Randomly perturbed self-similar sets

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Abstract: We are given a self-similar Iterated Function System (IFS) on the real line. This is a finite list of contracting similarity mappings $\mathcal{S} := \{S_1, \dots, S_m\}$. We fix a sufficiently large interval $\widehat{I}$ which is sent into itself by all mappings of $\mathcal{S}$. For an arbitrary $n \geq 1$, and $\mathbf{i} = (i_1, \dots, i_n) \in \{1, \dots, m\}^n$ the corresponding level n cylinder interval is

$$I_{i_1,\dots,i_n} := S_{i_1} \circ \cdots \circ S_{i_n}(\widehat{I}). \tag{1}$$

The collection of level n cylinder intervals is

$$\mathcal{I}_n := \{I_{i_1,\dots,i_n} : (i_1, \dots, i_n) \in \{1, \dots, m\}^n\}.$$

The attractor Λ of $\mathcal{S}$ is the set that remains if we iterate this system on $\widehat{I}$ infinitely many times:

$$\Lambda := \bigcap_{n=1}^{\infty} \bigcup_{I \in \mathcal{I}_n} I. \tag{2}$$

We say that Λ is self-similar set. For example the well-known triadic Cantor set is a self-similar set for the IFS $\{S_1(x) = x/3, S_2(x) = x/2 + 2/3\}$. Here $\widehat{I} = [0, 1]$ and then $I_1 = \left[0, \frac{1}{3}\right]$, $I_2 = \left[\frac{2}{3}, 1\right]$.

Open Problem Is there a self-similar set of positive Lebesgue measure and empty interior on the line?

We consider this problem for randomly perturbed self-similar sets, which are obtained in the following way: In the randomly perturbed case, the n-cylinder interval $\widetilde{I}_{i_1,\dots,i_n}$ corresponding to the indices $\mathbf{i} = (i_1, \dots, i_n) \in \{1, \dots, m\}^n$ is obtained by replacing S_{i_k} in formula (1) by a random and independent of everything translation $\widetilde{S}_{i_k}$ of S_{i_k} for all $k = 1, \dots, n$. Then we build the randomly perturbed attractor in an analogous way to formula (2) from the randomly perturbed cylinder intervals $\widetilde{I}_{i_1 \dots i_n}$. That is

$$\widetilde{\mathcal{I}}_n := \left\{\widetilde{I}_{i_1,\dots,i_n} : (i_1, \dots, i_n) \in \{1, \dots, m\}^n\right\},$$

and the randomly perturbed self-similar set is $\widetilde{\Lambda} := \bigcap_{n=1}^{\infty} \bigcup_{\widetilde{I} \in \widetilde{\mathcal{I}}_n} \widetilde{I}$.

First, I review results related to the Lebesgue measure and Hausdorff dimension of these randomly perturbed self-similar sets. Then, I turn to our new result (joint with M. Dekking, B. Szekely, and N. Szekeres) about the existence of interior points in these randomly perturbed self-similar sets.