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Riesz operators and Schur's lemma

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Abstract. We present sufficient conditions in order (the space of) a Riesz operator T in a Hilbert space H have a Jordan–Schur basis with respect to a scalar product equivalent to the original one. This is related to Schur's lemma for a compact operator, which is an extension of Schur's classical theorem on unitary triangularization in a finite dimensional space. The finite dimensional case is also studied.

1. Introduction

Let T be a bounded linear operator in the complex Banach space X. The aim of this paper is to establish, under suitable conditions, the existence of several kinds of bases of "finite types" for the given operator or for its restriction to a certain T-invariant subspace of X. The expression "finite types" refers to the fact that the results are most complete in the case of a finite dimensional inner product space X, but the infinite dimensional cases are also instructive. As a classical result in this area we cite Schur's lemma for a compact operator in a Hilbert space H (cf. [7, I. Lemma 4.1]). Many relevant concepts and results can also be found in the monograph [7]. Note that the related problem of spectral synthesis for related classes of Jordan-like operators has recently been studied by Seubert [15], [16], by (naturally) completely different methods.

It is well known that the restriction to a finite dimensional T-invariant subspace of X has a *Jordan basis*, i.e. a linear basis, in which the matrix of the restriction is the direct sum of Jordan blocks. If X is a Hilbert space, we shall say that the

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restriction to a certain T-invariant subspace of X has a *Schur basis*, if the basis is orthonormal and the (possibly inifinite) matrix of the restriction relative to this basis is upper triangular. In a Hilbert space we shall say that the restriction has an *orthogonal basis (consisting) of Jordan subspaces*, if the subspace of the restriction is the orthogonal sum of subspaces in each of which there is a (finite) Jordan (vector) basis for the restriction. Finally, we say that the restriction has a *Jordan–Schur basis*, if the restriction has an orthonormal basis in which the (infinite) matrix of the restriction is the orthogonal sum of upper triangular matrices of finite orders, which are in a 1-1 correspondence with the orders of corresponding Jordan blocks. Note that we shall recall these concepts exactly under the concrete conditions of their appearance and application.

To fix some *terminology*: we shall say that a bounded linear operator D in the Hilbert space H is *positive definite* (self-adjoint), if there is a number $m > 0$ such that for every $h \in H$ we have $(Dh, h) \geq m(h, h)$, where $(,)$ denotes the inner product in H. (Note that some writers call such an operator *strictly positive* or *positively bounded from below*). Defining

$$\langle h, k\rangle := (Dh, k) \qquad (h, k \in H),$$

we obtain a new (sometimes called the D-) scalar product $\langle , \rangle$ that is *equivalent* to the old one $(,)$ (cf., e.g., Rätz [12]).

The notion of a *Riesz operator* appeared first in Ruston [13], see also Dieudonné [2], West [19], [20] and many others. For a nice introduction and equivalent definitions see also Dowson [4]. One version: a bounded linear operator T in a Banach space X is a Riesz operator if and only if the following conditions hold: every non-zero point of the spectrum $\sigma(T)$ is isolated, and for every such point z of $\sigma(T)$ the (Riesz–Dunford) spectral projection $E(z) \equiv E(\{z\})$ has finite dimensional range.

For the notions of a *spectral operator* in the sense of Dunford, its *scalar part* and *resolution of the identity* we refer the reader to [5], [6] or [4].

The following deep result is classical. It is connected with the names of Lorch [9], Day [1], Mackey [11], Sz.-Nagy [17], Dixmier [3], Wermer [18], and its same nice proof is given in [6, XV.6.1] or [4, Th. 8.1].

Theorem A. *Let G be a bounded multiplicative abelian group of operators in the Hilbert space H. There is then a bounded positive definite self-adjoint operator B in H such that for every operator T in G the operator BTB^{-1} is unitary.*

The known proofs of this theorem are highly nonconstructive. They use, e.g., the concept of the Banach limit, or a fixed point theorem of Markov–Kakutani in order to find a function $f\colon H \times H \to \mathbb{C}$ satisfying

$$f(Tx, Ty) = f(x, y) \quad \text{for every } x, y \in H,\ T \in G.$$

This result is then used to prove the following (cf. [6, XV.6.2] or [4, Prop. 8.2])

Theorem B. *Let A be a bounded Boolean algebra of projections ($\equiv$ bounded idempotent operators) in H. There is a positive definite self-adjoint operator B in H such that for every E in the algebra A the operator BEB^{-1} is a self-adjoint projection in H.*

We shall repeat here the short but clever proof stemming from Dixmier [3].

