Subnormal operators, cyclic vectors and reductivity

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Abstract

Two characterizations of the reductivity of a cyclic normal operator in Hilbert space are proved: the equality of the sets of cyclic and *-cyclic vectors, and the equality $L^2(\mu) = \mathbf{P}^2(\mu)$ for every measure μ equivalent to the scalar-valued spectral measure of the operator, respectively. A cyclic subnormal operator is reductive if and only if the first condition is satisfied. Several consequences are also presented.

1 Introduction

Reductive normal operators were studied first by Halmos [HN] and Wermer [W], and important related properties for subnormal operators were investigated by Bram [BJ]. Dyer, Pedersen and Porcelli [D] proved (see also [A]) that every operator in a separable Hilbert space of dimension greater than 1 has a nontrivial invariant subspace if and only if each reductive operator is normal.

The general concept of cyclicity with respect to a set $\mathbf{A}$ of operators was studied in approximation problems connected with invariant subspaces (see, e.g., [NI, pp. 312-313]). The notions of cyclicity proper and *-cyclicity correspond to the simplest cases $\mathbf{A} := \{T\}$ and $\{T, T^*\}$, respectively, for a bounded linear operator $T \in B(H)$ in the Hilbert space H.

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For any vector h and any bounded linear operator T in a *separable Hilbert* space H let $R(h,T) \equiv R(h)$ denote the smallest T-reducing subspace containing h, and let $I(h,T) \equiv I(h)$ denote the smallest T-invariant subspace containing h. The vector h in H is called a **-cyclic vector* for the bounded linear operator T if $R(h,T) \equiv R(h) = H$. We shall write then $h \in *cyc(T)$ and, if the latter set is nonvoid, call T a **-cyclic* operator. The vector $h \in H$ is a *cyclic vector* for T if $I(h,T) \equiv I(h) = H$. We shall write then $h \in cyc(T)$ and, if the latter set is nonvoid, call T *cyclic*. Clearly, $cyc(T) \subset *cyc(T)$, and the inclusion *can be proper*. Bram [BJ] proved that *if the operator is normal*, and the latter set is nonvoid, so is the former, i.e. a normal operator N is *-cyclic if and only if it is cyclic.

It is known that this property does not hold for each operator $T \in B(H)$: denoting the unilateral shift of multiplicity 1 by S, the orthogonal sum $T := S \oplus S$ is not cyclic, but $T^* = S^* \oplus S^*$ is (cf. [HP, Problem 163]). It follows that the *subnormal* operator T is *-cyclic though not cyclic.

Feldman [F] proved a large number of deep results on the existence of *-cyclic and cyclic vectors, respectively, for subnormal operators. Among other useful facts on cyclicity, he showed that the adjoint of a pure subnormal operator is cyclic, and that a subnormal operator has a cyclic adjoint if and only if it is *-cyclic. We shall refer to his results wherever they are close to ours.

A normal operator N has been called *reductive* (or, equivalently, completely normal or, equivalently, having the so called property (P)) if its every invariant subspace is orthogonally reducing. In modern usage this definition (without the parentheses) applies to each operator T, and is equivalent to $Lat(T) = Lat(T^*)$, where Lat denotes the family of invariant subspaces.

Recall some characterizations of *reductive normal* operators. The normal operator N is reductive if and only if 1) *or* 2) *or* 3) holds:

1) For every pair $x, y \in H$ satisfying $< N^k x, y >= 0$ for every $k \in \mathbf{N}_0 := \{0, 1, 2, \dots\}$ we have $< E(b)x, y >= 0$ for every Borel set b, where E is the resolution of the identity for N, and $<, >$ denotes scalar product in H ([W, Lemma 2]).

2) $\mathbf{P}^\infty(\mu) = L^\infty(\mu)$ for some (and then *every*) scalar-valued spectral measure μ for N (see Sarason [SW, p.14] or [CT, Corollary 1.3 on p. 310]). Here $\mathbf{P}^\infty(\mu)$ denotes the weak* closure of the set of the polynomials in $L^\infty(\mu)$, and a *scalar-valued spectral measure μ for N* is any nonnegative Borel measure mutually absolutely continuous with E.

3) The adjoint N^* is in the closed (in the weak operator topology) sub-

algebra of $B(H)$ generated by N and the identity I (Sarason [SA]).

