results (Theorems 1-3) can be formulated with the help of the metric space $[\tilde{S}_n, \tilde{m}]$ in a natural way.

We shall consider, in particular, *lists* $L \in S_n$ of complex numbers *of the following type*:

$$L := (c_1; c_2, c_3, \ldots, c_n), \qquad c_1 \geq |c_j| \quad (j = 2, \ldots, n).$$

We shall call $c_1 \geq 0$ *the Perron-Frobenius (eigen)value of the list*, and we shall write $L \in PF_n$. The sets of such lists *having a nonnegative (positive) realizing matrix* will be denoted by $NNPF_n$ (PPF_n).

A list as

$$L' := (c_1+p_1; c_2+p_2, c_3+p_3, \ldots, c_n+p_n), \qquad c_1+p_1 \geq |c_j+p_j| \quad (j = 2, \ldots, n)$$

will be considered as the result of *a perturbation of the list* L, and the entry p_j as *the perturbation (value) of the entry* c_j.

As usual in this field, the expression *spectrum* (of a square matrix A) will always stand for a *spectral list*, i.e. a fixed sequence of all the eigenvalues (taken into account their respective *algebraic multiplicities*), and will be denoted by $spec(A)$, distinguishing it from the *set* $\sigma(A)$. As a slight abuse of language, $spec(A)$ is often called *the* spectrum of A. It is clear that (any such) spectral list of a real (nonnegative) matrix is a self-dual list (having a Perron-Frobenius eigenvalue if ordered in the required way).

Note that the pseudo-metric m in the space $\mathbf{C}^n$ majorizes the Hausdorff distance, and is regarded as a very useful measure in theoretical and practical perturbation problems (cf. [27, pp. 168-169]). For the basics on inverse spectrum problems for nonnegative matrices we refer the reader to [4] or [18].

2 Inverse spectrum theorems

Theorem 1. *Assume that the list* $L := (r_1, \ldots, r_v, z_1, \overline{z_1}, \ldots, z_c, \overline{z_c}) \in \mathbf{C}^n$ *is the spectrum of a matrix* $A \in \mathbf{R}^{n\times n}$*, where the numbers* r *are real, the numbers* z *are nonreal, and form pairs (as above) with their conjugates. (Hence* L *and the sublist* $D := (z_1, \ldots, \overline{z_c})$ *are self-conjugate.)*

Then for every positive η *there is a positive* $\delta = \delta(\eta)$ *with the following property: if* $|s_j| < \delta$ $(j = 1, \ldots, v)$*, the numbers* s_j *are either real or, provided they are perturbations of the same real number* r*, they can form several pairs of conjugate nonreal numbers, and if* $|w_j| < \delta$ $(j = 1, \ldots, c)$*, the numbers* w_j *are complex, then the list*

$$L_1 := (r_1 + s_1, \ldots, r_v + s_v, z_1 + w_1, \overline{z_1 + w_1}, \ldots, z_c + w_c, \overline{z_c + w_c})$$

is the spectrum of a matrix $R \in \mathbf{R}^{n\times n}$ *such that* $||R - A||_{l_\infty} < \eta$*.*

Proof. We shall formulate the proof based on real-type similarity to a real Jordan matrix. In addition, we call attention to the fact that we could base it with very little change on a real-type unitary similarity to a real upper block triangular matrix with the help of the real version of Schur's theorem (cf. [10, Section 2.3]). (In some applications the first, in others the second method will be of advantage.)

Consider a fixed *real, block upper triangular* Jordan matrix $J \equiv J(A)$ of A satisfying $AT = TJ$, in which the matrix T is real. (Note that in the second version we may take T to be a unitary matrix with real entries and J a real upper block triangular matrix with diagonal blocks of order 1 or 2.)

Since L is a spectral list of A, hence of J, we can and shall *reorder* L according to the chosen (fixed) structure of J in the following way. Denote a (complex type) Jordan block with eigenvalue u and of order g by $J(u, g)$.