Proof. For every $E \in A$ define $F(E) := I - 2E$. Easy calculation yields

$$F(E)^2 = I, \qquad F(E_1)F(E_2) = F[E_1(I - E_2) \vee E_2(I - E_1)],$$

where $D \vee E$ is defined as $D + E - DE$. Hence the family G of all $F(E)$ with $E \in A$ is a bounded multiplicative abelian group of operators in H. From Theorem A we obtain that there is a positive definite self-adjoint operator B in H such that for every operator $F(E)$ in G the operator $BF(E)B^{-1} =: U$ is unitary. From $F(E)^2 = I$ we obtain $U^2 = I$, hence $U^* = U^{-1} = U$. Thus U is self-adjoint, and the operator

$$BEB^{-1} = (1/2)B[I - F(E)]B^{-1} = (I - U)/2$$

is also self-adjoint. ■

An immediate *corollary* is that for any *scalar-type* spectral operator S (in Hilbert space) there is a bounded positive definite self-adjoint operator B in H such that the operator $N := BSB^{-1}$ is normal.

2. The infinite-dimensional case

A further consequence of Theorem B is formulated in the following theorem, for which we need some *preparations*. Let T be a Riesz operator in a Banach space X, and let $E(z)$ denote the (Riesz–Dunford) spectral projection for every non-zero point z of $\sigma(T)$.

In each of the subspaces $E(z)X$ *fix a Jordan basis* for the restriction operator $T|E(z)X$. It means that we fix $k \equiv k(z)$ *ordered bases* of $T|E(z)X$-invariant subspaces $S(z, j)$ of $E(z)X$ such that in each subspace $S(z, j)$ the restriction $T|S(z, j)$

has an upper Jordan block of dimension $d(z,j)$ as its matrix in the fixed ordered basis $(j=1,2,\dots,k(z))$. Then, clearly,

$$\sum_{j=1}^{k(z)} \dim S(z,j) = \sum_{j=1}^{k(z)} d(z,j) = \dim E(z)X.$$

With respect to this fixed Jordan basis define the *Jordan idempotents* $J_0(z,j)$ in $E(z)X$ $(j=1,2,\dots,k(z))$ as the skew projections of $E(z)X$ onto $S(z,j)$ parallel to the direct sum of all the remaining ones, i.e., to $S(z,1)\oplus\cdots\oplus S(z,j-1)\oplus S(z,j+1)\oplus\cdots\oplus S(z,k(z))$. These Jordan idempotents clearly satisfy

$$J_0(z,j)J_0(z,r) = \delta_{jr}J_0(z,j), \qquad J_0(z,1)+\cdots+J_0(z,k(z)) = I|E(z)X$$

(where δ is the Kronecker delta), and commute with the restriction $T|E(z)X$.

Further, define $J(z,j)\equiv J(T,z,j)$ by

$$J(z,j) := J_0(z,j)E(z)\colon X \longrightarrow S(z,j) \qquad (j=1,2,\dots,k(z)).$$

We shall call them the *Jordan idempotents in X with respect to the fixed Jordan basis*. They satisfy

$$J(z,j)J(w,r) = \delta_{zw}\delta_{jr}J(z,j) \quad [z,w\in\sigma(T)\setminus\{0\},\ j=1,\dots,k(z),\ r=1,\dots,k(w)],$$

and commute with the operator T. Indeed, for each pair (z,j) we have

$$\begin{aligned} J(z,j)T = J_0(z,j)E(z)T = J_0(z,j)TE(z) &= J_0(z,j)[T|E(z)X]E(z) \\ &= [T|E(z)X]J_0(z,j)E(z) = TJ(z,j). \end{aligned}$$

Consider the Boolean algebra $A_1\equiv A_1(T)$ of all bounded idempotent operators in X which commute with T. The *family of all Jordan idempotents in X* from above is a subset of A_1, hence we can define the Boolean algebra $A(T)\subset A_1(T)$ generated by the family.