Note that Scroggs [Sc] proved that $int[\sigma(N)] \neq \emptyset$ implies that N is *not* reductive, whereas Wermer [W, Theorem 7] showed that the conditions $int[\sigma(N)] = \emptyset$ and $\mathbf{C} \setminus \sigma(N)$ is connected *together* imply that N is *reductive.* Here *int* denotes interior of the set, and σ denotes spectrum.

Ross and Wogen noted in [R, p. 1538] that if a *cyclic normal* operator T is reductive, then $cyc(T) = *cyc(T)$. *One of the aims of this paper* is to show that this statement (clearly) holds without assuming normality and, what is more remarkable, *the converse* holds for (cyclic) normal and even *subnormal* operators. As an application, another characterization of the reductivity of a cyclic normal operator will be given in terms of the conditions

$$\mathbf{P}^2(\mu) = L^2(\mu)$$

(exact definitions and conditions see below in Section 2). Further, several consequences will be studied.

2 Normal operators

Theorem 2.1. *If the cyclic operator $T \in B(H)$ is reductive, then $cyc(T) = *cyc(T)$. In the converse direction: if $cyc(N) = *cyc(N)$ for a cyclic normal operator N, then N is reductive. Hence a cyclic normal operator N is reductive if and only if $cyc(N) = *cyc(N)$.*

Proof. Assume first that T is reductive and $c \in *cyc(T)$. Then the minimal T-reducing subspace containing c, i.e. the space $R(c,T)$ is the whole space H. Since T is reductive, the minimal T-invariant subspace containing c, i.e. the space $I(c,T)$ is T-reducing. Hence $I(c,T) = R(c,T) = H$, i.e. $c \in cyc(T)$.

Assume now that $cyc(N) = *cyc(N)$ for a cyclic *normal* operator N. Assume that X is an N-invariant subspace of H. Denote the smallest N-reducing subspace containing X by $R(X)$, and assume that $R(X) \neq X$. Then there is $x \in X$ such that $I(x,N) \subset X$, but the generated reducing subspace $R(x,N)$ is not contained in X. Denote the orthogonal projection of H onto $R(x,N)$ by P. Then PH is invariant for N and N^*, hence $PN = NP$. The restriction $N|R(x,N)$ is a *-cyclic normal operator for which $x \in *cyc(N|R(x,N))$.

By assumption, N is *-cyclic. Let y_0 be any vector in $*cyc(N)$. Then there is a compactly supported regular Borel measure μ on $\mathbf{C}$ and an isometric isomorphism $V : H \to L^2(\mu)$ such that $VNV^{-1} = N_\mu$ (multiplication by the independent variable on $K := support(\mu)$), and $Vy_0 = \mathbf{1}$ (the function equal to 1 $[\mu]$, i.e. μ-a.e., cf. [CF, IX.3.4]). Then $Vx \in L^2(\mu) \cap VR(x, N)$. Let $h := \{c \in K : [Vx](c) \neq 0\}$, and let $w \in L^2(\mu)$ be any vector with the property that $\{c \in K : w(c) \neq 0\} = K \setminus h$ (*both relations understood* $[\mu]$). Then the vectors x and $V^{-1}w$ are orthogonal in the space H, since their images Vx and w are orthogonal in $L^2(\mu)$. Further, the sum $Vx+w \in L^2(\mu)$ is not zero on K $[\mu]$, hence is a *-cyclic vector for N_μ. It follows that the vector $z := x + V^{-1}w$ is in the set $*cyc(N) = cyc(N)$.

Let y be any vector in $cyc(N)$, and let $\overline{sp}$ denote closed linear span *everywhere in the following.* Then

$$\overline{sp}[N^j y : j \in \mathbf{N}_0] = I(y, N) = H.$$

Assume that an arbitrary $T \in B(H)$ commutes with N. By a well-known theorem of Fuglede (see, e.g., [CS, p.81]), then T commutes with N^*. Hence the closure of the range $\overline{TH}$ is a reducing subspace for N, thus $N|\overline{TH}$ is normal, and

$$\overline{sp}[N^j Ty : j \in \mathbf{N}_0] \supset TI(y, N) = TH.$$

This implies that

$$I(Ty, N|\overline{TH}) = \overline{TI(y, N)} = \overline{TH}.$$

It shows that $Ty \in cyc(N|\overline{TH})$.