Let u be *a* **real** *spectral value* of A, and assume that

$$J(u) := J(u, k_1) \oplus J(u, k_2) \oplus \cdots \oplus J(u, k_s) \quad (k_1 + \cdots + k_s = k)$$

is the direct sum of *all* the Jordan blocks with eigenvalue u. Clearly, $J(u)$ is then the sum of u times the identity matrix I_k of order k and of a matrix in which every entry $(j, j+1) \quad (j = 1, \ldots, k-1)$ is either 1 or 0, and all other entries are 0. By assumption, we have the following (type of) perturbation values (for u) in the list L_1:

$$s_1, \overline{s_1}, \ldots, s_m, \overline{s_m}, s_{2m+1}, \ldots, s_k,$$

where the first $2m \leq k$ perturbation values

$$s_j = a_j + ib_j,\ \overline{s_j} = a_j - ib_j \qquad\qquad (a_j \in \mathbf{R}, b_j > 0)$$

are nonreal, and the remaining are real.

$J(u)$ can be written as

$$J(u) = [E_1 \oplus E_2 \oplus \cdots \oplus E_m \oplus diag(u, \ldots, u)] + N_0,$$

where the 2×2 matrices $E_j \quad (j = 1, \ldots, m)$ have the form

$$E_j \equiv E_j(x_j) := \begin{pmatrix} u & x_j \\ 0 & u \end{pmatrix} \qquad (x_j \in \mathbf{R}),$$

where $x_j = 0$ or 1 in the Jordan case, (but can be an arbitrary real number in the Schur case,) and the matrix N_0 of order k has every entry 0 except possibly the entries $(j, j+1) \quad (j = 2, 4, \ldots, 2m, 2m+1, \ldots, k-1)$ being either 1 or 0 in the Jordan case, (but an arbitrary real number in the Schur case). Consider the following matrix $F(u) \in \mathbf{R}^{k \times k}$:

$$F(u) := F_1 \oplus F_2 \oplus \cdots \oplus F_m \oplus diag(s_{2m+1}, \ldots, s_k).$$

Here we define each $F_j \equiv F_j(s_j, x_j)$ in the following way:

$$F_j := \begin{pmatrix} a_j & g_j \\ h_j & a_j \end{pmatrix},$$

where $-h_j(g_j + x_j) = b_j^2 > 0$, and the values of the pair h_j, g_j will be defined as follows. If $|x_j| \geq b_j > 0$, then $g_j := 0$, and define h_j in the evident way. Then

$$|h_j| = \frac{b_j^2}{|x_j + g_j|} \leq b_j.$$

If $|x_j| < b_j$, then there is $g_j \in [-b_j, b_j]$ such that $|x_j + g_j| = b_j$, hence the uniquely determined h_j satisfies $|h_j| = b_j$. Recapitulating: in both cases we have obtained that $|g_j|,\ |h_j| \leq b_j$, hence (using the notation of the norm in [10, 5.6])

$$||F_j(s_j, x_j)||_{l_\infty} \leq |s_j|.$$

Furthermore, the spectrum $\sigma(E_j + F_j)$ satisfies

$$\sigma(E_j + F_j) = \sigma \begin{pmatrix} u + a_j & x_j + g_j \\ h_j & u + a_j \end{pmatrix} = \{u + a_j + ib_j, u + a_j - ib_j\}.$$

Then we may have, for example,

$$J(u)+F(u)=$$

$$= \begin{bmatrix} u+a_1 & x_1+g_1 & \vdots & 0 & 0 & 0 & \dots & 0 \\ h_1 & u+a_1 & \vdots & 0 & 0 & 0 & \dots & 0 \\ \dots & & & & & & & \\ 0 & 0 & \vdots & u+a_m & x_m+g_m & 0 & \dots & 0 \\ 0 & 0 & \vdots & h_m & u+a_m & 1 & \dots & 0 \\ 0 & 0 & \vdots & 0 & 0 & u+s_{2m+1} & \dots & 0 \\ \dots & & & & & & & \\ 0 & 0 & \vdots & 0 & 0 & 0 & \dots & u+s_k \end{bmatrix}.$$

By construction, this matrix (or the corresponding matrix in the Schur case, where the entries to the upper right of the main block diagonal may be arbitrary,) has the spectral list

$$(u+s_1, u+\overline{s_1}, \dots, u+s_m, u+\overline{s_m}, u+s_{2m+1}, \dots, u+s_k).$$

Now let $c, d \in \mathbf{R}$, $u = c+id$ be **nonreal**, and $J(c,d,2p)$ be a *(real-type)* Jordan block of order $2p$ corresponding to the pair of conjugate nonreal eigenvalues u and $\overline{u}$ of A (cf. [8, pp. 369-370]), i.e.,

$$J(c,d,2p) := \begin{pmatrix} c & d & 1 & 0 & 0 & 0 & \vdots & 0 & 0 \\ -d & c & 0 & 1 & 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & c & d & 1 & 0 & \vdots & 0 & 0 \\ 0 & 0 & -d & c & 0 & 1 & \vdots & 0 & 0 \\ \dots & & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \vdots & c & d \\ 0 & 0 & 0 & 0 & 0 & 0 & \vdots & -d & c \end{pmatrix}.$$