Recall that an early significant result by Dunford [5, Theorem 20] was slightly extended, and can be formulated as follows (cf. [4, 6.12]): Let X be a weakly complete Banach space, and let the operator T in X have *totally disconnected spectrum*. The following are then *equivalent*:

1. T is a *spectral operator* in the sense of Dunford.

2. There is a number $K\ge 1$ such that for every clopen ($\equiv$closed-and-open) set c in the relative topology of $\sigma(T)$ the Riesz-Dunford spectral projection $E(c)$ [for T] has norm at most K.

Note that then there is a number $L \geq 1$ such that for every Borel set $b \subset \mathbb{C}$ the projection $E(b)$ in the resolution of the identity [for T] has norm not exceeding L.

These preliminaries applied to a Riesz operator in a *Hilbert space* yield the following.

Theorem 1. *Let T be a Riesz operator in the Hilbert space H, and apply the preceding notation. Assume that the Boolean algebra $A(T) \subset A_1(T)$ generated by the family of projections*

$$\{J(z,j) : z \in \sigma(T) \setminus \{0\}, j = 1, 2, \ldots, k(z)\}$$

is a bounded set (in the operator norm in H). Then T is a spectral operator, and there is a positive definite self-adjoint operator B in H such that for every E in the algebra $A(T)$ the operator BEB^{-1} is a self-adjoint projection in H.

Proof. It is clear that each Riesz operator has totally disconnected spectrum. Further, the spectral projection $E(z)$ for T corresponding to the one-point set $\{z\}$ $(z \neq 0)$ satisfies

$$E(z) = J(z,1) + J(z,2) + \cdots + J(z,k(z)).$$

Hence it is an element of the Boolean algebra $A(T)$, and the same is valid for the spectral projection $E(c)$ for any clopen set c *not* containing the point 0. It follows for any clopen set c *containing* the point 0, hence for *all clopen* sets c. By [6, 6.12], T is a spectral operator, the resolution of the identity E of which is then a norm-bounded family of projections. Theorem B implies then the last statement. ■

Example. The following example will show that there is a compact, hence Riesz, spectral operator T in a Hilbert space H for which the family

$$\{E(b) : b \text{ Borel set}\}$$

of all projections *in the resolution of the identity* is *bounded* in the operator norm, but the family

$$\{J(z,j) : z \in \sigma(T) \setminus \{0\}, j = 1, 2, \ldots, k(z)\}$$

of Jordan idempotents is *not bounded.*

Let $H := l^2(\mathbb{N})$, and let the doubly indexed vector sequence

$$\{b_r^i : i \in \mathbb{N},\ r = 1, 2\}$$

be the orthonormal basis in H consisting of the vectors with $(2i-1)$st, $2i$th components (in this order)

$$(1,0), \qquad (0,1),$$

(all other components equal to 0), and let H_i denote the 2-dimensional subspace spanned by $\{b_1^i, b_2^i\}$ $\quad (i \in \mathbb{N})$. In each H_i consider the (nonorthogonal) basis $\{b_1^i, v_2^i\}$, where the $(2i-1)$st, $2i$th components of the vectors (in this order) are

$$(1,0), \qquad (p_i, q_i) \qquad (p_i > 0, q_i > 0, p_i^2 + q_i^2 = 1) \qquad (i \in \mathbb{N}),$$

and all other components are 0. Compute the (cosine of the) angle between the two subspaces determined by the vectors $\{b_1^i, v_2^i\}$. The general unit vectors x from the first and y from the second subspace have the "essential components" (all others being 0)

$$x = (\alpha, 0) \quad (|\alpha| = 1), \qquad y = \beta(p_i, q_i) \quad (|\beta| = 1).$$

The cosine of the angle between these subspaces is defined (cf., e.g., [7, VI.5.4]) by

$$\sup_{|x|=|y|=1} |(x,y)| = \sup_{|\alpha\beta|=1} |\alpha\beta| p_i = p_i.$$

Note that, taking a positive increasing sequence $\{p_i\}$ converging to 1, the sequence of these cosines tends also to 1.