The orthogonal projection P of H onto the reducing subspace $R(x, N)$ commutes with N. By the preceding paragraph, then $x = Pz \in cyc(N|PH)$. It follows that

$$I(x, N) = I(x, N|PH) = PH = R(x, N),$$

a contradiction. This shows that $R(X) = X$, i.e. each N-invariant subspace of H is N-reducing. The proof is complete. $\square$

Remark. *The relation $x \in *cyc(N|R(x, N))$ is equivalent to*
$Vx \in *cyc(N_\mu|VR(x, N))$. Indeed, $VN = N_\mu V$ and the Fuglede-Putnam theorem imply $VN^* = N_\mu^* V$. Since V is a homeomorphism,
$VR(x, N) = V\overline{sp}[N^j N^{*k} x : j, k \in \mathbf{N}_0] = \overline{sp}[N_\mu^j N_\mu^{*k} Vx : j, k \in \mathbf{N}_0]$
$= R(Vx, N_\mu)$.
From this the stated equivalence follows.

Remark. *If μ is any scalar-valued spectral measure for the cyclic normal operator N, then N is unitarily equivalent to the operator N_μ of multiplication by the complex variable on the Hilbert space $L^2(\mu) \equiv L^2(\sigma(N), B, \mu)$,* *where B denotes the family of Borel sets. N is reductive if and only if N_μ is reductive. By Theorem 2.1, this holds if and only if $*cyc(N_\mu) = cyc(N_\mu)$. This is the case if and only if for any $f \in L^2(\mu)$*

$$f(z) \neq 0 \quad \mu - a.e. \quad \iff \quad clos\{pf : p \in \mathbf{P}\} = L^2(\mu).$$

Here clos denotes closure in $L^2(\mu)$, and $\mathbf{P}$ denotes the set of all polynomials in z. Taking $f(z) := \mathbf{1}$, we see that if N_μ is cyclic and reductive, then $\mathbf{P}^2(\mu)$, the closure of the polynomials in $L^2(\mu)$, is equal to $L^2(\mu)$.

As usual, call two such measures μ and m as above *equivalent* and write $\mu \sim m$ iff they are mutually absolutely continuous. This is the case if and only if the multiplication operators N_μ and N_m are unitarily equivalent: $N_\mu \cong N_m$. Recall that Bram [BJ, Theorem 6] has proved that *for every cyclic normal operator N_μ there is* a suitable measure m such that $\mu \sim m$ and $\mathbf{P}^2(m) = L^2(m)$.

Sarason [SW, p.14] asked for a characterization of a (fixed) measure μ satisfying $\mathbf{P}^2(\mu) = L^2(\mu)$. Note that the similarity of this question to his characterization of reductivity there is clear. Later, e.g., Trent [T] gave such a characterization. For the structure of the general space $\mathbf{P}^2(\mu)$ see, e.g., Conway [CS, Corollary V.4.4] and Thomson [Th, Theorem 5.8]. We prove now

Theorem 2.2. *A cyclic normal operator N is reductive if and only if (with the notation used in the preceding Remark) for every measure μ equivalent to the scalar-valued spectral measure for N we have*

$$\mathbf{P}^2(\mu) = L^2(\mu).$$

Proof. The only if statement has been proved in the preceding Remark. We prove the if statement now.

Assume that $f \in *cyc(N_m)$ for a fixed measure m equivalent to the scalar-valued spectral measure for N (or, equivalently, for N_m). It means that $f \in L^2(m)$, $|f| > 0$ $[m]$. Define the measure μ by $d\mu := |f|^2 dm$. It is clear that $\mu \sim m$, and is easy to check that

$$L^2(m) f^{-1} = L^2(\mu).$$

By assumption, the latter space is equal to $\mathbf{P}^2(\mu)$. Hence for every $x \in L^2(m)$ there is a sequence $\{p_n\}$ of complex polynomials such that

$$\int |p_n - x f^{-1}|^2 d\mu \to 0 \qquad (n \to \infty).$$

It follows that

$$\int |p_n f - x|^2 dm = \int |p_n - x f^{-1}|^2 |f|^2 dm \to 0 \qquad (n \to \infty).$$

This shows that $f \in cyc(N_m)$. Theorem 2.1 yields that N_m is reductive, hence so is N. □

3 Subnormal operators

For the basics on subnormal operators we refer to the monographs by Conway [CS], [CT]. If S is a (bounded) subnormal operator acting in the Hilbert space H, we shall denote (one fixed of) its minimal normal extension(s), acting in the Hilbert space $K \supset H$, by N. We shall apply the introduced notation to both operators S and N, and add the following one:

We shall write that *condition $C(S)$ holds* if and only if $cyc(S) = *cyc(S)$, and write similarly $C(N)$ for the operator N.