If the spectral value u (and its conjugate) in the list L are perturbed by the conjugate complex pairs

$$(c_1+id_1, c_1-id_1, \dots, c_p+id_p, c_p-id_p),$$

then consider the matrix $D(c,d,2p)$ of order $2p$ obtained by replacing in $J(c,d,2p)$ the diagonal 2×2 blocks by their modifications by the subscripts $1, 2, \dots, p$, and replacing all the 1's by 0's. Then

$$||D(c,d,2p)||_{l_\infty} = \max\{|c_j|, |d_j|;\ j = 1, 2, \dots, p\},$$

and the matrix $J(c,d,2p) + D(c,d,2p) \in \mathbf{R}^{2p\times 2p}$ has the perturbed spectral list (also in the Schur case)

$$(c+c_1+i(d+d_1), c+c_1-i(d+d_1), \dots, c+c_p+i(d+d_p), c+c_p-i(d+d_p)).$$

Denote the direct sum of *all* the Jordan matrices and of *all* the perturbation matrices of the pair of *nonreal* eigenvalues $(u, \overline{u})$ [or, equivalently, of the pair

$(c,d)]$, by $\hat{J}(u)$ and by $D(u)$, respectively. Here the order of the direct sum of the blocks corresponding to the pair is an even number. Consider the direct sum G of *all* the perturbation matrices $F(u)$ (for u real as defined previously) with *all* the perturbations $D(u)$ of the direct sums $\hat{J}(u)$ of Jordan blocks (for u nonreal as settled above). We have then

$$G = [\oplus_u F(u)] \oplus [\oplus_u D(u)].$$

Define the matrix Q of order n by $Q := TGT^{-1}$, and define the matrix $R := A + Q$. Then

$$T^{-1}RT = T^{-1}AT + T^{-1}QT = J(A) + G.$$

By construction, the spectrum of $T^{-1}RT$, hence of R, is the list L. Also by construction, the "perturbation matrix" G of $J(A)$ and the transformation matrix T have only real entries. It follows that the matrix $R = A + TGT^{-1}$ is real.

Finally, the formulae for the entries in the matrix sums for G show that if the modulus of each perturbation value is sufficiently small, say $< \delta$, then the modulus of each entry of the matrix G is also, i.e., $||G||_{l_\infty} < \delta$. A possible method for the calculation of a $\delta = \delta(\eta)$ will be given at the end of the next proof. ■

Theorem 2. *Assume that the list* $L := (p; r_1, \ldots, r_v, z_1, \overline{z_1}, \ldots, z_c, \overline{z_c})$ *is the spectrum of a positive matrix* N *of order* n *with Perron eigenvalue* p*, where the numbers* r *are real, the numbers* z *are nonreal, and form pairs (as above) with their conjugates.*

Then there is a positive δ *with the following property: if* $q \in \mathbf{R}$, $|q| < \delta$, $|s_j| < \delta$ $(j = 1, \ldots, v)$*, the numbers* s_j *are either real or, provided they are perturbations of the same real number, they can form several pairs of conjugate nonreal numbers, and if* $|w_j| < \delta$ $(j = 1, \ldots, c)$*, the numbers* w_j *are complex, then the list*

$$L_1 := (p + q; r_1 + s_1, \ldots, r_v + s_v, z_1 + w_1, \overline{z_1 + w_1}, \ldots, z_c + w_c, \overline{z_c + w_c})$$

is the spectrum of a positive matrix P.

Proof. Retracing the argument in the preceding proof, we obtain that the matrix Q has again only real entries, and the modulus of each entry of the real matrix Q is arbitrarily small, if every perturbation value q, s_j and w_j has modulus sufficiently small, say, $< \delta$. The entrywise positivity of N implies that the matrix $P := N + Q$ is entrywise positive.

A number δ can be determined in the following way. It may be advantageous to *assume here* that in the preceding proof J is required to have only real upper block triangular form with 2×2 or 1×1 diagonal blocks, and the transformation matrix T is a real unitary matrix (*the Schur case*).