In each subspace H_i consider the nonorthogonal normed (Jordan) basis consisting of the 2 vectors $\{b_1^i, v_2^i\}$ from above, and define the restriction $T|H_i$ by

$$Tb_1^i := s_i b_1^i, \quad Tv_2^i := s_i v_2^i,$$

where the sequence $\{s_i\}$ of nonzero complex numbers tends to 0. It means that the matrix of the restriction $T|H_i$ with respect to this ordered basis is diagonal of order 2 with the eigenvalue s_i. By a result of Lyantse [10] (cf. also [7, VI.5.4]), the norm of the idempotent projecting H_i onto the span of the first vector parallel to the span of the second one is the reciprocal of the sine of the above calculated angle, i.e., $1/\sqrt{1-p_i^2}$, which tends to ∞, if $p_i \to 1$.

Define the operator T as the orthogonal sum of the operators $T|H_i$ $(i \in \mathbb{N})$. T is then a compact Riesz operator, and the preceding considerations show that the operator norms of the Jordan idempotents $J(s_i, 1)$ tend to ∞, if $i \to \infty$. On the other hand, T is spectral, all the spectral projections in the resolution of the identity of T are orthogonal, hence their family is bounded in the operator norm of H, as stated. In the terminology of the Introduction we can say that $\{H_i : i \in \mathbb{N}\}$ is an orthogonal basis consisting of Jordan subspaces for the operator T.

Theorem 2. *Under the conditions of Theorem 1 define in H the B-scalar product $\langle , \rangle$ relative to the original one $(,)$ as follows:*

$$\langle x, y \rangle := (Bx, By) \qquad (x, y \in H).$$

Then the projections $E(\cdot)$ in the resolution of the identity for T are self-adjoint with respect to the B-scalar product $\langle , \rangle$, hence the scalar part $S := \int zE(dz)$ of the operator T is normal with respect to $\langle , \rangle$. Further, all the projections in the Boolean algebra A, in particular, the projections

$$J(z, j) \qquad [z \in \sigma(T) \setminus \{0\}, j = 1, 2, \ldots, k(z)]$$

are self-adjoint with respect to $\langle , \rangle$. Since the Jordan idempotents $J(z, j)$ are pairwise disjoint, the ranges of any pair of them are orthogonal with respect to $\langle , \rangle$.

Proof. By the preceding theorem, for every E in the algebra A the operator BEB^{-1} is a self-adjoint projection with respect to $(,)$ in H. It is easy to check that $\langle , \rangle$ is a scalar product in H, equivalent to $(,)$, for which, with the notation $x = B^{-1}h, y = B^{-1}k$, we have

$$\langle Ex, y \rangle = (BEB^{-1}h, BB^{-1}k) = (h, BEB^{-1}k) = (Bx, BEy) = \langle x, Ey \rangle \quad (x, y \in H),$$

i.e., E is self-adjoint with respect to $\langle , \rangle$. In particular, the same holds for the projections $J(z, j)$ and also for all the $E(\cdot)$ in the resolution of the identity for T. Let S^+ denote the adjoint of the operator S *with respect to the scalar product* $\langle , \rangle$. It is easy to check that $SS^+ = S^+S$, i.e. S is normal with respect to $\langle , \rangle$. ∎

Remark. Under the conditions of Theorem 2 the family of subspaces

$$\{E(0)H,\ J(z, j)H : z \in \sigma(T) \setminus \{0\}, j = 1, 2, \ldots, k(z)\}$$

forms a *basis (of H) from subspaces* which is *orthogonal with respect to* $\langle , \rangle$, in the terminology of [7, VI.5].

The next Corollary is *the main result* for a Hilbert space on the *existence of a Jordan–Schur basis* with respect to the equivalent scalar product $\langle , \rangle$.

Corollary. *Under the conditions and with the notation of the preceding two theorems the canonical (Jordan–Dunford) decomposition $T = S + Q$, where Q is a quasinilpotent operator commuting with S, has the following particular properties:*

The restriction $T|E(\mathbb{C} \setminus \{0\})H$ has a Jordan–Schur basis with respect to the scalar product $\langle , \rangle$ in the sense that the finite dimensional range spaces $E(z)H \quad (z \neq$

0) *are pairwise orthogonal to each other for distinct values of* z, *and in each subspace* $E(z)H$ *there is a* $\langle , \rangle$*-orthonormal basis such that the matrix of* $T|E(z)H$ *in this basis is the direct sum*

$$M[d(z,1)] \oplus M[d(z,2)] \oplus \cdots \oplus M[d(z,k(z))]$$

of upper triangular quadratic matrices M *of the indicated dimensions (orders). (Note that the orders are equal to the respective orders of the classical Jordan decomposition blocks for the matrix of* $T|E(z)H$*.) Further, we have the* $\langle , \rangle$*-orthogonal sum decomposition*

$$T = [S+Q]|E(\mathbb{C} \setminus \{0\})H \oplus Q|E(\{0\})H,$$

where the last operator is quasinilpotent.