The following facts concerning our problem are well known or can readily be proved with the help of Theorem 2.1 above.

Scholium. *Assume that the subnormal operator S is cyclic. The following statements are equivalent:*

1) S is reductive,

2) N is reductive,

3) $S = N$ and $C(N)$ holds,

4) $N^*H \subset H$ and $C(N)$ holds.

Each of them evidently implies that $C(S)$ holds.

Proof. A proof will be given only in the form of short references. 1) implies 3) by [CS, Proposition VIII.1.15, p.425] (see also [Th, Theorem 5.8]), and Theorem 2.1 above. 3) clearly implies 4), and 4) implies 2) by Theorem 2.1 above. If 2) holds, then pick any $x \in cyc(S)$. Since $N^k x = S^k x$ for $k \in \mathbf{N}_0$, the subspace

$$I(x, N) = I(x, S) = H$$

is N-invariant. By assumption 2), H is then also N^*-invariant. It follows that the space of the minimal normal extension is also H, i.e. the operator $S = N$ is normal and reductive, thus 1) holds. □

Consider the situation that the subnormal operator S *is cyclic and condition $C(S)$ holds*. Note that $C(S)$ is a condition that involves only S (and not the minimal normal extension N). Do then the statements in the Scholium follow?

Working in this direction we shall need the basic fact on *not necessarily reductive* cyclic subnormal operators that was proved by Bram [BJ, Lemma 4] and reproved by Yoshino [Y, Lemma 1]. We shall complete and formulate it here in a slightly more precise form, and shall give the short proof.

Proposition 3.1. *Assume that the subnormal operator S has the cyclic vector $x \in H$. Then the minimal normal extension $N \in B(K)$ is also cyclic, and $x \in *cyc(N)$. Further,*

$$cyc(S) \subset *cyc(N) \cap H \subset *cyc(S).$$

Proof. Consider the subspace

$$M := \overline{sp}[N^{*m}N^n x : m, n \in \mathbf{N}_0].$$

Since $x \in cyc(S)$, and $N^n x = S^n x$ for each $n \in \mathbf{N}_0$, we have $H \subset M$. Clearly, M is the reducing subspace $R(x, N)$. Since N is the minimal normal extension, we have $M = K$, hence $x \in *cyc(N) \cap H$. Bram [BJ, Theorem 6] proved that a normal operator N is *-cyclic if and only if it is cyclic.

Let $h \in *cyc(N) \cap H$. Then the induced reducing subspace $R(h, N)$ is equal to $K \supset H$. Denote the orthogonal projection of K onto H by P. Then $PR(h, N) = PK = H$. Hence (cf. [CT, p.31]) $H = P\overline{sp}[N^{*m}N^n h : m, n \in \mathbf{N}_0] \subset \overline{sp}[PN^{*m}N^n h : m, n \in \mathbf{N}_0] = \overline{sp}[S^{*m}S^n h : m, n \in \mathbf{N}_0] \subset H$. We have obtained $R(h, S) = H$, thus the proof is complete. □

Remark. *The last paragraph shows that $h \in *cyc(N) \cap H$ even implies that h is a strongly *-cyclic vector for S in the terminology of Feldman [F, p.381 or p.387]. This means that $\overline{sp}[S^{*m}S^n h : m, n \in \mathbf{N}_0] = H$.*

The following result is [BJ, Corollary 2, pp. 86-87] formulated in our terminology.