Consider the following 3 distinct norms defined and denoted exactly as in [10, 5.6] for *any square matrix* A with entries a_{ij} of order n:

$$||A||_\infty := \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|, \quad ||A||_1 := \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}|, \quad ||A||_{l_\infty} := \max_{1\le i,j\le n} |a_{ij}|.$$

We can estimate the modulus of the (i,j) entry Q_{ij} of the matrix Q (applying similar notation everywhere) by

$$|Q_{ij}| \le \sum_{k,m=1}^{n} |T_{ik}G_{km}(T^{-1})_{mj}| \le ||T||_\infty ||G||_{l_\infty} ||T^{-1}||_1.$$

Since T, T^{-1} are unitary, we obtain

$$\max[||T||_\infty, ||T^{-1}||_1] \leq \sqrt{n}.$$

Hence we obtain

$$||Q||_{l_\infty} \leq ||T||_\infty ||G||_{l_\infty} ||T^{-1}||_1 \leq n||G||_{l_\infty} < n\delta.$$

It follows that

$$n\delta < \min_{1 \leq j,k \leq n} N_{jk}$$

implies that $P := N + Q$ is a positive matrix. ■

Remark. A comparison of the following result to Guo [29], Corollary 3.2 may be instructive. The latter considers only real perturbations of real spectral lists, whereas Theorem 3 does (suitable) complex perturbations of complex spectral lists. The price of the greater generality is that the moduli of the admitted perturbations are constrained to decrease, roughly speaking, from q/n to $q/(n^2)$ [in the notation of the following proof].

Theorem 3. *Assume that the list* $L := (p; r_1, \ldots, r_v, z_1, \overline{z_1}, \ldots, z_c, \overline{z_c})$ *is the spectrum of a nonnegative matrix* N *of order* n *with Perron-Frobenius eigenvalue* p, *where the numbers* r *are real, the numbers* z *are nonreal, and form pairs (as above) with their conjugates.*

Then for every $q > 0$ *there is* $\delta = \delta(q) > 0$ *with the following property: if* $|s_j| < \delta$ $(j = 1, \ldots, v)$, *the numbers* s_j *are either real or, provided they are perturbations of the same real number* r, *they can form several pairs of conjugate nonreal numbers, and if* $|w_j| < \delta$ $(j = 1, \ldots, c)$, *the numbers* w_j *are complex, then the list*

$$L_1 := (p + q; r_1 + s_1, \ldots, r_v + s_v, z_1 + w_1, \overline{z_1 + w_1}, \ldots, z_c + w_c, \overline{z_c + w_c})$$

is the spectrum of a positive matrix P.

Proof. There is a nonnegative matrix B of order n with spectral list L which satisfies $Be = pe$ with the vector $e := (1, \ldots, 1)^t$. (For a constructive proof of this fact see [11, pp. 113-114], for a nonconstructive proof see [29, Lemma 2.2]. Let $p' := p + q$. For $q = p' - p > 0$ define the n-vector $k := (q/n, \ldots, q/n)^t$. Then $k^t e = q$, and the positive matrix $A := B + ek^t$ is entrywise not less than the matrix of order n with every entry equal to $q/n > 0$. Further, Brauer's theorem [1, Theorem 27] shows that a spectral list of A is the list

$$L' := (p'; r_1, \ldots, r_v, z_1, \overline{z_1}, \ldots, z_c, \overline{z_c}),$$

and $Ae = p'e$. To the spectral value p' (the spectral radius of A) we order the perturbation value 0, and then proceed with the perturbation of the positive matrix A as in the proofs of Theorems 1 and 2.

Given $q > 0$, a number $\delta = \delta(q)$ can be determined, applying the method in the proof of Theorem 2 for A, in the following way.

Define $\delta \equiv \delta(q) := \frac{q}{n^2}$, and consider perturbation values s_j, w_j of $spec(A)$ satisfying $|s_j|, |w_j| < \delta$. Then the corresponding perturbation matrix $G \equiv G(A)$ satisfies

$$||G||_{l_\infty} < \delta(q) = \frac{q}{n^2}.$$

Hence, applying the Schur similarity method, we obtain $||Q||_{l_\infty} < \frac{q}{n}$, and the perturbed matrix $P = A + Q$ in the proof is positive. ■

Example. Consider the following nonnegative matrix:

$$N := \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 9 & 0 & 17 & 0 & 7 & 0 \end{pmatrix}.$$