If, in addition, the restriction

$$T|E(\{0\})H = Q|E(\{0\})H$$

has a Jordan basis (of finite cardinality), then this part of the operator T, *hence all of* T, *also have a Jordan–Schur basis with respect to the scalar product* $\langle , \rangle$.

Proof. We have already proved the $\langle , \rangle$-orthogonality of the distinct spaces $J(z,j)H$. Essentially the same argument proves that each $J(z,j)H$ is $\langle , \rangle$-orthogonal to $E(\{0\})H$. In each space $J(z,j)H$ choose a Jordan basis (in which the matrix of $T|J(z,j)H$ is an upper Jordan block of order $d(z,j)$), and then apply the orthonormalization process to each such basis *separately*. If the additional assumption holds, then we can proceed in the subspace $E(\{0\})H$ as before. ∎

Remark. The last conclusion is also valid, if $T|E(\{0\})H$ has a Jordan–Schur basis of possibly infinite cardinality with respect to $\langle , \rangle$.

3. The finite dimensional case

Retain the assumptions and the notation of the last two theorems and of the corollary, and assume, *in addition*, that the underlying Hilbert space H with scalar product $(,)$ is *finite dimensional*. Then *every operator* T in H is a Riesz operator for which the Boolean algebra of projections $A \subset A_1$ is *finite*, hence a norm-bounded set. T is a spectral operator and, for every positive definite self-adjoint operator B (depending on T as above), we can define the $B \equiv B(T)$- (or, equivalently, $D \equiv D(T)$-) scalar product $\langle , \rangle$ relative to the original one $(,)$ as before:

$$\langle x, y \rangle_B \equiv \langle x, y \rangle := (Bx, By) = (Dx, y) \qquad (x, y \in H),$$

where we have set $D := B^2$. The operator D is clearly a positive definite operator corresponding in a 1-1 manner to B. The Borel set $\{0\}$ plays in this case no special role, hence all the statements of the corollary are valid.

Proposition. *In the finite-dimensional case one operator $D \equiv D(T)$ can effectively be determined by the method outlined in Theorems A and B.*

Proof. The generated Boolean algebra $A = A(T)$ of projections is in a 1-1 correspondence with the bounded multiplicative abelian group of operators $G := F(A)$ (cf. Theorem B). Hence the group G is finite, and we shall denote its elements by $F_1, F_2, \dots, F_r$. Define

$$(x, y)_G := r^{-1} \sum_{k=1}^{r} (F_k x, F_k y) \qquad (x, y \in H).$$

It is easy to check that $(,)_G$ is a new scalar product in H. For every $F_j \in G$ we have

$$(F_j x, F_j y)_G = r^{-1} \sum_{k=1}^{r} (F_j F_k x, F_j F_k y) = (x, y)_G \qquad (x, y \in H),$$

since left multiplication by F_j is an automorphism of the finite group G. Thus every $F_j \in G$ is a unitary operator with respect to the scalar product $(,)_G$. Hence $(,)_G$ can play the role of the scalar product $\langle , \rangle$ from the general considerations, and there is a positive definite operator B (or, as we have seen above, D) such that

$$(x, y)_G = (Bx, By) = (Dx, y) \qquad (x, y \in H).$$

In the space $H = \mathbb{C}^n$ choose any ordered *orthonormal* basis $\beta := (b_1, b_2, \dots, b_n)$ with respect to the old scalar product $(,)$. We have then

$${}_\beta[D]_\beta(j, k) = (Db_k, b_j) = (b_k, b_j)_G,$$

where ${}_\beta[D]_\beta(j, k)$ denotes the (j, k)-entry of the matrix of the operator D in the basis β, and $(b_k, b_j)_G$ is the (j, k)-entry of the Gram matrix of the vector sequence $(b_1, b_2, \dots, b_n)$ with respect to the scalar product $(,)_G$. ∎