Proposition 3.2. *Assume that S is subnormal on H, N is its minimal normal extension on $K \supset H$, and P denotes the orthogonal projection of K onto H. Then*

$$P[cyc(N^*)] \subset cyc(S^*) \subset *cyc(S).$$

Proof. Since N is normal, by [BJ, Theorem 6], the following three sets are simultaneously void (or not):

$$cyc(N^*), \qquad *cyc(N^*) \equiv *cyc(N), \qquad cyc(N).$$

Assume there is $k \in K \cap cyc(N^*)$ (otherwise there is nothing to prove), and let $g := Pk$. For every $n \in \mathbf{N}_0$ we have then

$$S^{*n}g = S^{*n}Pk = PN^{*n}k.$$

It follows that
$H \supset \overline{sp}[S^{*n}g :\ n \in \mathbf{N}_0] = \overline{sp}[PN^{*n}k :\ n \in \mathbf{N}_0] \supset P\overline{sp}[N^{*n}k :\ n \in \mathbf{N}_0]$
$= PK = H.$
This proves the first containment statement, and the second is evident. $\square$

Remark. *Feldman [F, Corollary 4.12] showed that if $*cyc(S)$ is nonvoid, so is $cyc(S^*)$. Moreover, [F, Corollary 3.2] shows that then the set $cyc(S^*)$ is dense in H, hence the same is valid for $*cyc(S)$.*

Proposition 3.3. *Assume that the subnormal operator S is cyclic, and is not normal. Then there is $g \in cyc(S)$ satisfying $N^*g \notin H$.*

Proof. By assumption, the Hilbert space K of the minimal normal extension N of S properly contains the space H of S. Since

$$K = \overline{sp}[N^{*n}h :\ n \in \mathbf{N}_0,\, h \in H],$$

there is $h \in H$ such that $N^*h \notin H$. By a result of Gehér ([G], see also [SF]), for every cyclic operator S we have $\overline{sp}[cyc(S)] = H$. If we had $N^*[cyc(S)] \subset H$, then we would also have $N^*H \subset H$, a contradiction. $\square$

The following result will describe the relations between generated reducing and invariant subspaces, respectively, for an orthogonal sum decomposition of a general operator.

Proposition 3.4. *Consider any operator $S \in B(H)$ and its decomposition with the help of orthogonal projections P_k satisfying $P_kS = SP_k$ ($k \in \mathbf{N}_0$) into the orthogonal sum*

$$S = S_0 \oplus S_1 \oplus \cdots \oplus S_n \oplus \ldots \qquad (S_k = S|P_kH,\ k \in \mathbf{N}_0).$$

Consider any vector $f \in H$ and its decomposition with the help of these orthogonal projections

$$f = f_0 \oplus f_1 \oplus \cdots \oplus f_n \oplus \cdots \in H,$$

where $f_k = P_kf$ ($k \in \mathbf{N}_0$). Then the generated reducing and invariant subspaces, respectively, satisfy
$R(f, S) \subset \oplus_{k=0}^{\infty} P_kR(f, S) = \oplus_{k=0}^{\infty} R(f_k, S_k),$

$I(f,S) \subset \oplus_{k=0}^{\infty} P_k I(f,S) = \oplus_{k=0}^{\infty} I(f_k,S_k)$.
Hence

$$P_k R(f,S) = R(f_k,S_k), \qquad P_k I(f,S) = I(f_k,S_k) \qquad (k \in \mathbf{N}_0).$$

Proof. We shall prove the statement for the reducing subspaces, the proof for the invariant subspaces being similar and even simpler. Denote

$$X(f) := \oplus_{k=0}^{\infty} R(f_k,S_k).$$

It is clear that

$$H \supset X(f) = \oplus_{k=0}^{\infty} R(f_k,S_k) \ni f_0 \oplus f_1 \oplus \cdots \oplus f_n \oplus \cdots = f.$$

The orthogonal sum of S-reducing subspaces is S-reducing, hence the subspace $X(f)$ is S-reducing. It follows that

$$f \in R(f,S) \subset X(f).$$

Hence we obtain

$$f_k = P_k f \in P_k R(f,S) \subset P_k X(f) = R(f_k,S_k) \quad (k \in \mathbf{N}_0).$$

We show that the subspace $P_k R(f,S)$ is a reducing subspace for S_k for each k. Indeed,

$$S_k P_k R(f,S) = S P_k R(f,S) = P_k S R(f,S) \subset P_k R(f,S),$$

since S leaves $R(f,S)$ invariant. The adjoint of an orthogonal sum is the orthogonal sum of the adjoints of the summands, hence we obtain

$$(S_k)^* P_k R(f,S) = S^* P_k R(f,S) = P_k S^* R(f,S) \subset P_k R(f,S).$$

The fact that $f_k \in P_k R(f,S)$ and the minimality of the reducing subspace $R(f_k,S_k)$ imply that $R(f_k,S_k) \subset P_k R(f,S)$. Hence