Its (complex) Jordan form is:

$$J = J(3,1) \oplus J(-3,1) \oplus J(i,2) \oplus J(-i,2).$$

Take $q := 1/10$. Then the wished new Perron eigenvalue is $p' = 31/10$, and the minimum of the entries of the corresponding positive matrix A is $1/60$. One possible pair T, T^{-1} transforming A to its real Jordan form $J(A)$ according to

$$A = TJ(A)T^{-1}$$

has the following approximate norms:

$$||T||_\infty = 822.96, \qquad ||T^{-1}||_1 = 0.677.$$

Using this transformation matrix T and the estimates above would imply that the perturbation matrix G of $J(A)$ should have

$$||G||_{l_\infty} \leq 0.0000299.$$

Applying the Schur unitary similarity method would imply that perturbation matrices G of a real Schur form of A satisfying

$$||G||_{l_\infty} < \frac{q}{n^2} \leq 0.00277$$

still yield a positive perturbed matrix $P = A + TGT^{-1}$. ■

Example. Consider the following nonnegative matrix:

$$N := \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}.$$

Its spectral list is $L := (\sqrt{3}; i, -i)$. It is known (see [17, p. 89] and [5]) that a spectral list of the form $(r; i, -i)$ is the list of a nonnegative matrix if and only if $r \geq \sqrt{3}$. This shows *all the possibilities for perturbing the spectral radius of* L so that the resulting list be again the list *of a nonnegative matrix*.

Further, the *application of the main result by* Laffey [14] shows that for all $t \geq 0$ the modified list $(\sqrt{3} + 2t, -t + i, -t - i)$ is the spectrum of a nonnegative matrix.

On the other hand, *our methods yield the following*: we have $N = TJT^{-1}$, where

$$T := \frac{1}{6} \begin{pmatrix} 2 & 2\sqrt{2} & 0 \\ 2 & -\sqrt{2} & \sqrt{6} \\ 2 & -\sqrt{2} & -\sqrt{6} \end{pmatrix}, \qquad J := \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

J is a real Jordan form of N, and the transformation matrix T and its inverse have norms such that approximately

$$||T||_\infty ||T^{-1}||_1 = 2.865251372.$$

An immediate perturbation of J by the matrix

$$Z := \begin{pmatrix} q & 0 & 0 \\ 0 & c & d \\ 0 & -d & c \end{pmatrix},$$

where $q > 0, c, d \in \mathbf{R}$, yields the real Jordan matrix $J+Z$, for which $T(J+Z)T^{-1}$ is $\frac{1}{3}$ times the circulant matrix with determining entries

$$\sqrt{3} + q + 2c, \quad 2\sqrt{3} + q - c + \sqrt{3}d, \quad q - c - \sqrt{3}d.$$

This method yields a nonnegative matrix with spectral list

$$(\sqrt{3} + q, c + i(1 + d), c - i(1 + d))$$

if and only if the 3 determining entries above are all nonnegative.

The method of the proof of Theorem 3 leads from N for every $q > 0$ to the positive matrix

$$A \equiv A(q) = \frac{1}{3} \begin{pmatrix} \sqrt{3} + q & 2\sqrt{3} + q & q \\ q & \sqrt{3} + q & 2\sqrt{3} + q \\ 2\sqrt{3} + q & q & \sqrt{3} + q \end{pmatrix}$$

with spectral list $(\sqrt{3} + q, i, -i)$. Applying the perturbation method in the Schur case, taking $\delta := \frac{q}{9}$ and any *complex* perturbation value w_1 with modulus smaller than $\frac{q}{9}$, the corresponding perturbed matrix $P = A + Q$ is positive and has the spectral list $(\sqrt{3} + q, i + w_1, -i + \overline{w_1})$. ∎

It is easy to see that the method of proofs of the above Theorems can also be used to prove the following extension of a nice theorem by Minc (cf. [18, pp.187-188] or [19]) for a *stronger type* of inverse spectral results, where not only the spectral list, but also *the Jordan structure is prescribed.*

Corollary 1. *Assume that the Jordan matrix (i.e. a direct sum of [complex type] Jordan blocks)*

$$J := \oplus_{k=1}^{m} J(a_k, \nu_k)$$

is similar to a positive matrix. Modify the matrix J to obtain the Jordan matrix $\tilde{J}$ by "connecting" some of the blocks belonging to identical eigenvalues, i.e. consider for the eigenvalue a instead of

$$J(a, n_1) \oplus \cdots \oplus J(a, n_s)$$

(where $n_1, \ldots, n_s$ are all the orders of Jordan blocks in J corresponding to the eigenvalue a) the direct sum $J(a, N_1) \oplus \cdots \oplus J(a, N_k)$, where the numbers N_j are sums of pairwise distinct groups of the numbers n_i satisfying

$$N_1 + \cdots + N_k = n_1 + \cdots + n_s,$$

and if the eigenvalue a is nonreal, carry out exactly the same "connecting" also for the conjugate $\overline{a}$. Then the Jordan matrix $\tilde{J}$ is also similar to a positive matrix.