Remark. In order to formulate the above results *in matrix terms*, recall the notation of [8, 0.10] of the (β, γ) bases representation ${}_\gamma[T]_\beta$ of an operator T, where β, γ are ordered bases in $\mathbb{C}^n$.

$${}_\gamma[T]_\beta := [[Tb_1]_\gamma, \dots, [Tb_n]_\gamma],$$

where $[Tb_k]_\gamma$ denotes the column of the coefficients of the image of the β-basis vector b_k with respect to the ordered basis γ. For any pairs of ordered bases we have then

$$_\gamma[T]_\gamma\, _\gamma[I]_\beta =_\gamma [I]_\beta\, _\beta[T]_\beta,$$

where I is the identity map in $\mathbb{C}^n$. As a relevant well-known example recall that if $_\gamma[T]_\gamma$ is arbitrary, and β is any (fixed ordered) *Jordan basis* for the operator T, then the intertwining *invertible* matrix $_\gamma[I]_\beta$ yields the connection with the corresponding Jordan form (matrix) $_\beta[T]_\beta$. As a second relevant known example, if $_\gamma[T]_\gamma$ is arbitrary, then Schur's theorem (see [14] or [8, 2.3.1]) shows that there is a *Schur basis* β such that the intertwining $_\gamma[I]_\beta$ is a *unitary* matrix, and $_\beta[T]_\beta$ is an upper triangular matrix.

Our result can be formulated in these matrix terms as follows:

Theorem 3. *If $_\gamma[T]_\gamma$ is arbitrary, then there is a positive definite operator B (or, as we have seen above, D) which defines a new scalar product*

$$\langle x, y\rangle = (x, y)_G = (Bx, By) = (Dx, y) \qquad (x, y \in H)$$

such that there is a Jordan–Schur basis β which is orthonormal with respect to the new scalar product $(,)_G$, satisfies

$$_\beta[T]_\beta = \bigoplus_{z\in\sigma(T)} \bigoplus_{j=1}^{k(z)} M[d(z,j)],$$

and for which the matrix $_\gamma[B]_\gamma\, _\gamma[I]_\beta$ is unitary.

Proof. We note here only that if $(b_1, b_2, \ldots, b_n)$ are the (column) vectors of the basis β, then

$$_\gamma[B]_\gamma\, _\gamma[I]_\beta =_\gamma [B]_\beta = [[Bb_1]_\gamma, \ldots, [Bb_n]_\gamma].$$

The basis β is orthonormal with respect to the scalar product $\langle,\rangle \equiv (B\cdot, B\cdot)$, which means that the columns of $_\gamma[B]_\gamma\, _\gamma[I]_\beta$ form a unitary matrix. ∎

Remark. Introduce the short notation for the matrices:

$$C :=_\gamma [B]_\gamma, \qquad V :=_\gamma [I]_\beta.$$

We have seen above that then the matrix $U := CV$ is unitary. Hence $V = C^{-1}U$ is the *polar form* of the matrix V. Both factors on the right-hand side are invertible, hence (cf. [8, 7.3.3]) they are determined uniquely by V:

$$C^{-1} = (VV^*)^{1/2}, \qquad U = CV.$$

This shows that knowledge of the intertwining matrix $V =_\gamma [I]_\beta$ determines the (matrix C of the) operator B.

Remark. Assume that the *infinite dimensional* Hilbert space H is a $(\,,\,)$-orthogonal sum of finite dimensional subspaces H_r, in each of which a pair (T_r, D_r) $(r \in \mathbb{N})$ satisfies the conditions of Theorem 3. Assume further that the orthogonal sum operator $T := \bigoplus_{r=1}^{\infty} T_r$ is bounded, i.e., $|T| = \sup\{|T_r| :\ r \in \mathbb{N}\} < \infty$, there is some $m > 0$ such that the operators D_r satisfy $(D_r x_r, x_r) \geq m(x_r, x_r)$ for every $x_r \in H_r$, and that the orthogonal sum operator $D := \bigoplus_{r=1}^{\infty} D_r$ is bounded. Then D is a positive definite operator in H, and T has an upper *triangular* matrix in a basis that is orthonormal with respect to the new scalar product $\langle x, y\rangle := (Dx, y) \quad (x, y \in H)$.

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