$$P_k R(f,S) = R(f_k,S_k) \quad (k \in \mathbf{N}_0).$$

□

Remark. *For the generated invariant subspaces the stated containment relation may be proper. Indeed, let* m *denote normalized Lebesgue measure on the unit circle* $\mathbf{T}$, *and let* c_k *be closed, pairwise disjoint arcs of* $\mathbf{T}$ *such that* $m(\mathbf{T} \setminus \cup_{k=0}^{\infty} c_k) = 0$, *and* $f := \mathbf{1} \in L^2(m)$. *Then polynomials are dense in* $L^2(m|c_k)$ *for every* $k \in \mathbf{N}_0$, *but not in* $L^2(m)$. *Denote by* S *the operator of multiplication by the variable in* $L^2(m)$, *and apply the notation of Proposition 3.4. It follows that*

$$I(\mathbf{1},S) = P^2(m) \neq L^2(m) = \oplus_{k=0}^{\infty} I(\mathbf{1}_k,S_k).$$

The next theorem is our main result on a cyclic subnormal operator S satisfying condition $C(S)$, and its proof is based on Proposition 3.1.

Theorem 3.5. *If the subnormal operator S is cyclic and condition $C(S)$ holds, then $S = N$. Hence S is reductive.*

Proof. It is known that S is a cyclic subnormal operator if and only if S is unitarily equivalent to the operator S_μ of multiplication by the variable z on the space $\mathbf{P}^2(\mu)$ for some compactly supported Borel measure μ on $\mathbf{C}$. Hence there are pairwise disjoint Borel sets $b_0, b_1, \dots$ in $\mathbf{C}$ such that the restricted measures $\mu_n := \mu | b_n$ $(n \in \mathbf{N}_0)$ satisfy (cf. [Th, Theorem 5.8], [CS, pp. 297-298])

1) $\mu = \sum_{n=0}^{\infty} \mu_n$,

2) $H \equiv \mathbf{P}^2(\mu) = L^2(\mu_0) \oplus \mathbf{P}^2(\mu_1) \oplus \mathbf{P}^2(\mu_2) \oplus \dots$,

3) for $n \in \mathbf{N}$ the subspace $\mathbf{P}^2(\mu_n)$ is either infinite dimensional and contains no nontrivial characteristic functions, or is the zero subspace.

Here the generalized Hardy spaces $\mathbf{P}^2(\mu), \mathbf{P}^2(\mu_n)$ are the closures of the spaces of the polynomials in $L^2(\mu), L^2(\mu_n)$, respectively. The operators $S_n := S_\mu | \mathbf{P}^2(\mu_n)$ are cyclic subnormal irreducible operators for every $n \in \mathbf{N}$, and $S_0 := S_\mu | L^2(\mu_0)$ is normal. The irreducibility of S_n $(n \in \mathbf{N})$ implies that for each such n every nonzero vector $f_n \in \mathbf{P}^2(\mu_n)$ is *-cyclic for S_n: otherwise some $R(f_n, S_n)$ would be an S_n-reducing subspace of $\mathbf{P}^2(\mu_n)$.

Assume that for a fixed $n \in \mathbf{N}$ the subspace $\mathbf{P}^2(\mu_n)$ is infinite dimensional. Then the invariant subspace theorem of S.Brown [BS] implies that there is a nonzero vector $g_n \notin cyc(S_n)$. [CT, Example 2.13, p.41] shows that the minimal normal extension of the operator S_μ is the operator N_μ acting in Ł$^2(\mu)$. By assumption and by Proposition 3.1, we have

$$cyc(S_\mu) = *cyc(N_\mu) \cap H = *cyc(S_\mu).$$

It is known that the middle set is equal to

$$\{h \in \mathbf{P}^2(\mu) : \ h \neq 0 \quad [\mu]\}.$$

Denote the orthogonal projection of $\mathbf{P}^2(\mu)$ onto $\mathbf{P}^2(\mu_n)$ by P_n, and multiplication by the characteristic function χ_n of the set b_n from above by M_n. Then (cf. [Th, Theorem 5.8])

$$P_n \mathbf{P}^2(\mu) = M_n \mathbf{P}^2(\mu) = \mathbf{P}^2(\mu_n).$$

We shall show that

$$cyc(S_n) = M_n[cyc(S_\mu)] = M_n[*cyc(S_\mu)] = *cyc(S_n),$$

and start with the left-hand side equality. Let $h \in cyc(S_\mu)$. If $f \in \mathbf{P}^2(\mu)$, then for some sequence $\{p_k\}$ of polynomials (in one variable)

$$\int |p_k h - f|^2 d\mu \to 0 \qquad (k \to \infty).$$

Then

$$\int |p_k \chi_n h - \chi_n f|^2 d\mu = \int |p_k h - f|^2 \chi_n d\mu \to 0 \qquad (k \to \infty).$$