Proof. It is sufficient to recall that for any Jordan matrix the replacement of the "canonical" (nondiagonal) 1's by arbitrary nonzero complex numbers yields a matrix similar to the Jordan matrix, and then apply the method of the proofs of Theorems 1 and 2. ■

Remark. Note that the above Corollary can obviously be formulated in the terms of *elementary divisors*, as it was done in the cited result of Minc.

The following result is based on a remarkable characterization of the nonzero spectrum of a positive matrix obtained in [3]. Recall that *the number of the needed zeros* in the cited result has remained a difficult unsolved problem.

Corollary 2. *Assume that the list*

$$L := (z_1; z_2, \ldots, z_m)$$

consisting of nonzero complex entries satisfies the following conditions:

(a) $z_1 > |z_j|$ *for each* $j > 1$,

(b) $z_1^k + \cdots + z_m^k > 0$ *for each* $k = 1, 2, 3, \ldots$.

Then (the list is self-conjugate, and) there are a nonnegative integer M and a positive number $\delta > 0$ with the following property: if the perturbation p_1 of z_1 is real, if the nonreal pairs z_j, $\overline{z_j}$ receive conjugate complex perturbations p_j, $\overline{p_j}$, and the remaining $z_k \in L$, $k \neq 1$ and the number 0 (in M copies) receive either real perturbations p_k or (possibly several) pairs of the same real number receive pairs of conjugate complex perturbations $p_f, \overline{p}_f$ in such a way that

$$|p_g| < \delta \qquad \text{(for every perturbation)},$$

then there is a positive matrix P with spectrum

$$spec(P) = (z_1 + p_1; z_2 + p_2, \ldots, z_m + p_m, p_{m+1}, \ldots, p_{m+M}).$$

Proof. By [3, Proposition, p. 313], there is a positive matrix A whose nonzero spectrum is the list L. Let M denote the number of zeros in the spectrum of A, and apply Theorem 2. ■

Remark. Note that Theorems 1-3 are *of the following type*. Loosely speaking, we assume that the list L is the spectrum of a real or nonnegative or positive matrix, and some additional conditions are satisfied. If the pseudo-distance m of the list L and of another list L' is sufficiently small, then the list L' is also the spectrum of a real or nonnegative or positive matrix.

Theorem 2 can, e.g., be exactly reformulated in the following way:

Theorem 2'. *Assume that a list L from the class PF_n in the Introduction has a positive realization. Then there is a neighbourhood (with respect to the pseudo-metric m in PF_n) of L from which every list has a positive realization.*

In other words: the set PPF_n is open (with respect to m) in the set PF_n (or, also, in S_n). Note that similar statements can be formulated with respect to the metric $\tilde{m}$ using the corresponding quotient spaces.

Proof. Indeed, if the number $\delta > 0$ is sufficiently small, then the considered type of perturbation *is the most general among those producing again a self-dual list*: no pair of complex numbers with *distinct real parts* yields, after perturbations of moduli smaller than δ, a pair of conjugate complex numbers or a pair of real numbers unless both the original and the perturbed pairs are real. ■

Example. Note that the statement of Theorem 2 does *not* hold if we assume only that the unperturbed list is the spectrum of a *primitive irreducible nonnegative* (instead of a *positive*) matrix. Indeed, the spectral list

$$L := (1, -1/2, -1/2)$$

is the spectrum of a primitive nonnegative matrix N of order 3, by Sulejmanova's well-known theorem, see, e.g., [18, pp. 183-184]. There is $\delta > 0$ such that for every $0 < p < \delta$ the list

$$L' := (1, -1/2 - p, -1/2)$$

is a perturbation of the spectrum of N *of the type allowed in Theorem 2.* The trace condition shows that L' is *not* the nonzero spectrum of *any* nonnegative matrix. ■

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