By the orthogonal sum representation 2), an arbitrary element $f_n \in \mathbf{P}^2(\mu_n)$ has the form $\chi_n f$, where $f \in \mathbf{P}^2(\mu)$. Since $d\mu_n = \chi_n d\mu$, we see that $\chi_n h$ is an element of $cyc(S_n)$. We have proved that

$$cyc(S_n) \supset M_n[cyc(S_\mu)].$$

To prove the converse containment relation, let $h_n \in cyc(S_n)$. Then for every $f_n \in \mathbf{P}^2(\mu_n)$ there is a sequence $\{p_k\}$ of polynomials (in one variable) such that

$$\int |p_k h_n - f_n|^2 d\mu_n \to 0 \qquad (k \to \infty).$$

It follows that $h_n \neq 0 \quad [\mu_n]$. Indeed, $\mu_n\{h_n = 0\} > 0$ would imply that the function (element) $e_n \in \mathbf{P}^2(\mu_n)$ equal to 1 on this set cannot be approximated as prescribed above. This shows that there is $h \in \mathbf{P}^2(\mu)$ satisfying

$$h_n = \chi_n h, \qquad h \neq 0 \quad [\mu].$$

Thus $h \in cyc(S_\mu)$, and

$$cyc(S_n) \subset M_n cyc(S_\mu).$$

It follows that we have equality here. The proof of the stated equality for the *-cyclic sets is even simpler.

We have thus obtained that

$$cyc(S_n) = *cyc(S_n) = \mathbf{P}^2(\mu_n) \setminus \{0\}.$$

This shows that for any nonzero vector $g_n \in \mathbf{P}^2(\mu_n) \setminus cyc(S_n)$ is impossible. It follows that each subspace $\mathbf{P}^2(\mu_n) \quad (n \in \mathbf{N})$ is the zero subspace, and the operator S_μ is normal. By Theorem 2.1, $S = N$ is reductive. $\square$

Corollary. *Let H be a (not necessarily separable) complex Hilbert space, and let $N \in B(H)$ be a reductive (sub)normal operator. Then there are reducing subspaces H_k $(k \in \omega)$ for N such that*

$$H = \oplus_{k\in\omega} H_k, \qquad N = \oplus_{k\in\omega}(N|H_k),$$

and the summands $N|H_k =: N_k$ satisfy

$$cyc(N_k) = *cyc(N_k) \neq \emptyset \qquad (k \in \omega).$$

In the converse direction: assume that for a subnormal (or normal) operator $N \in B(H)$ there is an orthogonal decomposition with the properties above. Then the operator N is not necessarily reductive.

Proof. Assume first that the operator N is reductive subnormal. It is known ([CS, p.425]) that then N is normal. It is also known that each normal operator, hence N, can be decomposed into an orthogonal sum of restrictions to *-cyclic subspaces H_k for N. The parts $N_k := N|H_k$ of the reductive operator N are clearly reductive and cyclic. Applying Theorem 2.1 to each part, we obtain

$$cyc(N_k) = *cyc(N_k) \neq \emptyset \qquad (k \in \omega).$$

In the converse direction: by the assumption and Theorem 2.1, each part N_k is reductive. The following remark will explicitly show that the orthogonal sum N of reductive normal operators N_k is not necessarily reductive. $\square$

Remark. *Concerning the last sentence see the remarkable paper by Wiggen [Wi]. A simple example there shows that there are unitary reductive operators such that their orthogonal sum is the (nonreductive unitary) bilateral shift of multiplicity 1. On the other hand, it is proved there that the statement*

"if $T \in B(H)$ is reductive, then $T \oplus T$ is reductive (on $H \oplus H$)"

is equivalent to the statement that each operator in $B(H)$ has a nontrivial invariant subspace in a Hilbert space H of dimension greater than 1.

The well-known remarkable result of Dyer, Pedersen and Porcelli [D] on reductive operators and the invariant subspace problem can be completed as follows.

Theorem 3.6. *Assume that the complex Hilbert space H is separable. The following are equivalent:*

1) each reductive operator $N \in B(H)$ is normal,

2) each reductive operator $N \in B(H)$ is subnormal,

3) each reductive operator $N \in B(H)$ satisfies $*cyc(N) \neq H \setminus \{0\}$,

4) each operator $T \in B(H)$ satisfies $cyc(T) \neq H \setminus \{0\}$,

5) each operator $T \in B(H)$ has a nontrivial invariant subspace.

Proof. 1) clearly implies 2) (in fact, they are known to be equivalent, see, e.g., [CS, p.425]). They imply that each reductive operator has a nontrivial invariant subspace, hence there exists a nonzero noncyclic vector for N. By Theorem 2.1, then 3) holds. If 3) is valid, but 4) is not, then there is an operator $T \in B(H)$ such that $cyc(T) = H \setminus \{0\}$. Hence for every nonzero $x \in H$ we have

$$I(x, T) = H = R(x, T).$$

It follows that T is reductive, and $*cyc(T) = H \setminus \{0\}$ contradicts 3). If 4) holds, then each operator $T \in B(H)$ has a nonzero noncyclic vector, hence a nontrivial invariant subspace, thus 5) is valid. A proof that 5) implies 1) is given in [A]. □

Remark. *It is clear that for any operator $T \in B(H)$ and any vector $x \in H$ we have*

$$R(x, T) = R(x, T^*).$$

We have seen that if T is reductive and $x \in H$, this implies that $I(x, T) = H \iff I(x, T^*) = H$, which means

$$cyc(T) = cyc(T^*).$$

The question may arise whether the last equality for a cyclic operator T implies that T is reductive. The negative answer is contained in the fact that this last equality holds (cf. [HP, Problem 164]) for each cyclic normal operator T (including the nonreductive ones).

The following example may illustrate some results of the paper.

Example Let μ be a nonnegative measure defined on the Borel sets of the unit circle

$$\mathbf{T} := \{z \in \mathbf{C} : \ |z| = 1\},$$

and let m denote normalized Lebesgue measure on $\mathbf{T}$ (satisfying $m(\mathbf{T}) = 1$). On the Hilbert space $H := L^2(\mathbf{T}, \mu)$ consider the operator $T := N_\mu$ of multiplication by the variable $z \in \mathbf{T}$. It is well-known to be unitary, and [NO, 4.8.1-2, p. 75] shows that for $f \in L^2(\mathbf{T}, \mu)$ we have

$$f \in *cyc(N_\mu) \iff |f(z)| > 0 \quad [\mu].$$

Further, the decomposition $\mu = \mu_a + \mu_s$ into absolutely continuous and singular parts (with respect to m) implies

$$f = f_a \oplus f_s \in L^2(\mathbf{T}, \mu_a) \oplus L^2(\mathbf{T}, \mu_s), \qquad d\mu = w dm + d\mu_s,$$

where w is the Radon-Nikodym derivative $d\mu_a/dm$. [NO, 4.8.1-2, p. 75] shows that

$$f \in cyc(N_\mu) \Longleftrightarrow |f(z)| > 0 \quad [\mu] \quad \text{and} \quad \log |f_a w^{1/2}| \notin L^1(\mathbf{T}, m).$$

Applying Theorem 2.1, N_μ is reductive if and only if for every $f \in L^2(\mathbf{T}, \mu)$

$$|f(z)| > 0 \quad [\mu] \Longrightarrow \log |f_a w^{1/2}| \notin L^1(\mathbf{T}, m).$$

Consider first the special case $\mu = m$. Then $f_a = f, \quad w \equiv 1$. Hence N_m is reductive if and only if $\log |f| \notin L^1(\mathbf{T}, m)$ for *every* $f \in L^2(\mathbf{T}, m)$ satisfying $|f(z)| > 0 \quad [m]$, which is clearly false.

Consider now an other case, when μ is 0 on an arc of $\mathbf{T}$, and is identical to m outside this arc. Then $w = 0$ on this arc, hence $\log |f_a w^{1/2}| \notin L^1(\mathbf{T}, m)$. It follows that the operator N_μ is reductive (cf. also [W, Theorem 7]).

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